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Finite three dimensional partial orders which are not sphere orders

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Abstract

Given a partially ordered set $\boldsymbol{P}=(X,P)$, a function F which assigns to each $x\in X$ a set $F(x)$ so that $x\leqslant y$ in P if and only if $F(x)\subseteq F(y)$ is called an inclusion representation. Every poset has such a representation, so it is natural to consider restrictions on the nature of the images of the function F. In this paper, we consider inclusion representations assigning to each $x\in X$ a sphere in $\mathbb{R}^d$, d-dimensional Euclidean space. Posets which have such representations are called sphere orders. When $d=1$, a sphere is just an interval from $\mathbb{R}$, and the class of finite posets which have an inclusion representation using intervals from $\mathbb{R}$ consists of those posets which have dimension at most two. But when $d\geqslant 2$, some posets of arbitrarily large dimension have inclusion representations using spheres in $\mathbb{R}^d$. However, using a theorem of Alon and Scheinerman, we know that not all posets of dimension $d+2$ have inclusion representations using spheres in $\mathbb{R}^d$. In 1984, Fishburn and Trotter asked whether every finite 3-dimensional poset has an inclusion representation using spheres (circles) in $\mathbb{R}^2$. In 1989, Brightwell and Winkler asked whether every finite poset is a sphere order and suggested that the answer was negative. In this paper, we settle both questions by showing that there exists a finite 3-dimensional poset which is not a sphere order. The argument requires a new generalization of the Product Ramsey Theorem which we hope will be of independent interest.

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1. Introduction

Given a partially ordered set (poset) $\boldsymbol{P}=(X,P)$, a function F which assigns to each $x\in X$ a set $F(x)$ is called an *inclusion representation* of $\boldsymbol{P}$ if $x\leqslant y$ in P if

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and only if $F(x)\subseteq F(y)$. Every poset has such a representation. For example, just take $F(x)=\{y\in X\colon\ y\leqslant x \text{ in } P\}$. In recent years, there has been considerable interest in inclusion representations where the images of the function F are required to be geometric objects of a particular type, with attention focused on circles and spheres. We refer the reader to [8] for a summary of results in this area and an extensive bibliography.

As is well known, the finite posets of dimension at most two are just those which have inclusion representations using closed intervals of the real line $\mathbb{R}$. Because a closed interval of $\mathbb{R}$ can also be considered as a sphere in $\mathbb{R}^1$, it is natural to ask which posets have inclusion representations using circular disks in $\mathbb{R}^2$. For historical reasons, these posets are called *circle orders*. Fishburn [5] showed that all interval orders are circle orders. Also, the so called *standard examples* of n-dimensional posets, the posets consisting of all 1-element and $(n-1)$-element subsets of $\{1,2,\dots,n\}$, ordered by inclusion, are circle orders. So among the circle orders are *some* posets of arbitrarily large dimension.

Call a poset $\boldsymbol{P}$ a *sphere order* if there is some $d\geqslant 1$ for which it has an inclusion representation using spheres in $\mathbb{R}^d$. Using the 'degrees of freedom' theorem of Alon and Scheinerman [1], it follows that not all posets of dimension $d+2$ have inclusion representations using spheres in $\mathbb{R}^d$. In particular, when $d=2$, we conclude that there are 4-dimensional posets which are not circle orders. In this case, an explicit example can be given, as Sidney et al. [23] have shown that the 4-dimensional poset consisting of the 14 proper nonempty subsets of $\{1,2,3,4\}$ ordered by inclusion is not a circle order.

In [22], Scheinerman and Wierman used a very nice Ramsey theoretic argument to show that the countably infinite 3-dimensional poset $\mathbb{Z}^3$ is not a circle order. They also noted that $\{1,2,\dots,n\}\times\{1,2,\dots,n\}\times\mathbb{N}$ is not a circle order when n is sufficiently large. In [4], El-Zahar and Fateen show that the three dimensional poset $\mathbf{4}^3$ is a circle order, a result which is harder than it may at first appear. Additional contributions along this line appear in Hurlbert [12], Lin [14] and Fon-Der–Flaass [10]. The last of these proves that $\{1,2\}\times\{1,2,3\}\times\mathbb{N}$ is not a sphere order.

