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GRAPHS AND PARTIALLY ORDERED SETS: RECENT RESULTS AND NEW DIRECTIONS

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ABSTRACT. We survey some recent research progress on topics linking graphs and finite partially ordered sets. Among these topics are planar graphs, hamiltonian cycles and paths, graph and hypergraph coloring, on-line algorithms, intersection graphs, inclusion orders, random methods and ramsey theory. In each case, we discuss open problems and future research directions.

1. INTRODUCTION

In recent years, there has been rapid growth in research activity centered on combinatorial problems for partially ordered sets, evidenced in part by the new AMS subject classification 06A07: Combinatorics of Partially Ordered Sets. In this article, we explore problems which relate graphs and partially ordered sets, summarizing recent results and indicating directions which look particularly promising for the future. The selection of topics must necessarily reflect the author's own perspectives; yet the goal is to highlight work which will be of wide interest to researchers and students from both areas.

We consider only finite simple graphs, i.e., graphs without loops or multiple edges. Also, we consider a *partially ordered set* (or *poset*) $\mathbf{P} = (X, P)$ as a structure consisting of a set X and a reflexive, antisymmetric and transitive binary relation P on X. We call X the *ground set* of the poset $\mathbf{P}$, and we refer to P as a *partial order* on X. In the remainder of this article, we will assume that the reader is familiar with the basic concepts for partially ordered sets, including maximal and minimal elements, chains and antichains, sums and cartesian products, comparability graphs and Hasse diagrams.

Although we are concerned primarily with *finite* posets, i.e., those posets with finite ground sets, we find it convenient to use the familiar notation $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{N}$ to denote respectively the reals, rationals, integers and positive integers equipped with the usual orders. Note that these four infinite posets are *total* orders; in each case, any two distinct points are comparable. Total orders are also called *linear* orders, or *chains*.

When $\mathbf{P} = (X, P)$ is a poset, a linear order L on X is called a *linear extension* of P when $x < y$ in L for all $x, y \in X$ with $x < y$ in P. A set $\mathcal{R}$ of linear extensions of P

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is called a *realizer* of $\mathbf{P}$ when $P = \cap\mathcal{R}$, i.e., for all x, y in X, $x < y$ in P if and only if $x < y$ in L, for every $L \in \mathcal{R}$. The minimum cardinality of a realizer of $\mathbf{P}$ is called the *dimension* of $\mathbf{P}$ and is denoted $\dim(\mathbf{P})$. Dimension for partially ordered sets plays a role which in many instances is analogous to chromatic number for graphs, and this analogy will be prominent in this article.

For additional background information on posets, the reader is referred to the author's monograph [109], the survey article [55] on dimension by Kelly and Trotter and the author's survey articles [106] and [111]. The articles [104], [108] and [110] also discuss combinatorial problems for posets. The recent survey article by Brightwell [10] is another excellent source for background material. For other perspectives on partially ordered sets, the reader is encouraged to consult the volumes edited by I. Rival [81], [82] and [83]. Also, the journal *Order* [84] is an excellent source for research articles on a wide range of topics on partially ordered sets.

2. Three Examples of Posets with Large Dimension

In this section, we briefly discuss three well known examples of posets with large dimension. These examples will help readers who are new to the subject of partially ordered sets with concepts discussed in subsequent sections.

For integers $n \geq 3$, $k \geq 0$, let $\mathbf{S}_n^k$ denote the height 2 poset with $n+k$ minimal elements $a_1, a_2, \ldots, a_{n+k}$, $n+k$ maximal elements $b_1, b_2, \ldots, b_{n+k}$ and $a_i < b_j$, for $j = i+k+1, i+k+2, \ldots, i-1$. In this definition, we interpret subscripts cyclically so that $n+1 = n$, $n+2 = 2$, etc. These posets are called *crowns*, and the following formula for their dimension is derived in [98].

Theorem 2.1. (Trotter) *Let $n \geq 3$ and $k \geq 0$ be integers. Then*

$$\dim(\mathbf{S}_n^k) = \left\lceil \frac{2(n+k)}{k+2} \right\rceil. \tag{1}$$

□

When $k = 0$, the crown $\mathbf{S}_n^0$ (also denoted $\mathbf{S}_n$), is called the *standard example* of an n-dimensional poset. To see that $\dim(\mathbf{S}_n) \leq n$, for each $i = 1, 2, \ldots, n$, take L_i as any linear extension of $\mathbf{S}_n$ with $a_i > b_i$ in L_i. It follows easily that $\{L_1, L_2, \ldots, L_n\}$ is a realizer. Conversely, if $\dim(\mathbf{S}_n) = t$ and $\{L_1, L_2, \ldots, L_t\}$ is a realizer, then for each $i = 1, 2, \ldots, n$, we may choose an integer $j_i \in \{1, 2 \ldots, t\}$ so that $a_i > b_i$ in L_{j_i}. Now suppose that $1 \leq i < k \leq n$. Then $a_k < b_i < a_i < b_k$ in L_i, so that $j_i \neq j_k$. Thus $t \geq n$.

The standard examples play a prominent role in many characterization problems for posets (see [5], [6], [98], [99] and [102], for example). Loosely speaking, standard examples play somewhat the same role in dimension theory for posets that the complete graphs play for graphs when discussing chromatic number. The presence of a large standard example as a subposet is enough to force the dimension to be large, but a poset may have large dimension without containing a large standard example as a subposet.

Note that $\mathbf{S}_n$ is isomorphic to the set of 1-element and $(n-1)$-element subsets of $\{1, 2, \ldots, n\}$ ordered by inclusion. More generally, for integers k, r and n, with

$1 \leq k < r \leq n-1$, let $\mathbf{P}(k, r; n)$ denote the poset consisting of all k-element and r-element subsets of $\{1, 2, \ldots, n\}$ ordered by inclusion. Also, let $\dim(k, r; n)$ denote the dimension of $\mathbf{P}(k, r; n)$. So $\mathbf{S}_n$ is isomorphic to $\mathbf{P}(1, n-1; n)$ and $\dim(1, n-1; n) = n$.

Our second example of a family of posets of large dimension is $\{\mathbf{P}(1, 2; n) : n \geq 3\}$. To see that $\lim_{n\to\infty} \dim(1, 2; n) = \infty$, suppose to the contrary that there exists a positive integer t so that $\dim(1, 2; n) \leq t$, for every $n \geq 3$. We obtain a contradiction when n is sufficiently large. Let $\mathcal{R} = \{L_1, L_2, \ldots, L_t\}$ be a realizer of $\mathbf{P}(1, 2; n)$. For each 3-element subset $\{i, j, k\} \subseteq \{1, 2, \ldots, n\}$ with $1 \leq i < j < k \leq n$, note that $\{j\} \| \{i, k\}$, so we may choose an integer $\alpha \in \{1, 2, \ldots, t\}$ so that $\{j\} > \{i, k\}$ in L_α. Then we have a coloring of the 3-element subsets of $\{1, 2, \ldots, n\}$ with t colors. If n is sufficiently large, then (by Ramsey's theorem) there exists a 4-element subset $S = \{i < j < k < l\} \subseteq \{1, 2, \ldots, n\}$ and an integer $\alpha \in \{1, 2, \ldots, t\}$ so that all 3-element subsets of S are mapped to α. This means that $\{j\} > \{i, k\} > \{k\} > \{j, l\} > \{j\}$ in L_α, which is a contradiction.

Note that each of the first two examples is a height 2 poset. So posets of bounded height can have arbitrarily large dimension. Our third example is quite different. In this family, large height is required for large dimension. For each $n \geq 3$, let $\mathbf{I}(n) = (I_n, P_n)$ denote the poset defined by setting I_n to be the family of all 2-element subsets of $\{1, 2, \ldots, n\}$ with $\{i, j\} < \{k, l\}$ in P_n when $1 \leq i < j < k < l \leq n$. We next show that $\lim_{n\to\infty} \dim(I_n, P_n) = \infty$. The argument is similar to the one used for the second example. Suppose to the contrary that $\dim(I_n, P_n) \leq t$, for all $n \geq 3$. We obtain a contradiction when n is large.

Let i, j, and k be distinct integers with $1 \leq i < j < k \leq n$. Then $\{i, j\} \| \{j, k\}$ in P_n, so if $\mathcal{R} = \{L_1, L_2, \ldots, L_t\}$ is a realizer of P_n, then we may choose $\alpha \in \{1, 2, \ldots, t\}$ so that $\{i, j\} > \{j, k\}$ in L_α. As before this is a coloring of the 3-element subsets of $\{1, 2, \ldots, n\}$ with t colors. If n is sufficiently large, there exists a 4-element subset $s = \{i < j < k < l\}$ and an integer $\alpha \in \{1, 2, \ldots, t\}$ so that all 3-element subsets of S are mapped to α. This implies that $\{i, j\} > \{j, k\} > \{k, l\} > \{i, j\}$ in L_α, which is a contradiction.

Posets in the family $\{\mathbf{I}_n : n \geq 3\}$ are called *canonical interval orders*, and we will discuss the general class of interval orders in further detail in Section 4. Also, in Section 10, we will discuss some surprisingly accurate estimates on the dimension of the posets in the second and third examples. For now, we are content with the knowledge that posets in these families may have large dimension.

