Solutions

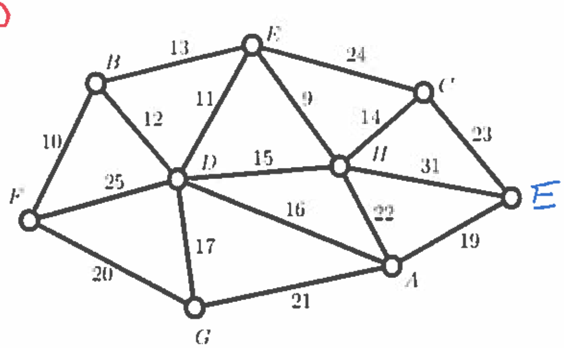
Student Name and ID Number

MATH 3012, Quiz 3, April 16, 2018, WTT

16

8+8

1. A graph with weights on edges is shown below. In the space to the right of the figure, list *in order* the edges which make up a minimum weight spanning tree using, respectively, Kruskal's Algorithm (avoid cycles) and Prim's Algorithm (build tree). For Prim, use vertex A as the root.



Kruskal	Prim
EH 9	AD 16
BF 10	DE 11
DE 11	EH 9
BD 12	DB 12
CH 14	BF 10
DA 16	CH 14
DG 17	DG 17
AE 19	AE 19

12

3×4

2. a. What is the probability that a hand of five cards from a standard deck of 52 cards would be classified as "two pairs". This means the five cards are of the form $\{x, x, y, y, z\}$.

$$\frac{\binom{13}{2}\binom{4}{2}\binom{4}{2}\underbrace{\binom{11}{1}\binom{4}{1}}_{\text{or } \binom{44}{1}}}{\binom{52}{5}}$$

b. In a Bernoulli trial experiment, the probability of success is 1/7. If 1000 trials are conducted, what is the probability that the number of successes is 143?

$$\binom{1000}{143}\left(\frac{1}{7}\right)^{143}\left(\frac{6}{7}\right)^{857}$$

c. Alice rolls a pair of dice until the score is either less than five—in which case she loses four matchsticks—or greater than eight—and in this case she wins three matchsticks. What is the expected value of game? *Note.* The answer can be positive, negative or zero. or more than eight.

$Pr(<5) = 1+3;\ 1+2;\ 1+1;\ 2+2;\ 2+1;\ 3+1$ So $\frac{6}{36}$

$Pr(>8) = 6+6;\ 6+5;\ 6+4;\ 6+3;\ 5+6;\ 5+5;\ 5+4;\ 4+6;\ 4+5;\ 3+6$ So $\frac{10}{36}$

$$\frac{S}{S+F} = \frac{\frac{10}{36}}{\frac{6}{36}+\frac{10}{36}} = \frac{10}{16} = \frac{5}{8} \qquad \frac{F}{S+F} = \frac{3}{8}$$

$$E = \frac{5}{8}\cdot 3 + \frac{3}{8}(-4) = \frac{15-12}{8} = \frac{3}{8}$$

20

3. A data file `digraph_data.txt` has been read for a digraph whose vertex set is $\{1, 2, \ldots, 7\}$. The weights on the directed edges are shown in the matrix below. The entry $w(i, j)$ denotes the length of the edge from i to j. If there is no entry, then the edge is not present in the graph. Apply Dijkstra's algorithm to find the distance from vertex 1 to all other vertices in the graph. Also, for each x, find a shortest path from 1 to x. Please show your work.

W	1	2	3	4	5	6	7
1	0		18	50	12	35	
2		0	4		30	23	60
3			0		41	18	61
4	27	30	28	0		45	20
5			2		0	21	79
6	82	55	3	82	2	0	
7		8	22	4	18	12	0

P(1,2) ∞   P(1,3) 18   P(1,4) 50 $\boxed{P(1,5)\ 12}$ P(1,6) 35   P(1,7) ∞

P(1,2) ∞ $\boxed{P(1,5,3)\ 14}$ P(1,4) 50   P(1,5,6) 33   P(1,5,7) 91

P(1,2) ∞   P(1,4) 50 $\boxed{P(1,5,3,6)\ 32}$ P(1,5,3,7) 75

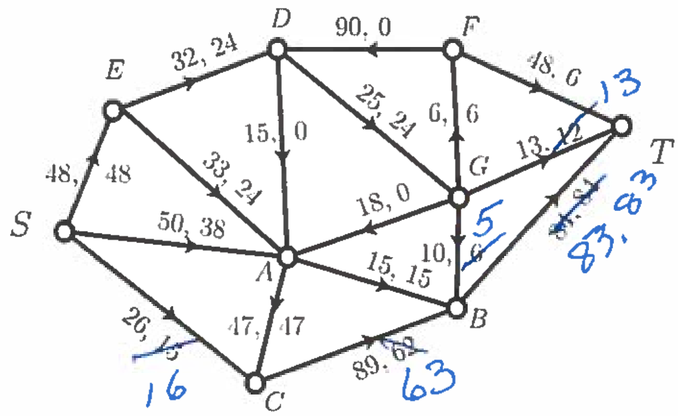
P(1,5,3,6,2) 87 $\boxed{P(1,4)\ 50}$ P(1,5,3,7) 75 $\boxed{P(1,4,7)\ 70}$

P(1,4,2) 80

$\boxed{P(1,4,7,2)\ 78}$

30

4. Consider the following network flow:



a. What is the current value of the flow?