These results leave open the following question:

Question 1. *Is every finite* 3*-dimensional poset a circle order*?

This question was raised by Fishburn and Trotter at the Banff meeting on ordered sets in 1984 but has also been posed by other researchers. Although the results in the preceding paragraph suggest that the answer is negative, some evidence supports a positive answer. As shown in [25], for every finite 3-dimensional poset $\boldsymbol{P}$ and every integer $n\geqslant 3$, $\boldsymbol{P}$ has an inclusion representation using regular n-gons in the plane. So it is natural to surmise that as $n\to\infty$, we may be able to pass to a limit and obtain the desired inclusion representation using circles.

Some of the motivation for questions involving inclusion representations for posets comes from the parallel concept of intersection graphs. For example, Maehara [15] showed that for every finite graph $\boldsymbol{G}=(V,E)$, there is some $d\geqslant 1$ so that $\boldsymbol{G}$ is the intersection graph of a family of spheres in $\mathbb{R}^d$. The corresponding question for posets was posed independently by Brightwell and Winkler [3] and by Meyer [16]. Brightwell and Winkler also conjectured that the answer is negative.

Question 2. *Is every finite poset a sphere order?*

This paper settles Question 1 and Question 2 with the following result.

Theorem 1.1. *There exists a finite* 3*-dimensional poset which is not a sphere order.*

Inclusion representations that use circles and spheres have other applications and have been studied for a variety of reasons. For example, Scheinerman [19] proved that a graph $G=(V,E)$ is planar if and only if the poset formed by its vertices and edges, ordered by inclusion, is a circle order. Knight [13] has studied representation problems using non-standard analysis, while Meyer [16–18] and Brightwell and Gregory [2] have investigated the modeling of time and space with spheres, an approach of interest to physicists.

Additional information on circle and sphere orders appears in Scheinerman [20], [21], while more general geometric objects are considered in Fishburn and Trotter [7], Sidney et al. [23], Tanenbaum [24], Urrutia [28] and other papers cited in Fishburn and Trotter [8].

The remainder of the paper is organized as follows. Section 2 provides basic notation and terminology. Section 3 outlines the proof. Section 4 gathers important Ramsey theoretic tools essential to our argument, tools which we feel will have applications beyond this paper. Section 5 includes some elementary but technical results. In Sections 6–11, we present the proof of Theorem 1.1. Section 12 discusses related problems and research directions.

2. Notation and terminology

Although we are concerned primarily with finite posets, we will use the letters $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$ to denote respectively the set of real numbers, the set of integers and the set of positive integers. Also let $\mathbb{R}_0$ denote the set of all positive real numbers. For positive integers n and t, let $\boldsymbol{n}$ denote the chain $0<1<\cdots<n-1$, and let $\boldsymbol{n}^t$ denote the cartesian product of t copies of $\boldsymbol{n}$, so that $(i_1,i_2,\dots,i_t)\leqslant(j_1,j_2,\dots,j_t)$ in $\boldsymbol{n}^t$ if $i_k\leqslant j_k$ in $\boldsymbol{n}$ for $k=1,2,\dots,t$.

Given a poset $\boldsymbol{P}=(X,P)$, recall that the the minimum cardinality of a family of linear extensions of P whose intersection is P is called the *dimension* of $\boldsymbol{P}$ and is denoted by $\dim(\boldsymbol{P})$. We refer the reader to [25] for additional background material on the subject of dimension for partially ordered sets and to [26,27] for more discussion of connections between graphs and posets. Here we will need only a few basic facts from dimension theory. The most important of these is that a finite poset has dimension at most t if and only if there is an integer n for which it is isomorphic to a subposet of $\boldsymbol{n}^t$. Hence, to prove Theorem 1.1, it then suffices to establish the following result.

