Student Name and ID Number

MATH 3012 Quiz 1, September 21, 2017, WTT

Important Note: These solutions have been prepared in the format of a "study guide." For this reason, the answers are more complete than what students are expected to provide on their exam papers. Also, spacing has been altered to accommodate additional comments on solutions. Point values are given at the very end.

1. Consider the 52-element set consisting of the upper and lower case letters of the English alphabet.

a. How many strings of length 16 can be formed if repetition of symbols is *not* permitted?

> This is the basic problem of counting the number of permutations of length n from an m letter alphabet. The answer is $P(m, n)$ in general, so in this specific instance, the answer is $P(52, 16)$.

b. How many strings of length 16 can be formed if repetition of symbols is permitted?

> With repetition allowed, the number of strings of length n from an alphabet of size m is m^n. So in this specific case, the answer is 52^{16}.

c. How many strings of length 16 can be formed using exactly three x's, four B's and nine A's?

> This is a "MISSISSIPPI" problem and the answer is the multinomial coefficient $\binom{16}{3,4,9}$. Alternatively, this answer can be written as $\frac{16!}{3!4!9!}$.

d. How many strings of length 16 can be formed using exactly three x's, four B's and nine A's, with the four B's occuring consecutively?

> We simply treat the four B's as a single character. Now the string will have length 13 and consist of three x's, one $BBBB$ and nine A's. So the answer is the multinomial coefficient $\binom{13}{3,1,9}$. As before, this can be written as $\frac{13!}{3!1!9!}$.

2. How many lattice paths from $(0, 0)$ to $(12, 15)$ pass through $(4, 9)$?

> The answer will the product of the number of ways to go from $(0, 0)$ to $(4, 9)$ and the number of ways to go from $(4, 9)$ to $(12, 15)$. For the first part, we have $\binom{13}{4}$. For the second part, we note that the number of ways to go from $(4, 9)$ to $(12, 15)$ is the same as the number of ways to go from $(0, 0)$ to $(8, 6)$, which is $\binom{14}{8}$. So the answer is $\binom{13}{4}\binom{14}{8}$. Note that the answer can be written four different ways since $\binom{13}{4} = \binom{13}{9}$ and $\binom{14}{8} = \binom{14}{6}$.

3. A wealthy donor intends to donate \$1,000,000 to Georgia Tech by partitioning the total into four awards which will then be distributed to the Schools of (1) Mathematics, (2) Computer Science, (3) Industrial Science and Engineering and (4) Electrical Engineering. Each of the schools will receive an amount which is a positive multiple of \$10,000.

a. In how many different ways can the donor distribute this sum?

Since $1,000,000 = 10^6$ and $10,000 = 10^4$, essentially there are $10^{6-4} = 10^2 = 100$ objects, with each object being an amount of \$10,000, say a single bill of that denomination. The restriction that each school will receive some award turns this into the classic problem of distributing 100 non-distinct objects among four distinct cells, with each cell non-empty. So the answer is: $\binom{99}{3}$.

b. In how many different ways can the donor distribute this sum if the School of Mathematics will receive at least \$300,000?

To insure that Math receives at least \$300,000, we set aside \$290,000 and then require that beyond that, require that Math gets a positive amount. This implies that 29 objects will be set aside, so that 71 remain. Now the answer is: $\binom{70}{3}$.

c. In how many different ways can the donor distribute this sum if the School of Mathematics will receive at least \$300,000 and at most \$500,000

One of the distributions we've just counted is "bad" if Math gets more than 50 objects. This would result from a set aside of 50 with the further requirement that Math gets at least one object. Setting aside 50 means 50 remain, so the number of bad distributions is $\binom{49}{3}$ and the final answer is: $\binom{70}{3} - \binom{49}{3}$.

4. Use the Euclidean algorithm to find $d = \gcd(306, 1190)$.

Using long division, we calculate:

$$\begin{aligned} 1190 &= 3 \cdot 306 + 272 \\ 306 &= 1 \cdot 272 + 34 \\ 272 &= 8 \cdot 34 + 0 \end{aligned}$$

The last positive remainder is the greatest common divisor, so $34 = \gcd(306, 1190)$.