3

48+38+15 = 101

b. What is the capacity of the cut $V = \{S, A, E, C\} \cup \{T, B, D, F, G\}$.

3

32+15+89 = 136

c. Carry out the labelling algorithm, using the pseudo-alphabetic order on the vertices and list below the labels which will be given to the vertices.

6

S (*, +, ∞)
A (S, +, 12)
C (S, +, 11)
E (A, −, 12)
B (C, +, 11)
D (E, +, 8)
G (B, −, 6)
~~E (G, , 6)~~
T (G, +, 1)

d. What is the augmenting path identified by the labelling algorithm.

3

S, C, B, G, T

e. The augmenting path informs us how the flow should be increased. Make these changes by marking directly on the diagram for the network.

3

f. What is the value of the new flow?

3

101+1 = 102

g. Carry out the labelling algorithm on the updated network flow, again using the pseudo-alphabetic order on the vertices and list below the labels which will be given to the vertices. Hint. The algorithm should halt without the sink being labelled. Again, use two columns.

6

S(*, +, ∞)
A(S, +, 12)
C(S, +, 10)
E(A, −, 12)
B(C, +, 10)
D(E, +, 8)

G(B, −, 5)

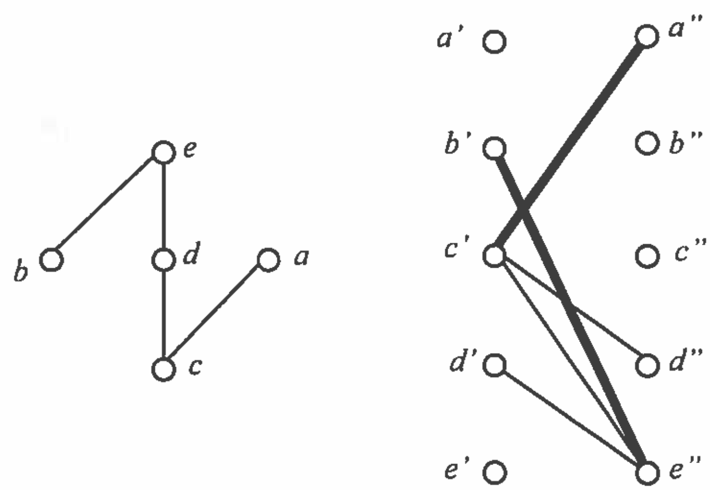
h. Find a cut whose capacity is equal to the value of the current flow.

3 {S, A, C, E, B, D, G} ∪ {F, T}   capacity 6+13+83 = 102

5.

10

2x5



A poset P is shown above on the left while the bipartite graph G associated with P is shown on the right. Some of the edges in the graph have been darkened.

a. What matching in G is associated with the chain partition $\{c, d, e\} \cup \{a\} \cup \{b\}$ of the poset P.

c', d''   d', e''   with or without commas

c'd''   d'e''

b. What chain partition of P is associated with the maximum matching given by the darkened edges

{c, a} ∪ {b, e} ∪ {d}

12

1x12

6. True–False. Mark in the left margin. Note: The first five of these questions are asked for the application of network flows (and bipartite matchings in particular) to solve the Dilworth problem for a poset P. In these five questions, the symbol G is used to represent the balanced bipartite graph associated with P.

F 1. When $x < y$ in P, the edge $y'x''$ is in G. Backwards

F 2. For every $x \in P$, the edge $x'x''$ is in G. Never

F 3. The width of P is the same as the maximum size of a matching in G.

T 4. When $x'y''$ is an edge in the maximum matching, then x and y are consecutive elements in the same chain in the associated chain partition.

T 5. When the labelling algorithm halts, a maximum antichain in P can be obtained by selecting a point x from each chain in the chain partition associated with the maximum matching so that x' is labelled and x'' is unlabelled.

F 6. To implement Kruskal's algorithm, it is not necessary to sort the edges by weight. One can simply take the edges in any order and take the first one avoiding a cycle when added to those edges already chosen.

DO NOT COUNT 7. ~~Dijkstra's algorithm finds shortest paths having the minimum number of edges.~~ Depends on code

F 8. The key idea behind the Ford-Fulkerson algorithm for network flows is to find at each step an augmenting path which maximizes the increase in the amount of the flow.

T 9. All network flow problems are also linear programming problems.

F 10. All linear programming problems are also network flow problems.

F 11. All linear programming problems posed with integral constraints have integral solutions.

F 12. Let X be a finite set. Then any function mapping the subsets of X to $[0, 1]$ is a probability measure on X.

F 13. The expected value of a random variable is always non-negative.

FUN 14. Weakly convergent generating functions spanning Dilworth partitions admit Kruskal flows with irrational coefficients having distinct odd arrays.