Theorem 2.1. *There exists an integer n_0 so that if $n \geqslant n_0$, the finite 3-dimensional poset $\boldsymbol{n}^3$ is not a sphere order.*

Given a partial order P on a set X, the *dual* of P, denoted by P^d, is the partial order on X defined by $x<y$ in P^d if and only if $x>y$ in P. If $\boldsymbol{P}=(X,P)$, we denote the poset (X,P^d) by $\boldsymbol{P}^d$ and refer to it as the dual of $\boldsymbol{P}$. As is well known, $\dim(\boldsymbol{P})=\dim(\boldsymbol{P}^d)$ for every poset $\boldsymbol{P}$. A poset is said to be *self dual* if it is isomorphic to its dual. Note that if $\boldsymbol{P}$ is a product of chains, then it is self-dual.

For positive integers n, d and t, we consider inclusion representations of the poset $\boldsymbol{n}^t$ using spheres from $\mathbb{R}^d$. We use the letters u, v, w, x, y, z, B and T to denote elements of $\boldsymbol{n}^t$. For example, the coordinates of x for $t=3$ would be $(x(1),x(2),x(3))$. Also, we write, for example, $x=(5,4,7)$ to indicate the element in $\boldsymbol{n}^3$ with $x(1)=5$, $x(2)=4$ and $x(3)=7$.

Given an inclusion representation F of $\boldsymbol{n}^3$, using spheres in $\mathbb{R}^d$, the center of the sphere $F(x)$ will be denoted by $c(x)$. We never refer explicitly to the coordinates of $c(x)$, as we wish to emphasize that our argument is independent of the value of d. Also, we let $r(x)$ denote the radius of the sphere $F(x)$.

We will use the symbol s (with various subscripts) to denote points in $\mathbb{R}^d$ which may or may not be centers of spheres in our representation. We denote the Euclidean distance between points s_1 and s_2 from $\mathbb{R}^d$ by $\rho(s_1,s_2)$. When x and y are points in $\boldsymbol{n}^3$, we abbreviate $\rho(c(x),c(y))$ by $\rho(x,y)$. Accordingly, the inclusion rule may be stated as follows:

$$x \leqslant y \text{ in } \boldsymbol{n}^3 \text{ if and only if } r(y)-r(x) \geqslant \rho(x,y). \tag{1}$$

In other words, one sphere is contained in another when the difference in their radii is at least as large as the distance between the centers. Technically speaking, we should write $\rho_F(x,y)$ because the distance between $c(x)$ and $c(y)$ depends on F. However, in our proof, once an inclusion representation F is determined, we make at most two modifications to the representation, and both leave the distance between centers invariant.

Given two points s_1 and s_2 in $\mathbb{R}^d$, let $L(s_1,s_2)$ denote the line they determine. The line $L(c(x),c(y))$ will be abbreviated by $L(x,y)$.

Given three non-collinear points s_1, s_2 and s_3, let $\phi(s_1,s_2,s_3)$ denote the angle at s_1 determined by $L(s_1,s_2)$ and $L(s_1,s_3)$. Also let $\gamma(s_1,s_2,s_3)$ denote the angle formed at s_3 by $L(s_1,s_3)$ and $L(s_2,s_3)$. Then let $p(s_1,s_2,s_3)$ denote the unique point on $L(s_1,s_3)$ which is closest to s_2, and let $h(s_1,s_2,s_3)=\rho(s_2,p(s_1,s_2,s_3))$ (see Fig. 1). As usual, when discussing centers, we will just write $\phi(x,y,z)$, $\gamma(x,y,z)$, $p(x,y,z)$ and $h(x,y,z)$.

The proof of our main theorem uses a sequence of 'large constants' which we define inductively by setting $N_0=10^6$ and $N_{i+1}=10^6 N_i$ for $i \geqslant 0$. The purpose of these constants is to control the magnitude of errors used in approximations. For example

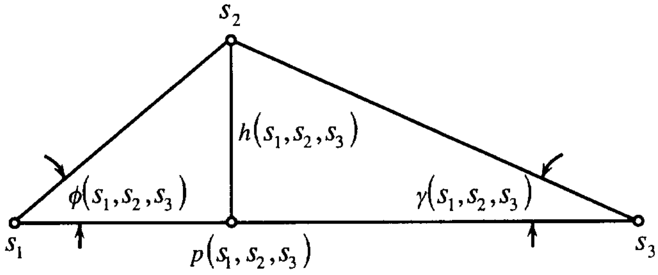

Fig. 1.

if we know that e_1, e_2, e_3 and e_4 are positive numbers with $e_1 < (1 + 1/N_{i+1})e_2$ and $e_3 < (1 + 1/N_{i+1})e_4$ for some $i \geqslant 0$, then $e_1e_3 < (1 + 1/N_i)e_2e_4$. In other words, the accuracy of various estimates will deteriorate as we combine expressions, but we will need to control the degree to which this occurs. In almost all cases, such inequalities will be quite generous.