3. Planar Graphs and Poset Dimension

Perhaps the single most striking result relating graphs to posets in the past 10 years is W. Schnyder's characterization [91] of planar graphs in terms of poset dimension. With a graph $\mathbf{G} = (V, E)$, we associate an *incidence poset* $\mathbf{P} = (X, P)$ with $X = V \cup E$. The height of $\mathbf{P}$ is two, with all elements of V being minimal and all elements of E being maximal. Furthermore, $x < e$ in P if and only if vertex x is an endpoint of edge e, for all $x \in X$ and all $e \in E$. The incidence poset of a graph $\mathbf{G}$ is also called the *vertex-edge* poset of $\mathbf{G}$. For example, the poset $\mathbf{P}(1, 2; n)$ defined in Section 2 is just the incidence poset of the complete graph $\mathbf{K}_n$ on n vertices.

With this background, here is Schnyder's theorem.

Theorem 3.1. (Schnyder) *A graph* $\mathbf{G}$ *is planar if and only if the dimension of its incidence poset is at most* 3. □

Yannakakis [115] showed that testing $\dim(\mathbf{P}) \le t$ is NP-complete, for every fixed $t \ge 3$. Curiously, Schnyder's theorem then seems at first glance to equate a problem (planarity testing) for which there exist linear time agorithms with an NP-complete problem. The rub is that we are testing dimension in a very special class of posets, namely height two posets in which each maximal element is comparable with exactly two minimal elements.

Moreover, Schnyder's theorem has a beautiful proof and requires a number of interesting lemmas which are of independent interest. Interestingly, the *easy* part of the theorem is to show that if the dimension of the incidence poset of a graph is at most 3, then the graph is planar. In fact, this part of Schnyder's theorem appears in a earlier paper [4] by Babai and Duffus. The challenging part of Schnyder's theorem is to prove that the incidence poset of a planar graph has dimension at most 3.

Given a planar graph $\mathbf{G}$ and a plane drawing D of $\mathbf{G}$ without edge crossings, it is natural to consider the poset of vertices, edges and faces determined by D. Here we intend that an edge e is less than a face F if and only if e is one of the edges which form the boundary of F. We call this poset, the *vertex-edge-face* poset of D. Brightwell and Trotter [16] then prove the following upper bound on the dimension of this poset.

Theorem 3.2. (Brightwell and Trotter) *Let* D *be a plane drawing of a planar graph* $\mathbf{G} = (V, E)$*, and let* $\mathbf{P}$ *be the vertex-edge-face poset of* D*. Then* $\dim(\mathbf{P}) \le 4$. □

Although we are limiting our attention to simple graphs in this article, the preceding theorem also applies to planar multi-graphs—with loops and multiple edges allowed. The proof of the preceding theorem is inductive and the following theorem of Brightwell and Trotter [14] is required as the base step. Of course, this theorem was proved before Theorem 3.2.

Theorem 3.3. (Brightwell and Trotter) *Let* D *be a drawing of a 3-connected planar graph* $\mathbf{G}$*, and let* $\mathbf{P}$ *be the vertex-edge-face poset of* D*. Then* $\dim(\mathbf{P}) = 4$*. Furthermore, deleting any vertex or any face from* $\mathbf{P}$ *leaves a 3-dimensional subposet.* □

This second theorem includes the difficult part of Schnyder's theorem as a special case. To see this, observe that it is enough to prove that if $\mathbf{G}$ is a maximal planar graph, then its vertex-edge poset has dimension at most 3. By inspection, this is true if $\mathbf{G}$ has at most 3 vertices. However, if $\mathbf{G}$ has 4 or more vertices, then it is 3-connected, and in this case, we conclude that its vertex-edge-face poset has dimension 4. However, the removal of a single face from this poset leaves a 3-dimensional subposet. In particular, the vertex-edge poset of $\mathbf{G}$ has dimension at most 3.

Recall that Steinitz' theorem [96] characterizes graphs which arise from convex polytopes in $\mathbb{R}^3$. These are exactly the 3-connected planar graphs. So Theorem 3.3 can be reformulated in terms of convex polytopes.

Theorem 3.4. (Brightwell and Trotter) *Let* $\mathbf{M}$ *be a convex polytope in* $\mathbb{R}^3$*, and let* $\mathbf{P}$ *be the vertex-edge-face poset of* $\mathbf{M}$*. Then* $\dim(\mathbf{P}) = 4$*. Furthermore, deleting any vertex or any face from* $\mathbf{P}$ *leaves a 3-dimensional subposet.* □

A poset $\mathbf{P} = (X, P)$ is called *t-irreducible* if $\dim(X, P) = t$, and removing any element from X leaves a $t-1$-dimensional subposet. Theorem 3.4 then provides an unexpected strategy for constructing 4-irreducible height two posets. Just take a convex polytope $\mathbf{M}$ in $\mathbb{R}^3$ and consider the poset $\mathbf{Q}$ of formed by the vertices and faces of $\mathbf{M}$ ordered by inclusion.

There are several interesting open problems emerging from this work.

Problem 3.5. *Let* n *be a positive integer and let* $\mathbf{G} = (V, E)$ *be a graph of genus* n*. Then let* D *be an embedding of* $\mathbf{G}$ *on a sphere with* n *handles, and let* $\mathbf{P} = (X, P)$ *be the set of all vertices, edges and faces of* D *ordered by inclusion. Find the maximum value* $d(n)$ *of the dimension of* $\mathbf{P}$. □

Problem 3.6. *Let* k *and* n *be fixed positive integers. Is it NP-complete to answer whether* $\dim(\mathbf{P}) \le k$ *if* $\mathbf{P}$ *is a height 2 poset with every maximal element comparable to at most* n *minimal elements?* □

Problem 3.7. *Yannakakis' NP-completeness argument for dimension shows that testing* $\dim(\mathbf{P}) \le t$ *is NP-complete, for every fixed* $t \ge 4$ *even when* $\mathbf{P}$ *is restricted to be a height 2 poset. Is it NP-complete to test* $\dim(\mathbf{P}) \le 3$*, when* $\mathbf{P}$ *has height 2?* □

Finally, although I am not ready to formulate it as a precise conjecture, it seems clear that there should be some way to extend Schnyder's theorem to higher dimensions. It is not just in terms of convex polytopes. As pointed out in [14], for every $k \ge 3$, there is a convex polytope in $\mathbb{R}^4$ containing a set S of k vertices each pair of which is an edge of the polytope. For this reason there is no upper bound on the dimension of incidence posets of polytopes in $\mathbb{R}^d$, for any $d \ge 4$.

Nevertheless, there is an appropriate generalization of Schnyder's theorem waiting to be found.

4. Chromatic Number and Hamiltonian Paths

In 1959, P. Erdös [25] used probabilistic methods to show that for every pair g, r of positive integers, there exists a graph $\mathbf{G}$ with the girth of $\mathbf{G}$ at least g and $\chi(\mathbf{G}) > r$. An elegant constructive proof of this result was provided by Nešetřil and Rödl [76], while polished probabilistic proofs have been provided by several authors. For example, see the arguments presented in [3] and [7].

But if we just want triangle-free graphs with large chromatic number, there are many easy constructions. Here is one such. For an integer $n \ge 3$, let $\mathbf{G}_n$ denote the graph whose vertex set consists of all 2-element subsets of $\{1, 2, \ldots, n\}$, with $\{i, j\}$ adjacent to $\{j, k\}$ in $\mathbf{G}_n$ whenever $1 \le i < j < k \le n$. This triangle-free graph is also known as the *shift graph*, and the formula for its chromatic number is a folklore result of graph theory: $\chi(\mathbf{G}_n) = \lceil \lg n \rceil$. Amusingly, several researchers in graph theory have told me that this result is due to Andras Hajnal, but Andras says that it is not. In any case, it is an easy exercise.

Because the topic will be addressed in several sections in this article, we pause to introduce a natural generalization of a shift graph. Fix integers n and k with $0 \le k < n-1$. We call an ordered pair (A, B) of $(k+1)$-element sets a *(k,n)-shift pair* if there exists a $(k+2)$-element subset $C = \{i_1 < i_2 < \cdots < i_{k+2}\} \subseteq \{1, 2, \ldots, n\}$

so that $A = \{i_1, i_2, \ldots, i_{k+1}\}$ and $B = \{i_2, i_3, \ldots, i_{k+2}\}$. We then define the (k,n)-*shift graph* $\mathbf{S}(k,n)$ as the graph whose vertex set consists of all $(k+1)$-element subsets of $\{1, 2, \ldots, n\}$ with a $(k+1)$-element set A adjacent to a $(k+1)$-element set B exactly when (A, B) is a (k,n)-shift pair. Note that the $(0,n)$-shift graph is just a complete graph on n vertices. For historical reasons, it is customary to call a $(1,n)$-shift graph just a shift graph; similarly, a $(2,n)$-shift graph is called a *double shift graph*.

If $\mathbf{P} = (X, P)$ is a poset, a subset $D \subseteq X$ is called a *down set*, or an *order ideal*, if $x \leq y$ in P and $y \in D$ always imply that $x \in P$. The following result appears in [41] but may have been known to other researchers in the area.