5. Use your work in the preceding problem to find integers a and b so that $d = 306a + 1190b$.

We rewrite the first two equations as follows:

$$\begin{aligned} 272 &= 1 \cdot 1190 - 3 \cdot 306 \\ 34 &= 1 \cdot 306 - 1 \cdot 272 \end{aligned}$$

Substitute the second into the first to obtain:

$$\begin{aligned} 34 &= 1 \cdot 306 - 1 \cdot 272 \\ &= 1 \cdot 306 - 1(1 \cdot 1190 - 3 \cdot 306) \\ &= (1+3) \cdot 306 - 1 \cdot 1190 \\ &= 4 \cdot 306 - 1 \cdot 1190 \end{aligned}$$

So a correct answer is $a = 4$ and $b = -1$. *Note.* There are infinitely many correct answers, and they have the form $a = 4+1190n$, $b = -1-306n$, where n is an integer (positive, negative or zero). The solution we found is just the single case $n = 0$.

6. For a positive integer n, let t_n count the number of ternary strings of length n that do not contain 02 or 201 as a substring. Note that $t_1 = 3$, $t_2 = 8$, $t_3 = 20$. Develop a recurrence relation for t_n and use it to compute t_4 and t_5.

This will be the single most challenging problem on the test. First, lets explain the values of t_1, t_2 and t_3. We note that $t_1 = 3$ since there are 3 ternary strings of length 1 and they are all "good." There are 9 strings of length 2 but the string 02 is bad. The remaining 8 are good, so $t_2 = 8$. There are 27 strings of length 3 but the following 7 strings are bad:

002, 102, 202, 020, 021, 022, 201

So $t_3 = 27 - 7 = 20$.

Now we turn our attention to obtaining a recursive formula for t_n. Consider an integer $n \geq 4$ and the family $\mathcal{F}$ of all good strings of length n. We partition this family as $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1 \cup \mathcal{F}_2$, where for $i = 0, 1, 2$, $\mathcal{F}_i$ is the set of all good strings of length n which end in i.

First consider the subfamily $\mathcal{F}_0$. The first $n-1$ positions of a string in this family is a good string of length $n-1$. Furthermore, a good string of length $n-1$ remains good if a 0 is added at the end. Therefore, $|\mathcal{F}_0| = t_{n-1}$.

Now consider the subfamily $\mathcal{F}_2$. Again the first $n-1$ positions of a string in this familly is a good string of length $n-1$, but now there are some good strings of length $n-1$ which turn bad when a 2 is added at the end. These are the good strings of length $n-1$ which end with a 0. By the argument in the preceding paragraph, the number of such strings is t_{n-2}. Therefore $|\mathcal{F}_2| = t_{n-1} - t_{n-2}$.

Finally, we consider the subfamily $\mathcal{F}_1$. Once again the first $n-1$ positions of a string in this familly is a good string of length $n-1$, but there are some good strings of length $n-1$ which turn bad when a 1 is added at These are the good strings of length $n-1$ which end with 20. In front of the 20 is a good string of length $n-3$ which does not end in a 0. By the argument of the previous paragraph, the number of such strings is $t_{n-3} - t_{n-4}$. Therefore, $|\mathcal{F}_1| = t_{n-1} - (t_{n-3} - t_{n-4}) = t_{n-1} - t_{n-3} + t_{n-4}$.

As a consequence of these calculations, we have:

$$\begin{aligned} t_n &= |\mathcal{F}| \\ &= |\mathcal{F}_0| + |\mathcal{F}_1| + |\mathcal{F}_2| \\ &= t_{n-1} + t_{n-1} - t_{n-3} + t_{n-4} + t_{n-1} - t_{n-2} \\ &= 3t_{n-1} - t_{n-2} - t_{n-3} + t_{n-4}. \end{aligned}$$

Furthermore, it is clear from the argument that $t_0 = 1$. Therefore, we compute:

$$\begin{aligned} t_4 = 3t_3 - t_2 - t_1 + t_4 &= 3 \cdot 20 - 8 - 3 + 1 = 50 \quad \text{and} \\ t_5 = 3t_4 - t_3 - t_2 + t_1 &= 3 \cdot 50 - 20 - 8 + 3 = 125 \end{aligned}$$

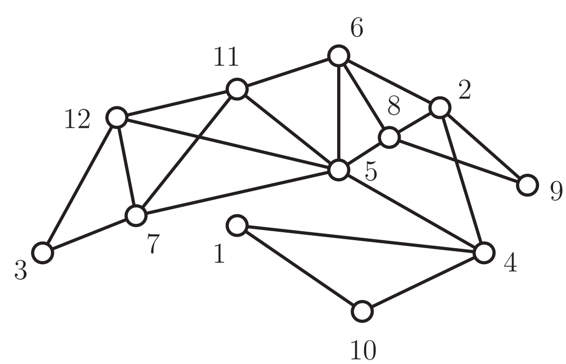
Note. Some of you may feel uncomfortable with the assertion that it is clear that we must have $t_0 = 1$. An alternative way to proceed is to calculate the value of t_4 directly.