In the closing stages of the argument, we will use the following 'shorthand' notation: When e_1 and e_2 are positive quantities, and we write $e_1 \ll e_2$, it will always be the case that $N_ie_1 < e_2$ for some $i > 0$. Also, when we write $e_1 \lesssim e_2$, it will always be the case that $e_1 < e_2(1+1/N_i)$ for some $i > 0$. The notation $e_2 \gg e_1$ is just an alternative for $e_1 \ll e_2$, while $e_2 \gtrsim e_1$ means the same as $e_1 \lesssim e_2$. Similarly, when we write $e_1 \approx e_2$, it will always be the case that $e_1 < e_2(1 + 1/N_i)$ and $e_2 < e_1(1 + 1/N_i)$ for some $i > 0$. In such cases, we may also write $e_1 \lesssim e_2 \lesssim e_1$.

Important Note. The notations $e_1 \ll e_2$, $e_1 \lesssim e_2$ and $e_1 \approx e_2$ are just shorthand and are not intended as formal definitions. For example, there is no specific value of M so that we write $e_1 \ll e_2$ if and only if $Me_1 < e_2$. Instead, when we write $e_1 \ll e_2$, it is intended to remind us that at some point earlier in the argument, we have determined that there is some $i > 0$ for which $N_ie_1 < e_2$. Whenever these shorthand notations are used, the actual inequalities will be enough to justify the application of the 'transitive law', at least when combining statements a bounded number of times. For example, whenever we write $e_1 \lesssim e_2$ and $e_2 \lesssim e_3$, the precise inequalities will be sufficiently strong that we are justified in writing $e_1 \lesssim e_3$. In the same spirit, when we write $e_1 \lesssim e_2$ and $e_3 \lesssim e_4$, the precise inequalities will be sufficiently generous that we could also write $e_1 + e_3 \lesssim e_2 + e_4$ and $e_1e_3 \lesssim e_2e_4$.

When arguing to a contradiction using quantities compared with this shorthand notation, we must be careful to avoid such traps as believing that

$$e_1 < e_2 \lesssim e_3 < e_4 < e_5 < e_1$$

results in a contradiction, because it only leads to the conclusion that the five quantities are approximately the same. So to obtain a contradiction, we will always show (at least) something like

$$e_1 \lesssim e_2 \quad \text{and} \quad 2e_2 \lesssim e_1.$$

3. Outline of the proof

The basic idea of the proof is straightforward. We will assume that we have an inclusion representation of $\boldsymbol{n}^3$ using spheres in $\mathbb{R}^d$ and argue to a contradiction — provided n is sufficiently large. The issue as to how large n must be in order to reach this contradiction will be independent of the value of d.

After applying Ramsey theoretic results to uniformize the spheres in our representation, we will be able to assume that the centers of the spheres lie very close to a line which passes through the center $c(B)$ of the bottom point $B=(0,0,0)$. Each sphere will have as its radius a value which is almost exactly the same as the distance from its center to $c(B)$. Given any two points $x, y \in \boldsymbol{n}^3$, both distinct from B, the center of one will be much closer to $c(B)$.

For distinct points x and y from $\boldsymbol{n}^3$, we define

$$\mathrm{gap}(x,y)=r(y)-r(x)-\rho(x,y).$$

When $x<y$, $\mathrm{gap}(x,y)>0$, and when x is incomparable to y, $\mathrm{gap}(x,y)<0$. However, as a consequence of our Ramsey theoretic arguments, $\rho(x,y)$, $|r(y)-r(x)|$ and $\max\{r(x),r(y)\}$ will all be approximately equal, so we will need to pay careful attention to the magnitude of the error terms.