Theorem 4.1. *Let $n \geq 4$. Then the chromatic number of the double shift graph $\mathbf{S}(2,n)$ is the least t so that there are at least n down sets in the Boolean lattice $\mathbf{2}^t$.* □

As is well known, the problem of counting the number of down sets in the Boolean lattice $\mathbf{2}^t$ is a classic problem and is traditionally called Dedekind's problem. Although no closed form expression is known, relatively tight asymptotic formulas have been given. For our purposes, the estimate provided by Kleitman and Markovsky [69] suffices. Theorem 4.1, coupled with the estimates from [69] permit the following surprisingly accurate estimate on the chromatic number $\chi(\mathbf{S}(2,n))$ of the double shift graph [41].

(2) $$\chi(\mathbf{S}(2,n)) = \lg \lg n + (1/2 + o(1)) \lg \lg \lg n.$$

In Section 2, we gave two examples of families of posets with large dimension, but now that we have introduced the double shift graph, the following observation can be made [41].

Proposition 4.2. *For each $n \geq 3$,*

$$\dim(1,2;n) \geq \chi(\mathbf{S}(2,n)), \text{ and } \dim(\mathbf{I}_n) \geq \chi(\mathbf{S}(2,n)).$$

□

In [41], Füredi, Hajnal, Rödl and Trotter show that the same asymptotic formula holds for $\dim(1,2;n)$ and for $\dim(\mathbf{I}_n)$ as does for $\chi(\mathbf{S}(2,n))$.

Theorem 4.3.

(3) $$\dim(1,2;n) = \lg \lg n + (1/2 + o(1)) \lg \lg \lg n \text{ , and}$$

(4) $$\dim(\mathbf{I}_n) = \lg \lg n + (1/2 + o(1)) \lg \lg \lg n.$$

□

We will comment more about the estimates in the preceding theorem in Sections 9 and 10.

For $k \geq 1$, there is an interesting interpretation of the chromatic number of the (k,n)-shift graph, although it does not yet seem to yield a clean asymptotic formula. For a positive integer t, let $[t]$ denote a t-element antichain. Then for for each $k \geq 0$, we define an operator D^k as follows. For a poset $\mathbf{P}$, $D^0(\mathbf{P}) = \mathbf{P}$, and for $k \geq 0$, $D^{k+1}(\mathbf{P}) = \mathbf{2}^{D^k(\mathbf{P})}$. Then Felsner [28] showed that:

Theorem 4.4. *For each integer $k \geq 1$, the chromatic number $\chi(\mathbf{S}(k,n))$ of the (k,n)-shift graph is the least $t \geq 1$ for which there are n antichains in $D^{k-1}([t])$.* □

It is relatively easy to see that for each $k \geq 1$, there exist positive constants c_k and c'_k so that

(5) $$c_k \log^{(k)} n \leq \chi(\mathbf{S}(k,n)) \leq c'_k \log^{(k)} n, \quad \text{for all } n \geq k+1.$$

Quite likely, for each $k \geq 3$, there exists a positive constant c''_k so that

(6) $$\lim_{n\to\infty} \frac{\log^{(k)} n}{\chi(\mathbf{G}(k,n))} = c''_k.$$

A graph is called a *cover graph* if it can be oriented so as to form the Hasse diagram of a poset. The problem of determining whether a graph is a cover graph is now known to be NP-complete, but this recognition problem has an interesting history. It is a subtle affair, and the original NP-completeness proof of Nešetřil and Rödl was flawed. Jens Thostrup spotted the problem, which was corrected by Nešetřil and Rödl in [77]; meanwhile, Brightwell [9] gave an elegant argument for this result.

In another direction, Pretzel and Youngs [79] developed methods for generating graphs which are not cover graphs in terms of flow differences for orientations. Additional details on this concept are given in Pretzel's survey article [80], and a brief summary appears in [10].

A cover graph is always a triangle-free graph, so we may ask whether there are cover graphs with large chromatic number. However, this question is trivial to answer, as the graphs constructed by Nešetřil and Rödl in [76] are cover graphs. Moreover, as pointed out in [76], the cover graph $\mathbf{G}$ of a poset $\mathbf{P}$ of height h has chromatic number at most h, and this bound is best possible, for all $h \geq 1$.

On the other hand, shift graphs are cover graphs of a special kind of poset known as an interval order. A poset $\mathbf{P} = (X, P)$ is called an *interval order* if there is a function F assigning to each element $x \in X$ a connected subset $F(x)$ of the real line $\mathbb{R}$ so that for all $x, y \in X$, $x < y$ in P if and only if $u < v$ in $\mathbb{R}$, for every $u \in F(x)$ and $v \in F(y)$. The fact that the posets in the family $\{\mathbf{I}_n = (I_n, P_n) : n \geq 3\}$ introduced in Section 2 are interval orders is evidenced by the obvious assignment: $F(\{i,j\}) = [i,j]$.

Fishburn [32] showed that a finite poset $\mathbf{P} = (X, Y)$ is an interval order if and only if X does not contain a 4-element subset $\{x, y, z, w\}$ so that $x < y$, $z < w$, $x \not< w$ and $x \not< y$. In other words, a poset is an interval order if and only if it does not contain two 2-element chains with both points in one chain incomparable with both points in the other. This *forbidden subposet* which characterizes interval orders is denoted $\mathbf{2}+\mathbf{2}$. The concepts of interval graphs and interval orders are quite natural, and there has been an enormous amount of research on them. We refer the reader to Fishburn's monograph [33] for a sampling of this research and for an extensive bibliography of work in this area. Another source of information on interval graphs, interval orders and their generalizations is the author's survey article [107].

In [31], Felsner and Trotter investigated the following extremal problem: What is the maximum chromatic number $f(n)$ of the cover graph of an interval order of height n. The shift graph shows that $\lceil \lg n \rceil \leq f(n)$, and this lower bound is improved in [31] to $\lceil 1 + \lg n \rceil \leq f(n)$. From above, the trivial bound $f(n) \leq n$ is far from best possible, and it is shown in [31] that $f(n) \leq \lceil 2 + \lg n \rceil$. It is still not known which of the two bounds is the right answer. However, the research to produce the upper bound led quite naturally to a new problem which is perhaps more interesting than the original one. The new problem can be formulated as an extremal problem for families of subsets. Call a sequence $S_1, S_2, \ldots, S_t$ of sets an *α–sequence* if:

1. $S_2 \not\subseteq S_1$; and
2. For all i, j with $1 \leq i < j - 1 < t$, $S_j \not\subseteq S_i \cup S_{i+1}$.

Problem 4.5. (Felsner and Trotter) *Given a positive integer n, determine the largest integer $t = \alpha(n)$ for which there exists an α–sequence $S_1, S_2, \ldots, S_t$ of subsets of $\{1, 2, \ldots, n\}$.* □

For example, $\alpha(3) = 6$, and this is witnessed by the family $\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,3\}, \{1,2,3\}\}$. Felsner and Trotter [31] show that

$$2^{n-2} \leq \alpha(n) \leq 2^{n-1} + \lfloor \frac{n+1}{2} \rfloor \tag{7}$$

and they conjecture that the upper bound in this inequality is tight. If this conjecture is valid, then the corresponding α–sequence must satisfy a number of special properties, and these properties in turn lead to an tantalizing hamiltonian path problem for cubes. When first posed, I thought this problem would be relatively easy, but it now seems that this initial assessment was premature.

Let n be a positive integer. Call a listing $A_1, A_2, \ldots, A_{2^n}$ of all the subsets of $\{1, 2, \ldots, n\}$ an *order preserving* hamiltonian path in the n-cube if:

1. $A_1 = \emptyset$;
2. For all i with $1 \leq i < 2^n$, $|A_i \Delta A_{i+1}| = 1$; and
3. For all i, j with $1 \leq i < j \leq 2^n$, if $A_j \subset A_i$, then $j = i + 1$.

Problem 4.6. (Felsner and Trotter) *Does the n-cube always admit an order preserving hamiltonian path?* □

Strictly speaking, the "order preserving" condition in this problem means *to the extent possible*, i.e., before visiting a particular set, all subsets—with at most one exception—must first be visited. And if there is an exception, then this set must be visited next. For example, when $n = 3$ the sequence

$$\emptyset, \{1\}, \{1,2\}, \{2\}, \{2,3\}, \{3\}, \{1,2,3\}, \{1,3\} \tag{8}$$

is an order preserving hamiltonian path in the 3-cube.

Up to relabeling the elements in the ground set, there is a unique order preserving hamiltonian path in the n-cube, when $n \leq 4$. There are 10 order preserving hamiltonian paths in the 5-cube and 123 in the 6-cube. For $n = 7$ and $n = 8$, there is at least *one* such path.

Efforts to resolve this conjecture have produced significant results on one of the oldest problems linking graphs and posets, determining whether there is a hamiltonian cycle between the middle two levels of a Boolean lattice $\mathbf{2}^{2k+1}$.

Problem 4.7. (The middle two levels problem) *Let k be a positive integer and let $n = 2k + 1$. Then let $\mathbf{G}_n$ denote the cover graph of the poset formed by all k-element and $k+1$-element subsets of $\{1, 2, \ldots, n\}$ ordered by inclusion. Does $\mathbf{G}_n$ have a hamiltonian cycle?* □

The origins of this problem are somewhat unclear, but it was first communicated to me by I. Havel [50].