There are 9 bad strings of the form $02**$ and nine more bad strings of the form $*02*$. There are also nine bad strings of the form $**02$, but one of them, namely 0202 has already been counted. So there are $9+9+8 = 26$ strings of length 4 which are bad because they contain the substring 01.

There are 3 bad strings of the form $201*$ and none of these contains 02 so they have not yet been counted. There are also 3 bad strings of the form $*201$, but one of them, namely 0201 has already been counted. So there are $3 + 2 = 5$ bad strings containing 201 but not 02.

Altogether, there are 81 ternary strings of length 4. We have counted $26 + 5 = 31$ bad strings so $t_4 = 81 - 31 = 50$, just as the calculation above gives *provided* we take $t_0 = 1$.

7. Use the greedy algorithm developed in class (always proceed to the lowest legal vertex) to find an Euler circuit in the graph G shown below (use node 1 as root):



The first pass of the algorithm yields:

$$(1, 4, 2, 6, 5, 4, 10, 1)$$

We scan this sequence left-to-right and find the first vertex incident with an edge not yet traversed. This is vertex 2. We start again from this vertex and obtain:

$$(2, 8, 5, 7, 3, 12, 5, 11, 6, 8, 9, 2)$$

The second pass is inserted in the first to form the combined sequence:

$$(1, 4, 2, 8, 5, 7, 3, 12, 5, 11, 6, 8, 9, 2, 6, 5, 4, 10, 1)$$

Again, we scan to find the first vertex incident with an edge not yet traversed. This is vertex 7. From this vertex, we get:

$$(7, 11, 12, 7)$$

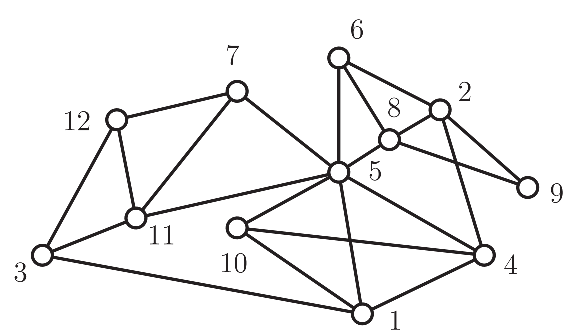
We insert this sequence to obtain:

$$(1, 4, 2, 8, 5, 7, 11, 12, 7, 3, 12, 5, 11, 6, 8, 9, 2, 6, 5, 4, 10, 1)$$

Now we scan and discover that there are no vertices incident with unscanned edges. Also, we see that we have traversed all edges and we now have the desired Euler circuit.

8. *Note.* The wording of the following problem has been changed in these study notes.

Consider the following graph:



(a) Find a clique of size 4.

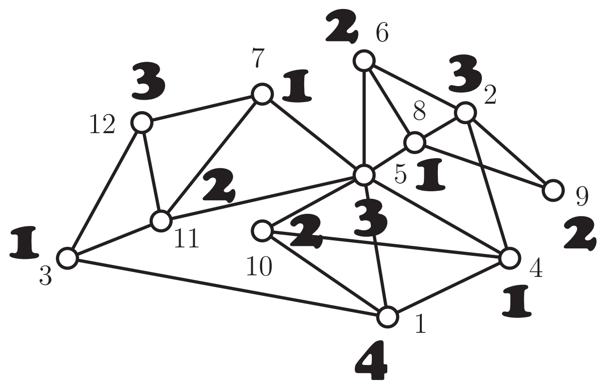
There is only one clique of size 4 and it consists of the vertices in $\{1, 4, 5, 10\}$. This is a set and the order of the elements does not matter.

(b) Find an induced cycle of size 5.

I am fairly certain that there is only one *induced* cycle of size 5 in this graph. This is the subgraph determined by the vertices in $\{1, 3, 12, 7, 5\}$. Note that there are other cycles of size 5, but they are not induced cycles as the problem specified.

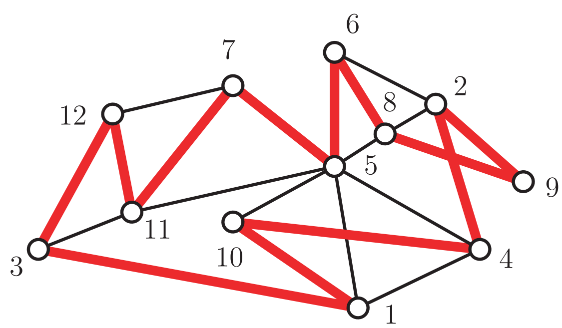
(c) Show that $\chi(G) \leq 4$ by producing a proper coloring using the elements of $\{1, 2, 3, 4\}$ as colors.