For three distinct points x, y and z, let

$$\Delta(x,y,z)=\rho(x,y)+\rho(y,z)-\rho(x,z).$$

Clearly, $\Delta(x,y,z)\geqslant 0$, and $\Delta(x,y,z)>0$ when the centers are not collinear.

The proof of our main theorem focuses on a 2-element chain $x<z$ and the quantity $\mathrm{gap}(x,z)$. We will obtain upper bounds on $\mathrm{gap}(x,z)$ by considering a point incomparable to both x and z. For example, suppose v is such a point. Then

$$r(z)-r(x)=(r(v)-r(x))+(r(z)-r(v))<\rho(x,v)+\rho(v,z),$$

so that

$$\mathrm{gap}(x,z)<\Delta(x,v,z).$$

Since this bound holds for any point incomparable to both x and z, we may consider several candidate points and take the best bound they produce.

To obtain a lower bound, we consider an integer k and a chain C of $2k+1$ points having x as its bottom element and z as its top element. Let $C=\{x=u_1<u_2<\cdots<u_{2k+1}=z\}$ be such a chain. Then

$$\begin{aligned} r(z)-r(x) &= r(u_{2k+1})-r(u_1) \\ &= \sum_{i=1}^{2k}[r(u_{i+1})-r(u_i)] \\ &> \sum_{i=1}^{2k}\rho(u_{i+1},u_i) \end{aligned}$$

$$= \sum_{i=1}^{k} [\rho(u_{2i+1}, u_{2i-1}) + \Delta(u_{2i-1}, u_{2i}, u_{2i+1})]$$

$$\geqslant \rho(u_1, u_{2k+1}) + \sum_{i=1}^{k} \Delta(u_{2i-1}, u_{2i}, u_{2i+1}).$$

$$= \rho(x,z) + \sum_{i=1}^{k} \Delta(u_{2i-1}, u_{2i}, u_{2i+1}).$$

Setting

$$\Delta(x,C,z) = \sum_{i=1}^{k} \Delta(u_{2i-1}, u_{2i}, u_{2i+1}),$$

we conclude that

$$\text{gap}(x,z) > \Delta(x,C,z).$$

In all cases, we will obtain a contradiction by carefully choosing a point v, with v incomparable to both x and z, and a chain C having x and z as its bottom and top elements so that

$$\Delta(x,v,z) < \Delta(x,C,z).$$

The chain C will often consist of x, z and one intermediate point, but there are cases that need several intermediate points.

Although our argument depends heavily on Ramsey theory to assure that the representation is suitably regular, we must avoid any dependence on the dimension of the space from which the spheres in the representation are taken.

4. Extensions of the product Ramsey theorem

Given a finite set S and an integer k with $0 \leqslant k \leqslant |S|$, we denote the set of all k-element subsets of S by $\binom{S}{k}$. Given integers t and k and finite sets $S_1, S_2, \ldots, S_t$, we call an element of $\binom{S_1}{k} \times \binom{S_2}{k} \times \cdots \times \binom{S_t}{k}$ a *grid* (also, a $\boldsymbol{k}^t$ grid). When g is a $\boldsymbol{k}^t$ grid and $g = T_1 \times T_2 \times \cdots \times T_t$, we call the set T_j the jth *factor set* of g for each $j = 1, 2, \ldots, t$. Also, if $T_j = \{i_{j,1} < i_{j,2} < \cdots < i_{j,t}\}$, we refer to $i_{j,s}$ as the sth element of the jth factor set of g.

Using the natural order, a set of integers is also a chain, so given sets $S_1, S_2, \ldots, S_t$ of integers, we can consider $S_1 \times S_2 \times \cdots \times S_t$ as a poset. This poset is just a product of chains and has the form $\boldsymbol{n}_1 \times \boldsymbol{n}_2 \times \cdots \times \boldsymbol{n}_t$, where $n_i = |S_i|$ for $i = 1, 2, \ldots, t$. Similarly, if $T_i \subseteq S_i$ for each $i = 1, 2, \ldots, t$, then $T_1 \times T_2 \times \cdots \times T_t$ is then a subposet of $S_1 \times S_2 \times \cdots \times S_t$. If $|T_i| = k$ for all $i = 1, 2, \ldots, t$, then this subposet is also a grid. However, in material to follow, we will also associate with a $\boldsymbol{k}^t$ grid g a particular chain in the subposet which it determines. Specifically, with a $\boldsymbol{k}^t$ grid $g = T_1 \times T_2 \times \cdots \times T_t$, we associate the k-element chain $x_1 < x_2 < \cdots < x_k$ where $x_s(j)$ is the sth element of the factor set of g.