Research on the middle two levels problem has been concentrated in two directions. First, researchers have produced families of perfect matchings in the bipartite graph $\mathbf{G}_n$ in the hopes that two of these matchings could be combined to form a hamiltonian cycle (see [22], [65] and [19], for example). A second approach has been to attempt to prove the existence of "long" cycles in $\mathbf{G}_n$. In [31], Felsner and Trotter use their results on α–sequences to show that $\mathbf{G}_n$ contains a cycle passing through more than $N/4$ vertices, where N denotes the total number of vertices of $\mathbf{G}_n$. Subsequently, Savage and Winkler [86] have extended this by showing that $\mathbf{G}_n$ contains cycles with more than $8N/10$ vertices. Unfortunately, this approach does not seem capable of settling the middle two levels problem, but with some additional work, it may yield an argument which shows that $\mathbf{G}_n$ contains cycles with $(1 - o(1))N$ vertices.

5. Graph Coloring and Trees

Now we know that a graph need not contain a large complete subgraph in order to have large chromatic number, but it is still tempting to try to say something about what must be contained in graphs of large chromatic number. Here are two elementary results of this flavor; note the role played by "induced" in the statements of these results. The first result is a trivial consequence of Ramsey's theorem. The second takes a little argument and is credited to L. Lovász.

Proposition 5.1. *For every pair of positive integers k and d, there exists an integer $r = r(k, d)$ so that if $\mathbf{G}$ is any graph with $\chi(\mathbf{G}) > r$, then either $\mathbf{G}$ contains a complete subgraph on k vertices or $\mathbf{G}$ contains an induced star $\mathbf{S}_d$ with d endpoints.* □

Proposition 5.2. *For every pair of positive integers k and n, there exists an integer $r' = r'(k, n)$ so that if $\mathbf{G}$ is any graph with $\chi(\mathbf{G}) > r'$, then either $\mathbf{G}$ contains a complete subgraph on k vertices or $\mathbf{G}$ contains an induced path $\mathbf{P}_n$ on n vertices.* □

A. Gyárfás [44] and D. P. Sumner [97] made the following beautiful conjecture, which has resisted all attempts to resolve it for more than 15 years.

Conjecture 5.3. (Gyárfás and Sumner) *For every positive integer k and every tree $\mathbf{T}$, there exists an integer $r = r(k, \mathbf{T})$ so that if $\mathbf{G}$ is any graph with $\chi(\mathbf{G}) > r$, then either $\mathbf{G}$ contains a complete subgraph on k vertices or $\mathbf{G}$ contains an induced copy of $\mathbf{T}$.* □

The conjecture is trivially true when $k = 1$ and $k = 2$, but for $k \geq 3$, progress has been painstakingly slow. Gyárfás, Szemerédi and Tuza [47] proved the existence of $r(3, \mathbf{T})$ when $\mathbf{T}$ is a radius 2 tree. Kierstead and Penrice [61] extended this result by proving the existence of $r(k, \mathbf{T})$, for all $k \geq 3$, when $\mathbf{T}$ is a radius 2 tree. Quite recently, A. Scott [93] has proven a topological version of the conjecture:

Theorem 5.4. (Scott) *For every tree $\mathbf{T}$, there exist a positive integer $t = t(\mathbf{T})$ so that for every positive integer k, there exists an integer $r = r(k, \mathbf{T})$ so that if $\mathbf{G}$ is* *any graph with* $\chi(\mathbf{G}) > r$, *then either* $\mathbf{G}$ *contains a complete subgraph on* k *vertices or* $\mathbf{G}$ *contains an induced subgraph* $\mathbf{T}'$ *which is a homeomorph of* $\mathbf{T}$ *with each edge subdivided at most* t *times.* □

Scott's theorem settles the Gyárfás/Sumner conjecture for certain trees with radius larger than 2. For example, the conjecture holds for arbitrary subdivisions of a star. However, the general conjecture remains open—even for radius 3 trees.

Recently, N. Sauer [85] has advanced a particularly attractive conjecture, which is weaker than the Gyárfás/Sumner conjecture. To state Sauer's conjecture requires the following definition. A class $\mathcal{G}$ of graphs is *vertex-ramsey* if for every graph $\mathbf{G} \in \mathcal{G}$, there exists a graph $\mathbf{H} \in \mathcal{G}$ so that whenever the vertices of $\mathbf{H}$ are 2-colored, there exists a monochromatic induced subgraph which is isomorphic to $\mathbf{G}$.

Conjecture 5.5. (Sauer) *For each integer* $k \geq 2$ *and each tree* $\mathbf{T}$, *the class of graphs which exclude* $\mathbf{T}$ *and a complete graph on* k *vertices as induced subgraphs is not vertex ramsey.* □

It takes just a little thought to be convinced that if the Gyárfás/Sumner conjecture holds, then so does the Sauer conjecture. In [58], Kierstead proves Sauer's conjecture for a class of trees obtained by attaching additional leaves to subdivisions of stars. In [67], Kierstead and Zhu prove Sauer's conjecture for a very special class of radius 3 trees. These results are quite technical, and I am tempted to speculate that the Gyárfás/Sumner conjecture is false. If that turns out to be the case, I do hope that the Sauer conjecture is still true.

Now we may ask: Is there an analogue of the Gyárfás/Sumner conjecture for posets? Here, we will replace chromatic number by dimension and ask the somewhat more general question: What must be contained in a poset $\mathbf{P}$ of large dimension? Now the notion of containment is itself just a partial order, so the following abstraction makes good sense. Let P denote the set of all finite posets, and let f be a function which assigns to each $\mathbf{P} \in P$ a non-negative integer $f(\mathbf{P})$. Then let $\subseteq$ be a binary relation on P satisfying the usual requirements of an inclusion relation, i.e., the reflexive, antisymmetric and transitive properties. Suppose further that f is monotonic with respect to $\subseteq$, i.e., if $\mathbf{P} \subseteq \mathbf{Q}$, then $f(\mathbf{P}) \leq f(\mathbf{Q})$. We then say that a poset $\mathbf{P}$ is *f-unavoidable* for $\subseteq$ if there exists an integer $t = t(\mathbf{P})$ so that if $\mathbf{Q}$ is any poset with $f(\mathbf{Q}) \geq t$, then $\mathbf{P} \subseteq \mathbf{Q}$. We denote the family of f-unavoidable posets for $\subseteq$ by $\mathcal{U}(f, \subseteq)$. Note that, if $\mathbf{Q} \in \mathcal{U}(f, \subseteq)$ and $\mathbf{P} \subseteq \mathbf{Q}$, then $\mathbf{P} \in \mathcal{U}(f, \subseteq)$.

The problem is then to determine $\mathcal{U}(\dim, \subseteq)$ for various natural containment relations $\subseteq$. One way to interpret $\subseteq$ is to say that $\mathbf{P} \subseteq \mathbf{Q}$ when $\mathbf{Q}$ contains a subposet which is isomorphic to $\mathbf{P}$. But this is not a particularly interesting interpretation of $\subseteq$. Here's why. First, posets of height two can have arbitrarily large dimension, so no poset in $\mathcal{U}(\dim, \subseteq)$ can have height greater than two. Second, it is easy to modify the Nešetřil/Rödl construction in [76] to produce posets with large dimension whose comparability graphs have large girth. This shows that no poset whose comparability graph contains a cycle can belong to $\mathcal{U}(\dim, \subseteq)$.

Considering the family $\{\mathbf{P}(1, 2; n) : n \geq 3\}$, we see that a poset $\mathbf{P} = (X, P)$ may have large dimension without containing any point x with $|D(x)| > 2$, where

$D(x) = \{y \in P : y < x \text{ in } P\}$. Dually, $\mathbf{P}$ may have large dimension without containing a point x with $|U(x)| > 2$, where $U(x) = \{y \in P : y > x \text{ in } P\}$.

A poset is called a *fence* if its comparability graph is a path. Our preceding remarks show that if $\mathbf{P} \in \mathcal{U}(\dim, \subseteq)$), then every component of $\mathbf{P}$ is a fence. On the other hand, an interval order can have large dimension, but an interval order does not contain a fence of 5 or more points as a subposet. Also, an interval order does not contain $\mathbf{2} + \mathbf{2}$ as a subposet.

Now let $\mathbf{N}$ denote the four element fence whose comparability graph is a path on 4 vertices. As noted by Brightwell and Trotter in [15], it follows that $\mathbf{P} \in \mathcal{U}(\dim, \subseteq)$ consists of all finite posets $\mathbf{P}$ which satisfy the following two conditions:

1. Any component of $\mathbf{P}$ is a subposet of $\mathbf{N}$; and
2. $\mathbf{P}$ has at most one non-trivial component.

The same conclusion holds if we weaken the definition of $\subseteq$ by saying that $\mathbf{P} \subseteq \mathbf{Q}$ when either $\mathbf{Q}$ or the dual of $\mathbf{Q}$ contains a subposet isomorphic to $\mathbf{P}$. So in order to get a more interesting problem, we need a less restrictive notion of containment.

Here is one natural way to do just that. First, we say that a poset $\mathbf{Q} = (Y, Q)$ contains the poset $\mathbf{P} = (X, P)$ as a *suborder* if there exists an injection $f \colon X \to Y$ so that for all $x_1, x_2 \in X$, if $x_1 < x_2$ in P, then $f(x_1) < f(x_2)$ in Q. Under this definition, an n-element chain contains any other n-element poset as a suborder. Now let $\mathbf{P} \subseteq \mathbf{Q}$ mean that either $\mathbf{Q}$ or the dual of $\mathbf{Q}$ contains $\mathbf{P}$ as a suborder.