There are many correct answers. One such coloring is given in the diagram immediately below:



(d) Show that the graph has a Hamiltonian cycle starting with $(1, 3, 12, 11)$.

I am fairly certain that this graph has (essentially) only one Hamiltonian cycle. This is illustrated by the darkened edges in the diagram immediately below:



9. True–False. Mark in the left margin.

FALSE 1. $P(10, 3) = 760$. *Note.* $P(10, 3) = 10 \cdot 9 \cdot 8 = 720$.

TRUE 2. $C(10, 3) = 120$. *Note.* $C(10, 3) = \binom{10}{3} = (10 \cdot 9 \cdot 8)/3 \cdot 2 \cdot 1 = 720/6 = 120$.

FALSE 3. Any connected graph with an even number of edges has an Euler circuit. *Note.* Just because there is an even number of edges in a graph, there can still be vertices of odd degree. In fact, there can be arbitrarily many vertices of odd degree. For example, consider the complete bipartite graph $K_{2n+1,2n}$. The number of edges is $(2n + 1)(2n)$, which is even. But there are $2n$ vertices of odd degree.

TRUE 4. Any sequence of 37 distinct real numbers either contains an increasing subsequence of length 5 or a decreasing subsequence of length 10. *Note.* The Erdős-Szekeres theorem asserts that a sequence of $nm + 1$ distinct real numbers has either an increasing subsequence of length $n + 1$

or a decreasing subsequence of length $m+1$. In this problem, we note that $37 = 4 \cdot 9 + 1$, so we set $n = 4$ and $m = 9$ to get the desired conclusion.

TRUE 5. There is a connected graph with 100 vertices and 100 edges which does not have a Hamiltonian cycle. *Note.* Consider a graph which consists of a triangle with a path of 97 vertices attached.

FALSE 6. If G is a graph on 20 vertices and every vertex has a least 12 neighbors, then G has a Hamiltonian cycle. *Note.* Dirac's theorem says that a graph on 20 vertices has a Hamiltonian cycle if each vertex has degree at least $\lceil 20/2 \rceil = 10$. Since $12 \geq 10$, such a graph would have a Hamiltonian cycle.

FALSE 7. The number of lattice paths from $(0,0)$ to $(12,12)$ which do not go above the diagonal is the Catalan number $\binom{12}{6}/7$. *Note.* The number of latttice paths from $(0,0)$ to (n,n) which do not go above the diagonal is the Catalan number $\binom{2n}{n}/(n+1)$. In this specific instance, $n = 12$ so the correct answer $\binom{24}{12}/13$.

FALSE 8. If G is a graph and $\chi(G) = 3$, then $\omega(G) = 3$. *Note.* The 5-cycle C_5 has chromatic number 3 and max clique size 2, i.e., $\chi(G_5) = 3$ and $\omega(C_5) = 2$.

TRUE 9. $\log n = O(\sqrt{n})$. *Note.* Setting $C = 1$, we observe that $\log n \leq C\sqrt{n}$ as n tends to infinity. Of course, this is a *very* weak inequality, as the next problem makes clear.

TRUE 9. $\log n = o(\sqrt{n})$. *Note.* As n tends to infinity, the ratio $\log n/\sqrt{n}$ tends to 0.

TRUE 10. $587n^{587} = o(2^n)$. *Note.* As n tends to infinity, the ratio $587n^{587}/2^n$ tends to 0.

FALSE 11. $2^n = O(587n^{587})$. *Note* There is no constant C so that $2^n \leq C587n^{587}$ when n is arbitrarily large.

FUN 12. Connected tilings of recursive multinomial graphs admit Euler circuits with vertices having clique size larger than their chromatic number, except when they have Hamiltonian certificates whose correctness can be estimated in polynomial time by a biased referee.

Point Totals

1. 12 points. 4 parts, each worth 4 points.
2. 7 points.
3. 9 points. 3 parts, each worth 3 points.
4. 7 points.
5. 7 points.
6. 15 points.
7. 15 points.
8. 16 points. 4 parts, each worth 4 points.
9. 12 points. 12 questions, each worth 1 point.

Total. 100 points for all 9 questions.