The following Product Ramsey Theorem, stated here in poset form, will be used extensively in making certain uniformizing assumptions about the inclusion representation. We refer the reader to [11] for the proof and additional material on Ramsey theory.

Theorem 4.1. *Given positive integers m, k, r and t, there exists an integer n_0 so that if $n \geqslant n_0$ and f is any map which assigns to each $\mathbf{k}^t$ grid of $\mathbf{n}^t$ a color from $\{1,2,\dots,r\}$, then there exists a subposet $\mathbf{P}$ isomorphic to $\mathbf{m}^t$ and a color $\alpha \in \{1,2,\dots,r\}$ so that $f(g)=\alpha$ for every $\mathbf{k}^t$ grid g from $\mathbf{P}$.*

We will refer to the least n_0 for which the conclusion of the preceding theorem holds as the Product Ramsey number $\mathrm{PR}(m,k,r,t)$.

Recall that $x \leqslant y$ in $\mathbf{n}^t$ if and only if $x(i) \leqslant y(i)$ for $i=1,2,\dots,t$. So it does not follow that $x(i)<y(i)$ for $i=1,2,\dots,t$ when $x<y$ in $\mathbf{n}^t$. Nevertheless, the following elementary proposition allows us to assume that if $x \neq y$, then $x(i) \neq y(i)$ for $i=1,2,\dots,t$. We view this proposition as a 'spacing' tool in that it allows us to assume that distinct points have all coordinates distinct and separated by some reasonable amount.

Proposition 4.2. *Let m, n and G be positive integers with $n \geqslant Gm^t$. Then the function $I:\mathbf{m}^t \to \mathbf{n}^t$ defined (cyclically) by*

$$I(x)(i)=G\sum_{j=1}^{t} x(i+j-1)(m-1)^{t-j-1}$$

is an embedding. Furthermore,

1. *If $x,y \in \mathbf{m}^t$, $i \in \{1,2,\dots,t\}$ and $x(i)<y(i)$, then $I(x)(i)<I(y)(i)$.*
2. *If $x,y \in \mathbf{m}^t$ and $x \neq y$, then $|I(x)(i)-I(y)(i)| \geqslant G$ for $i=1,2,\dots,t$.*

In what follows, we refer to the preceding result as the 'spacing proposition' and we call the integer G the *gap size* of the embedding I.

Let $\mathbf{P}$ be a poset and let f map $\mathbf{P}$ into $\mathbb{R}$. We say f is *monotonic* if it is either order-preserving or order-reversing. Now consider an order-preserving function f which maps $\mathbf{n}^t$ (or a subposet of $\mathbf{n}^t$) to $\mathbb{R}$. We say that f is *dominated* by coordinate α if for all x and y from its domain, $f(x)<f(y)$ whenever $x(\alpha)<y(\alpha)$. Dually, given an order-reversing function f, we say that f is *dominated* by α if for all x and y from its domain, $f(x)>f(y)$ whenever $x(\alpha)<y(\alpha)$.

In [9], Fishburn and Graham used the Product Ramsey Theorem to obtain the following result.

Theorem 4.3. *Given integers m and t, there exists an integer n_0 so that if $n \geqslant n_0$ and f is any injective function from $\mathbf{n}^t$ to $\mathbb{R}$, then there exist a coordinate $\alpha \in \{1,2,\dots,t\}$ and a subposet $\mathbf{P}$ isomorphic to $\mathbf{m}^t$ so that the restriction of f to $\mathbf{P}$ is monotonic and dominated by coordinate α.*

We stated the preceding theorem (and all to follow) in terms of injective functions, because all the functions we consider may be assumed to be injective. If this assumption is dropped, then a modestly more complicated concept of domination is needed, and the conclusions of the theorems have additional cases. However, the basic principles we discuss here apply to arbitrary functions.