We call a poset $\mathbf{T}$ a *tree* if its cover graph is a tree. Note that the comparability graph of a tree need not be a tree, e.g., a chain is a tree, its cover graph is a path and its comparability graph is a complete graph. However, the cover graph and the comparability graph of a tree of height at most two are identical. Furthermore, if $\mathbf{T}$ is a graph tree, then its incidence poset is a poset tree of height 2. Furthermore in this poset tree, every maximal point e has exactly two points (its endpoints) in its down set. The following result is proved in [15].

Theorem 5.6. (Brightwell and Trotter) *For each positive integer* n, *there exists an integer* $t = t(n)$ *so that if* $\mathbf{T}$ *is a graph tree on* n *nodes,* $\mathbf{T}'$ *is the incidence poset of* $\mathbf{T}$, *and* $\mathbf{P}$ *is any poset with* $\dim(\mathbf{P}) > t$, *then* $\mathbf{T}' \subseteq \mathbf{P}$, *i.e., either* $\mathbf{P}$ *or its dual contains* $\mathbf{T}'$ *as a suborder.* □

From this result, it follows immediately that $\mathcal{U}(\dim, \subseteq)$ consists of those posets $\mathbf{P}$ for which there is a graph forest $\mathbf{F}$ so that either $\mathbf{P}$ or its dual is the incidence poset of $\mathbf{F}$. Brightwell and Trotter [15] actually show that $t(n) = O(cn^6)$. From below, the results of Erdös, Kierstead and Trotter [26] on the dimension of random posets of height two imply that $t(n) = \Omega(n \log n)$. We will say more about this work in Section 9.

6. On-line Coloring and Partitioning Problems

It is natural to consider an on-line optimization problem, such as on-line graph coloring, as a two-person game involving a *Builder* and a *Colorer*. The game is played in in a series of rounds with the players alternating turns. Each instance of on-line graph coloring also involves two parameters: an integer t and a graph $\mathbf{G}$. If

G has n vertices, the game lasts at most n rounds. In Round i, where $1 \le i \le n$, Builder presents the vertex v_i of $\mathbf{G}$ and describes all edges joining v_i with vertices in $\{v_j : 1 \le j < i\}$. This information is complete and correct. In particular, if the game lasts all n rounds, then Builder must have correctly specified the entire graph.

After receiving the information for the new vertex v_i, Colorer must then assign to v_i a color from the set $\{1, 2, \ldots, t\}$ so that this color is distinct from those previously assigned to neighbors of v_i. These assignments are permanent.

The $(t, \mathbf{G})$ game ends at Round i and Builder is the winner if Colorer has no legitimate choice of a color for the new vertex v_i. If on the other hand, Colorer is able to respond with a legitimate color for each of the n vertices of $\mathbf{G}$, then Colorer is the winner. The *on-line chromatic number* of a graph $\mathbf{G}$ is then the least t for which Colorer has a winning strategy for the $(t, \mathbf{G})$ game—regardless of the strategy employed by Builder.

As an example, the on-line chromatic number of $\mathbf{P}_4$ (a path on 4 vertices) is 3, even though $\mathbf{P}_4$ is bipartite. Graphs which do not include $\mathbf{P}_4$ as an induced subgraph are perfect. Furthermore, they are optimally colored by the Greedy (First Fit) algorithm, regardless of the order in which the vertices are presented. On the other hand, Gyárfás and Lehel [46] showed that for every r, there exists a bipartite graph $\mathbf{G}_r$ which does not contain $\mathbf{P}_6$ as an induced subgraph and has on-line chromatic number at least r. For graphs which exclude $\mathbf{P}_5$ as an induced subgraph, the coloring problems are particularly interesting.

Problem 6.1. *For each positive integer k, let $r(k)$ and $r'(k)$ denote respectively the maximum chromatic number and the maximum on-line chromatic number of a graph which does not contain a complete subgraph on $k+1$ vertices and does not contain $\mathbf{P}_5$ as an induced subgraph.*

This is a case where there is a surprisingly small gap between the off-line and on-line versions. The best known bounds are:

$$\Omega(\frac{k^2}{\log k}) \le r(k) \le 2^k \quad \text{and} \tag{9}$$

$$r(k) \le r'(k) \le (4^k - 1)/3. \tag{10}$$

The upper bound on $r(k)$ is due to Gyárfás and Lehel [46], and the lower bound is an easy colorollary to Kim's new lower bound [68] on the ramsey number $R(3, k)$. The upper bound on $r'(k)$ is due to Kierstead, Penrice and Trotter [63].

Kierstead, Penrice and Trotter [62] then proceeded to prove the following theorem, which in my opinion is one of the deepest results in the subject of on-line graph coloring.

Theorem 6.2. (Kierstead, Penrice and Trotter) *For each positive integer k and each radius 2 tree $\mathbf{T}$, there exist an integer $r = r(k, \mathbf{T})$ so that if $\mathbf{G}$ is any graph which does not contain a complete subgraph on k vertices and does not contain $\mathbf{T}$ as an induced subgraph, then the on-line chromatic number of $\mathbf{G}$ is at most r.* □

As just one illustration of the power of this theorem, it provides as an easy corollary the solution to a long standing on-line partitioning problem. Recall that the well

known theorem of Dilworth [18] asserts that a poset of width n can be partitioned into n chains. The graph version of this result is that a comparability graph is perfect, i.e., if the independence number is n, then the graph can be partitioned into n complete subgraphs (see [48], for example).

We now consider the on-line versions of these problems, beginning with the on-line chain partitioning problem. Here we are presented with a poset one point at a time and asked to partition it (on-line) into chains. In [56], Kierstead showed that there is an on-line algorithm which will partition a poset of width n into $(5^n - 1)/4$ chains. However, it is not known whether there is some polynomial $p(n)$ so that a poset of width n can be on-line chain partitioned into $p(n)$ chains. Quite recently, Felsner [29] showed that a width 2 poset can be partitioned on-line into 5 chains. This result seems to suggest some improvement in Kierstead's bound is possible.

Now consider the problem of partitioning (on-line) a comparability graph into complete subgraphs. J. Schmerl [90] conjectured that there exists a function $f(n)$ so that there exists an on-line algorithm which will partition a comparability graph with independence number n into $f(n)$ complete subgraphs—regardless of the number of vertices. The problem is that Kierstead's on-line algorithm [56] for partitioning a poset into chains makes specific use of the partial order—not just the comparability graph. This is also true of Felsner's algorithm [29] for partitioning a width two poset into 5 chains

Schmerl's conjecture can be reformulated as follows: There a function $g(k)$ so that a co-comparability graph of maximum clique size k has on-line chromatic number at most $g(k)$. An affirmative answer to this conjecture follows easily from Theorem 6.2, since a co-comparability graph cannot contain the subdivision of a $\mathbf{K}_{1,3}$ as an induced subgraph. Of course, the subdivision of $\mathbf{K}_{1,3}$ is just a radius 2 tree.

There are a host of other challenging problems in on-line graph coloring, but the one which stands out in my mind as the most important is to provide tighter estimates on the maximum on-line chromatic number $c_3(n)$ of a 3-colorable graph $\mathbf{G}$ on n vertices. From below, Alon [1] and Vishwanathan [113] proved that $c_3(n) = \Omega(\log^2 n)$. From above, Lovàsz, Saks and Trotter proved that

$$c_3(n) = O\Big(\frac{n \log\log\log n}{\log\log n}\Big), \tag{11}$$

and Kierstead [59] removed the $\log\log\log n$ term from the numerator of this expression. Quite recently, Kierstead [59] has shown that there exist a constant c so that $c_3(n) < n^{2/3} \log^c n$, and this result may represent a real breakthrough in on-line coloring.

7. Hypergraphs and Fibers

Let $\mathbf{P} = (X, P)$ be a poset. Chains and antichains which contain two or more points are called *non-trivial.* Lonc and Rival [72] called a subset $A \subseteq X$ a *co-fiber* if it intersects every non-trivial maximal chain in $\mathbf{P}$. Let $a(\mathbf{P})$ denote the least t so that $\mathbf{P}$ has a co-fiber of cardinality t. Then let $a(n)$ denote the maximum value of $a(\mathbf{P})$ taken over all n-element posets. In any poset, the set A_1 consisting of all maximal elements which are not minimal elements and the set A_2 of all minimal elements which are not maximal are both co-fibers. As $A_1 \cap A_2 = \emptyset$, it follows that $a(n) \leq \lfloor n/2 \rfloor$. On the other hand, the fact that $a(n) \geq \lfloor n/2 \rfloor$ is evidenced by a height 2 poset with $\lfloor n/2 \rfloor$ minimal elements each of which is less than all $\lceil n/2 \rceil$ maximal elements. So $a(n) = \lfloor n/2 \rfloor$ (this argument appears in [72]).

Dually, a subset $B \subseteq X$ is called a *fiber* if it intersects every non-trivial maximal antichain. Let $b(\mathbf{P})$ denote the least t so that $\mathbf{P}$ has a fiber of cardinality t. Then let $b(n)$ denote the maximum value of $b(\mathbf{P})$ taken over all n-element posets. Trivially, $b(n) \geq \lfloor n/2 \rfloor$, and Lonc and Rival asked whether equality holds.

In [21], Duffus, Sands, Sauer and Woodrow showed that if $\mathbf{P} = (X, P)$ is an n-element poset, then there exists a set $F \subseteq X$ which intersects every 2-element maximal antichain so that $|F| \leq \lfloor n/2 \rfloor$. However, Sands then constructed a 17-point poset in which the smallest fiber contains 9 points. This construction was generalized by R. Maltby [75] who proved that for every $\epsilon > 0$, there exist a n_0 so that for all $n > n_0$ there exists an n-element poset in which the smallest fiber has at least $(8/15 - \epsilon)n$ points.

From above, the following theorem [20] shows that $b(n) \leq 2n/3$.

Theorem 7.1. (Duffus, Kierstead and Trotter) *Let $\mathbf{P} = (X, P)$ be a poset and let $\mathcal{H}$ be the hypergraph of non-trivial maximal antichains of $\mathbf{P}$. Then the chromatic number of $\mathcal{H}$ is at most* 3. □

The fact that $b(n) \leq 2n/3$ follows from the observation that if $X = B_1 \cup B_2 \cup B_3$ is a 3-coloring of $\mathcal{H}$, then the union of any two of $\{B_1, B_2, B_3\}$ is a fiber. Most recently, Lonc [71] has obtained the following intersesting result which provides a better upper bound for posets with small width.

Theorem 7.2. (Lonc) *Let $\mathbf{P} = (X, P)$ be a poset of width* 3 *and let $|X| = n$. Then $\mathbf{P}$ has a fiber of cardinality at most $11n/18$.* □

My guess is that $\lim_{n\to\infty} b(n)/n = 2/3$.

8. Intersection Graphs, Inclusion Orders, Hypergraphs and Fibers

Intersection graphs are one of the most widely studied topics in graph theory, and there are (at least) two common themes to this research. One theme is to to study intersection graphs where the sets are restricted to come from some particular class. As an example, interval graphs are just the intersection graphs of a family of intervals of the real line. Other examples include intersection graphs of line segments or disks in the plane. In such instances, we are restricting the set of graphs under consideration.

A second theme is to consider classes of sets defined in terms of a parameter. Usually, for each graph $\mathbf{G}$, we obtain $\mathbf{G}$ as the intersection graph of one of the families, provided the parameter is sufficiently large. For example, for each positive integer n, consider the set $\mathcal{G}_n$ of all graphs representable as the intersection graph of a family of subsets of $\{1, 2, \ldots, n\}$. It is then natural to define the *intersection number* of a graph $\mathbf{G}$ as the least n so that $\mathbf{G} \in \mathcal{G}_n$. Clearly, this concept is well defined.

Here's another example. For a graph $\mathbf{G} = (V, E)$, define the *interval number* of $\mathbf{G}$ as the least t for which $\mathbf{G}$ is the intersection graph of a family of sets $\{S_x : x \in V\}$ where each S_x is the union of t pairwise disjoint intervals of the real line $\mathbb{R}$. The concept of interval number has been studied extensively. In [49], Harary and Trotter show that the interval number of the complete bipartite graph $\mathbf{K}_{m,n}$ is $\lceil (mn+1)/(m+n) \rceil$. In [42], Griggs shows that the interval number of a graph on n vertices never exceeds $\lceil (n+1)/4 \rceil$. In [43], Griggs and West show that if the maximum degree $\Delta(\mathbf{G})$ is d, then the interval number of $\mathbf{P}$ is at most $\lceil (d+1)/2 \rceil$; this bound is tight whenever $\mathbf{G}$ is regular and triangle-free. And in [88], Scheinerman and West show that the interval number of a planar graph is at most 3.

For posets, there are natural analogues for each of these problems, and we have already seen an example of the first—the notion of an interval order. Alternatively, we can consider another order associated with a set of intervals, namely the inclusion order. As is well known, a finite poset is isomorphic to a collection of intervals of the real line ordered by inclusion if and only if it has dimension at most two. This is just one example of a *geometric containment order*, i.e., a poset obtained by partially ordering some naturally occurring family of geometric objects by inclusion.

Here are two surprisingly difficult open problems in this area (there are many more). A subset $S \subset \mathbb{R}^d$ is called a *d-sphere* if there exists a point $\mathbf{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ and a positive real number r so that $S = \{\mathbf{y} = (y_1, y_2, \ldots, y_d) \in \mathbb{R}^d : \sum_{i=1}^{d} (x_i - y_i)^2 \leq r^2\}$. A poset $\mathbf{P} = (X, P)$ is called a d-sphere order if there exists a function F assigning to each $x \in X$ a d-sphere $F(x)$ so that $x \leq y$ in P if and only if $F(x) \subseteq F(y)$. For historical reasons, 2-sphere orders are called *circle orders*, although it might have been more accurate to call them *disk orders*. The following problem is due to Brightwell and Winkler [17].

Problem 8.1. *Is it true that for every finite poset $\mathbf{P}$, there exists a positive integer d so that $\mathbf{P}$ is a d-sphere order, i.e., $\mathbf{P}$ is isomorphic to a family of d-spheres ordered by inclusion?* □

The use of the euclidean metric in defining a sphere is important in this problem. If we use another metric, the question may be trivial to answer. For example, it is an easy exercise to show that a n-dimensional poset is the inclusion order of a family of cubes in $\mathbb{R}^{n-1}$. However, I would speculate that there exists a finite 3-dimensional poset $\mathbf{P}$ which is not a d-sphere order for any $d > 1$. But this is a challenging problem, and many people have misjudged its difficulty in the past—including this author.

Now, a finite poset $\mathbf{P}$ of dimension at most two can be represented as a family of intervals of the real line ordered by inclusion. So a finite 2-dimensional poset $\mathbf{P}$ is also a circle order, and in fact, we can require that the centers of all the circles used in the representation lie on a straight line.

On the other hand, not all finite 4-dimensional posets are circle orders—in fact, most of them are not. This statement is an immediate consequence of the work of Alon and Scheinerman [2] on "degrees of freedom".

The situation with finite 3-dimensional posets remains unsettled.

Problem 8.2. *Let $\mathbf{P}$ be a finite poset with* $\dim(\mathbf{P}) \leq 3$. *Is $\mathbf{P}$ a circle order?* □

This problem is particularly vexing because there are partial results which support both positive and negative answers. On the positive side, it is an easy exercise to show that for each $n \geq 3$, a finite 3-dimensional poset $\mathbf{P}$ is isomorphic to a family of regular convex n-gons in the plane ordered by inclusion. By keeping $\mathbf{P}$ fixed and letting n tend to infinity, it seems reasonable that we are essentially representing $\mathbf{P}$ by circles.

On the other hand, there are countably infinite 3-dimensional posets which are not circle orders. Scheinerman and Weirman [88] gave the first such proof; they showed that $\mathbb{Z}^3$ was not a circle order. Subsequently, Hurlbert [53] gave a somewhat simpler proof. In [38], Fon-der-Flaass proves that the cartesian product $\mathbf{2} \times \mathbf{3} \times \mathbb{N}$ is not a d-sphere order, for any $d \geq 1$. But each of these arguments depends in a fundamental way on the poset being infinite, and there is no notion of compactness waiting to come to our rescue.

Here are some other results on circle orders and related topics. A region R of the plane is called a *angular region* if it is bounded by two one-way infinite rays emanating from a common point. The inclusion orders of a family of angular regions are called *angle orders.* In [37], Fishburn and Trotter show that every interval order is an angle order. In [34], Fishburn shows that every interval order is a circle order, and in [35], Fishburn shows that there exist circle orders which are not are angle orders. In [87], Scheinerman shows that a graph $\mathbf{G}$ is planar if and only if its incidence poset is a circle order (note the connection with Schnyder's theorem discussed in Section 3). And in [13], Brightwell and Scheinerman show that the dual of a circle order need not be a circle order. Of course, this last result concerns infinite posets, as the dual of a finite circle order is obviously a circle order.

Several other inclusion orders involving families of sets and parameters have been studied. For example, Madej and West [74] define the *interval inclusion number* of a poset $\mathbf{P} = (X, P)$, denoted $i(\mathbf{P})$, as the least t for which $\mathbf{P}$ is the inclusion order of a family $\{S_x : x \in V\}$ where each S_x is the union of at most t pairwise disjoint intervals of $\mathbb{R}$. Madej and West prove that $i(\mathbf{P}) \leq \lceil \dim(\mathbf{P})/2 \rceil$ and that this inequality is best possible. On the other hand, they show that interval orders have interval inclusion number at most two, even though they can have arbitrarily large dimension.

Given a finite poset $\mathbf{P} = (X, P)$, the least n for which $\mathbf{P}$ is the inclusion order of a family of subsets of $\{1, 2, \ldots, n\}$ is called the 2-dimension of $\mathbf{P}$, and is denoted $\dim_2(\mathbf{P})$. The reason for this notation is that $\dim_2(\mathbf{P})$ is just the least n for which $\mathbf{P}$ is isomorphic to a subposet of the cartesian product $\mathbf{2}^n$. More generally, J. Novak [78] defined $\dim_k(\mathbf{P})$ as the least n so that $\mathbf{P}$ is isomorphic to a subposet of $\mathbf{k}^n$. For a poset $\mathbf{P} = (X, P)$, Trotter proves the following bounds in [100] and [103]:

$$\dim_2(X, P) \leq |X|; \tag{12}$$

$$\dim_3(X, P) \leq \lceil (|X|)/2 \rceil, \text{ for } |X| \geq 6, \text{ and} \tag{13}$$

$$\dim_4(X, P) \leq \lfloor |X|/2 \rfloor, \text{ for } |X| \geq 7. \tag{14}$$

All three of these inequalities are best possible, and a full characterization of posets $\mathbf{P} = (X, P)$ for which $\dim_2(X, P) = |X|$ is given in [100]. The third inequality is just a bit stronger than the well known inequality of Hiraguchi [51] (see also [52]): $\dim(X, P) \leq \lfloor |X| \rfloor$, when $|X| \geq 4$.

And on this note, we pause to remark that the concept of dimension was introduced more than 50 years ago in Dushnik and Miller's classic paper [24], but the study of the combinatorial properties of dimension has its roots in three important papers, of which Hiraguchi's 1951 paper [51] is one. The other two are Dushnik's 1950 paper [23] on sets of arrangements and Dilworth's 1950 paper [18] on chain partitions. Dushnik's paper determines the dimension of $\mathbf{P}(1, r; n)$ (the set of all all 1-element and all r-element subsets of a n-element set ordered by inclusion) when $r \geq 2\sqrt{n}$. Dilworth's paper shows that the dimension of the distributive lattice $\mathbf{L} = \mathbf{2}^{\mathbf{P}}$, the set of all down sets of $\mathbf{P}$ ordered by inclusion, is just the width of $\mathbf{P}$. It is interesting to note that Dilworth's famous chain partitioning theorem began life as a lemma used to prove a dimension theoretic result for distributive lattices!

In [101], Trotter generalizes Dilworth's dimension formula for distributive lattices by showing that if $\mathbf{L} = \mathbf{2}^{\mathbf{P}}$, then $\dim_k(\mathbf{L})$ is the minimum number of chains of cardinality at most $k - 1$ required to cover X. In particular, $\dim_2(\mathbf{2}^{\mathbf{P}}) = |X|$.

Quite recently, Schumacher [92] has provided an improved bound on the 2-dimension of crowns. This work concentrates on crowns whose comparability graphs are cycles and improves on bounds obtained previously by Stahl [95]. Both papers correct the author's flawed formula for the 2-dimension of crowns given in [100].

Dushnik's work on arrangements has been greatly extended with a flurry of work on related problems. Recall that for integers k, r and n, with $1 \leq k < r \leq n - 1$, $\dim(k, r; n)$ denotes the dimension of the poset $\mathbf{P}(k, r; n)$. Spencer [94] gave asymptotic bounds for $\dim(1, r; n)$, for fixed r with n tending to infinity.

Theorem 8.3. (Spencer)

$$\lg \lg n \leq \dim(1, 2; n) \leq \lg \lg n + (1/2 + o(1)) \lg \lg \lg n; \tag{15}$$

$$\dim(1, r; n) \leq r 2^r \lg \lg n \quad \textit{when } 2 \leq r. \tag{16}$$

□

The upper bound on $\dim(1, 2; n)$ given in the first inequality in Theorem 8.3 coupled with the lower bound in Proposition 4.2 completes the proof of the estimate for $\dim(1, 2; n)$ given in Theorem 4.3.

A surprisingly large number of combinatorial problems have connections with computations of $d(k, r; n)$, especially for the case $k = 1$ and $r = 2$. In [105], Trotter extended the range of values covered by Dushnik's formula. This paper also shows that $d(1, 2; 13) = 5$, but $d(1, 2; n) \leq 4$ when $n \leq 12$. In [57], Kierstead uses some new and quite clever techniques to prove the following lower bounds for the case $k = 1$.

Theorem 8.4. (Kierstead)

$$2^{r-2} \lg \lg n \leq \dim(1, r; n), \tag{17}$$

$$\textit{if } \; 2 \leq r \leq \lg \lg n - \lg \lg \lg n;$$

$$\frac{(r + 2 - \lg \lg n + \lg \lg \lg n)^2 \lg n}{32 \lg(r + 2 - \lg \lg n + \lg \lg \lg n)} \leq \dim(1, r; n), \tag{18}$$

if $\lg\lg n - \lg\lg\lg\lg n \le r \le 2^{\lg^{1/2} n}$; *and*

(19) $$\frac{(r+2-\lg\lg n+\lg\lg\lg n)^2 \lg n}{32\lg(r+2-\lg\lg n+\lg\lg\lg n)} \le \dim(1,r;n) \le \frac{2k^2\lg^2 n}{\lg^2 k},$$

if $2^{\lg^{1/2} n} \le k \le 2\sqrt{n}-2$. □

The first of Kierstead's bounds is a surprisingly tight lower bound for $\dim(1,r;n)$, when n is large compared to r. Compare it with Spencer's upper bound given by the second inequality in Theorem 8.3. In Section 9, we will comment about one of the consequences of the second inequality of Theorem 8.4, a painful lesson for the author.

In the past two years, There has been an interesting series of papers providing estimates for $\dim(k,r;n)$ when $k \ge 2$. In [11], Brightwell, Kierstead, Kostochka and Trotter show that:

(20) $$d(k,k+s;n) \le \dim(1,2s;n) + 18s\log n \quad \text{for all } s > 0.$$

The proceeding formula shows that $\dim(k,k+1;2k+1) = O(\log k)$, but Kostochka [70] has improved this to $\dim(k,k+1;2k+1) = O(\log k/\log\log k)$. Perhaps this is the right answer, but from below, we know only that $\dim(k,k+1;2k+1) = \Omega(\log\log k)$. This is an easy corollary to Spencer's lower bound on $\dim(1,2;n)$.

In [54], Hurlbert, Kostochka and Talysheva show that $\dim(2,n-2;n)$ is $(n-1)$-irreducible, and they determine the exact value of $\dim(2,r;n)$, for almost all values of r, provided n is sufficiently large. Füredi [39] then shows that $\dim(k,n-k;n) = n-2$, whenever $3 \le k \le n^{1/3}/6$.

9. Random Graphs and Random Posets

The use of random methods has been one of the fastest growing areas of research in combinatorics over the past 20 years or so. Researchers in posets have also taken advantage of the power of these methods. Here are just a few examples.

Given a poset $\mathbf{P} = (X,P)$ and a point $x \in X$, define the *degree* of x in $\mathbf{P}$, denoted $\deg_{\mathbf{P}}(x)$, as the number of points in X which are comparable to x. This is just the degree of the vertex x in the associated comparability graph. Then define $\Delta(\mathbf{P})$ as the maximum degree of $\mathbf{P}$. Finally, define $D(k)$ as the maximum dimension of a poset $\mathbf{P}$ with $\Delta(\mathbf{P}) \le k$. Rödl and Trotter were the first to prove that $D(k)$ is well defined. Their argument showed that $D(k) \le 2k^2+2$. Subsequently, Füredi and Kahn [40] showed that $D(k) = O(k\log^2 k)$.

We pause to explain one key detail of Füredi and Kahn's argument for an upper bound on $D(k)$. Using a clever application of the Lovász Local Lemma [27], they show that

(21) $$D(k) = O\Big(\frac{k}{\log k}\dim(1,\log k; k\log k)\Big).$$

Füredi and Kahn then used an elementary probabilistic argument to show that $\dim(1,r;n) = O(r^2\log n)$, and thus $\dim(1,\log k; k\log k) = O(\log^3 k)$. These results then imply their upper bound on $D(k)$. For several years, the best lower bound stood at $D(k) \ge k+1$. This bound comes from the standard examples.

Then in 1991, Erdös, Kierstead and Trotter [26] investigated dimension for random posets of height 2. The model was defined as follows. For a fixed positive integer n, they considered a poset $\mathbf{P}_n$ having n minimal elements $a_1, a_2, \ldots, a_n$ and n maximal elements $b_1, b_2, \ldots, b_n$. The order relation was defined by setting $a_i < b_j$ with probability $p = p(n)$; also, events corresponding to distinct min-max pairs were independent.

In this paper, estimates are given on the expected value of the dimension of the resulting poset. The following theorem summarizes these lower bounds.

Theorem 9.1. (Erdös, Kierstead and Trotter)

1. *For every $\epsilon > 0$, there exists $\delta > 0$ so that if*

$$\frac{\log^{1+\epsilon} n}{n} < p \le \frac{1}{\log n},$$

then

$$\dim(\mathbf{P}) > \delta\, pn\log pn, \text{ for almost all } \mathbf{P}.$$

2. *For every $\epsilon > 0$, there exist $\delta, c > 0$ so that if*

$$\frac{1}{\log n} \le p < 1 - n^{-1+\epsilon},$$

then

$$\dim(\mathbf{P}) > \max\{\delta n,\, n - \frac{cn}{p\log n}\}, \text{ for almost all } \mathbf{P}.$$

□

Here is an easy corollary.

Corollary 9.2. (Erdös, Kierstead and Trotter) *For every $\epsilon > 0$, there exists $\delta > 0$ so that if*

$$n^{-1+\epsilon} < p \le \frac{1}{\log n},$$

then

$$\dim(\mathbf{P}) > \delta\Delta(\mathbf{P})\log n, \text{ for almost all } \mathbf{P}.$$

□

In particular, this corollary shows that

(22) $$\Omega(k\log k) = D(k) = O(k\log^2 k).$$

So what is the correct answer for the power of the $\log k$ term in the formula for $D(k)$? At one time, I suspected that the correct answer was the lower bound so that $D(k) = \theta(k\log k)$. I even went so far as to suggest that the way to prove this was to improve the trivial estimate $\dim(1,\log k; k\log k) = O(\log^3 k)$, As the only lower bound I knew was $\dim(1,\log k; k\log k) = \Omega(\log^2 k)$, I offered \$100 for improving the exponent in this estimate. In hindsight, I should have been more careful.

As a special case of Theorem 8.4, we note that

$$\dim(1, \log k; k \log k) = \Omega(\log^3 k / \log\log k). \tag{23}$$

Accordingly, Kierstead managed to collect the \$100 prize without settling the underlying question. Of course, Kierstead's result does tell us that a really new idea will be necessary to improve the upper bound on $D(k)$—if this is at all possible.

However, I think it is much more likely that the lower bound can be improved. To attempt this, another model for random posets should be investigated. In this model, we take a set of k random matchings in a complete bipartite graph $\mathbf{K}_{n,n}$. The techniques in [26] will not apply when k is very small relative to n, say when $k = o(\log n)$, but it is precisely in this range that the random poset may yield an improvement for $D(k)$.

Finally, we should mention that several other models of random posets have been investigated, including:

1. Random k-dimensional partial orders on n points—take k random linear orders on $\{1, 2, \ldots, n\}$ and let P be their intersection;
2. Transitive closures of random graphs on n vertices—take a random graph with vertex set $\{1, 2, \ldots, n\}$ and set $i < j$ in P when there exists an integer $t \geq 2$ and a sequence $i = i_1, i_2, \ldots, i_t = j$, with $i_1 < i_2 < \cdots < i_t$ in $\mathbb{N}$ with $i_j i_{j+1}$ an edge in the random graph, for every $j = 1, 2, \ldots, t-1$; and
3. Labelled partially ordered sets with ground set $\{1, 2, \ldots, n\}$—take all such posets as equally likely.

We encourage the reader to consult the survey articles by Winkler [114] and Brightwell [8] for additional details. Brightwell's paper is more recent and provides a good bibliography of papers in this area.

10. Fractional Dimension and an Analogue to Brooks' Theorem

Many researchers in combinatorics have investigated fractional versions of integer valued parameters, and often the resulting LP relaxation sheds light on the original problem. In [12], Brightwell and Scheinerman proposed to investigate fractional dimension for posets. This concept has already produced some interesting results, but many appealing questions have been raised. Here's a brief overview.

Let $\mathbf{P} = (X, P)$ be a poset and let $\mathcal{F} = \{M_1, \ldots, M_t\}$ be a multiset of linear extensions of P. Brightwell and Scheinerman [12] call $\mathcal{F}$ a *k-fold realizer* of P if for each incomparable pair (x, y), there are at least k linear extensions in $\mathcal{F}$ which reverse the pair (x, y), i.e., $|\{i : 1 \leq i \leq t, x > y \text{ in } M_i\}| \geq k$. The *fractional dimension* of $\mathbf{P}$, denoted by $\mathrm{fdim}(\mathbf{P})$, is then defined as the least real number $q \geq 1$ for which there exists a k-fold realizer $\mathcal{F} = \{M_1, \ldots, M_t\}$ of P so that $k/t \geq 1/q$ (it is easily verified that the least upper bound of such real numbers q is indeed attained). Using this terminology, the *dimension* of $\mathbf{P}$ is just the least t for which there exists a 1-fold realizer of P. It follows immediately that $\mathrm{fdim}(\mathbf{P}) \leq \dim(\mathbf{P})$, for every poset $\mathbf{P}$.

Brightwell and Scheinerman [12] proved that if $\mathbf{P}$ is a poset and $\Delta(\mathbf{P}) = k$, then $\mathrm{fdim}(\mathbf{P}) \leq k + 2$. They conjectured that this inequality could be improved to $\mathrm{fdim}(\mathbf{P}) \leq k + 1$. This was proved by Felsner and Trotter [30], and the argument

yielded a much stronger conclusion, a result with much the same flavor as Brooks' theorem for graphs.

Theorem 10.1. (Felsner and Trotter) *Let k be a positive integer, and let $\mathbf{P}$ be any poset with $\Delta(\mathbf{P}) = k$. Then $\mathrm{fdim}(\mathbf{P}) \leq k+1$. Furthermore, if $k \geq 2$, then $\mathrm{fdim}(\mathbf{P}) < k+1$ unless one of the components of $\mathbf{P}$ is isomorphic to $\mathbf{S}_{k+1}$, the standard example of a poset of dimension $k+1$.* □

Felsner and Trotter [30] derive several other inequalities for fractional dimension, and these lead to some challenging problems as to the relative tightness of inequalities similar to the one given in the preceding theorem. However, there remains an even more appealing problem, one posed in [12].

Problem 10.2. *What is the least positve real number d so that if $\mathbf{P}$ is an interval order, then $\mathrm{fdim}(\mathbf{P}) \leq d$?* □

Brightwell and Scheinerman [12] proved that $d \leq 4$, and they conjectured that $d = 4$. We believe this conjecture is correct but confess that our intuition is not really tested.

Now let us return for a moment to the problem of determining the dimension (not the fractional dimension) of the canonical interval order $\mathbf{I}_n$ consisting of all intervals with integer endpoints from $\{1, 2, \ldots, n\}$. As commented on in Section 4, it was known that that $\dim(\mathbf{I}_n)$ was at least as large as the chromatic number of the double shift graph, and a relatively precise estimate was known for this parameter. However, progress on upper bounds for $\dim(\mathbf{I}_n)$ was much slower.

At the risk of over simplifying matters, the problem reduced (more or less) to the following technical question. What is the chromatic number of the graph $\mathbf{OS}_n$ whose vertex set consists of all 4-element subsets of $\{1, 2, \ldots, n\}$ with $\{i_1, i_2, i_3, i_5\}$ adjacent to $\{i_2, i_4, i_5, i_6\}$, whenever $1 \leq i_1 < i_2 < \cdots < i_6$. We call $\mathbf{OS}_n$ the *overlapping double shift graph.* It is easy to see that the dimension of the canonical interval order $\mathbf{I}_n$ is at least as large as the chromatic number of the overlapping double shift graph.

Füredi, Hajnal, Rödl and Trotter [41] were able to show that asymptotically, the overlapping double shift graph has the same chromatic number as the double shift graph. With some attention to technical details, they then showed that the asymptotic formula for $\dim(\mathbf{I}_n)$ was the same as the maximum value $d(n)$ of the dimension among all interval orders of height n, and these two estimates were the same as the estimate for the chromatic number of the double shift graph.

Now perhaps the same kind of approach can be made to work for fractional dimension. So a natural starting point is to consider the fractional chromatic number of the double shift graph. However (and we consider this result surprising), it is an easy exercise to show that the fractional chromatic number of the double shift graph is at most 3. Again it is easy to see that the fractional dimension of the canonical interval order $\mathbf{I}_n$ is at least as large as the fractional chromatic number of the overlapping double shift graph. However, we believe that the fractional chromatic number of the overlapping double shift graph is 4. If this is correct, then there is indeed a fundamental difference between the fractional problems for double shift graphs and interval orders—while these two problems were ultimately shown to be equivalent in the integer valued versions.

Regrettably, we have not been able to make much progress on determining the fractional chromatic number of the overlapping double shift graph, so in an effort to build a body of techniques and insights, we began to attack a ramsey theoretic problem for probability spaces, a problem which we believe to be closely related. Fix an integer $k \geq 0$, and let $n > k+1$. Now suppose that Ω is a probability space containing an event E_S for every $(k+1)$-element subset $S \subset \{1, 2, \ldots, n\}$. We abuse terminology slightly and use the notation $\text{Prob}(S)$ rather than $\text{Prob}(E_S)$.

Now let $f(\Omega)$ denote the minimum value of $\text{Prob}(A\overline{B})$, taken over all (k, n)-shift pairs (A, B). Note that we are evaluating the probability that A is true and B is false. Then let $f(n, k)$ denote the maximum value of $f(\Omega)$ and let $f(k)$ denote the limit of $f(n, k)$ as n tends to infinity.

Even the case $k = 0$ is non-trivial, as it takes some work to show that $f(0) = 1/4$. However, there is a natural interpretation of this result. Given a sufficiently long sequence of events, it is inescapable that there are two events, A and B with A occuring before B in the sequence, so that

$$\text{Prob}(A\overline{B}) < \frac{1}{4} + \epsilon.$$

The $\frac{1}{4}$ term in this inequality represents coin flips. The ϵ is present because, for finite n, we can always do slightly better than tossing a fair coin.

For $k = 1$, Trotter and Winkler [112] show that $f(1) = 1/3$. Note that this is just the fractional chromatic number of the double shift graph. This result is also natural and comes from taking a random linear order L on $\{1, 2, \ldots, n\}$ and then saying that a 2-element set $\{i, j\}$ is *true* if $i < j$ in L. Trotter and Winkler conjecture that $f(2) = 3/8$, $f(3) = 2/5$, and are able to prove that $\lim_{k\to\infty} f(k) = 1/2$. They originally conjectured that $f(k) = (k+1)/(2k+4)$, but they have since been able to show that $f(4) \geq \frac{27}{64}$ which is larger than $\frac{5}{12}$.

As an added bonus to this line of research, we are beginning to ask natural (and perhaps quite important) questions about patterns appearing in probability spaces. I consider this a particularly fruitful area for future research.

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