Here is one elementary consequence of coordinate domination.

Proposition 4.4. *Let f, g and h be monotonic injective functions from $\boldsymbol{n}^t$ to $\mathbb{R}_0$, each dominated by a coordinate. If $h(x)=f(x)g(x)$ for all x in $\boldsymbol{n}^t$, then two of the three functions are dominated by the same coordinate.*

Proof. We provide the proof when f is order-preserving and g is order-reversing, all other cases being similar.

Suppose the conclusion fails and f, g and h are dominated by distinct coordinates, say f by coordinate 1, g by coordinate 2 and h by coordinate 3. Then consider the points $x_1=(1,3,2,0,0,\ldots,0)$, $x_2=(2,2,3,0,0,\ldots,0)$, and $x_3=(3,1,1,0,0,\ldots,0)$. Observe that $h(x_1)<h(x_2)<h(x_3)$, $x_1(3)=2$, $x_2(3)=3$ and $x_3(3)=1$. Thus h cannot be dominated by coordinate 3, regardless of whether it is order-preserving or order-reversing. □

Note that if f is a monotonic function from $\boldsymbol{n}^t$ to $\mathbb{R}_0$ and f is dominated by coordinate α, then the reciprocal of f is also dominated by coordinate α, as is the square of f.

One central concept in our proof is the notion of how fast a function changes. Now a sequence, even a strictly increasing sequence, does not have to change very much at all, but in this case, differences can change dramatically.

To provide further motivation, we present the following proposition.

Proposition 4.5. *For positive integers m and N with $N>2$, there exists an integer n_0 so that if $n\geqslant n_0$ and $a_1<a_2<\cdots<a_n$ is any strictly increasing sequence of real numbers, then there exists a subsequence $a_{p_1}<a_{p_2}<\cdots<a_{p_m}$ so that for all i,j,k,l with $1\leqslant i<j<k<l\leqslant m$, either*

$$a_{p_j}-a_{p_i}>N(a_{p_l}-a_{p_k}),$$

or

$$N(a_{p_j}-a_{p_i})<a_{p_l}-a_{p_k}.$$

We will be studying functions defined on $\boldsymbol{n}^t$ in what follows. Setting $u_i=(i,i,\ldots,i)$, the values of $f(u_i)$ form a long sequence, and we will want (at least) to control the behavior of f on a long subchain in a manner indicated by the conclusions of Proposition 4.5.

With these comments in mind, we present the basic definitions which will describe how a function changes. Let $\boldsymbol{P}=(X,P)$ be a poset and let N be any real number with $N>2$. We say an order-preserving function $f:\boldsymbol{P}\rightarrow\mathbb{R}_0$ is ACM(N) if $f(y)>Nf(x)$

whenever $f(y)>f(x)$. Dually, we say that an order-reversing function $f:\boldsymbol{P}\to\mathbb{R}_0$ is RAM(N) if $Nf(y)<f(x)$ whenever $f(y)<f(x)$. The notation in these two definitions are acronyms for *advances conservatively in magnitude* and *retreats aggressively in magnitude*, respectively. In both cases, the parameter N provides a lower bound for the *rate* at which the function changes.

We say a function $f:\boldsymbol{P}\to\mathbb{R}_0$ is NC(N) if $f(x)<f(y)(1+1/N)$ for all $x,y\in X$. The notation NC(N) is an abbreviation for *nearly constant*, and again the parameter N provides a tolerance for this assertion. Evidently, for a fixed value of N, the three properties ACM(N), RAM(N) and NC(N) are mutually exclusive. However, a function can be NC(N) without being monotonic.

When a function is nearly constant, we still need to describe how its differences behave. Accordingly, when f is an NC(N) order-preserving function, we say that f is AC(N) if $N(f(y)-f(x))<f(z)-f(y)$ whenever $f(x)<f(y)<f(z)$. This notation is an abbreviation for *advances conservatively*, although now we drop the reference to magnitude. Similarly, we say that an order-preserving NC(N) function f is AA(N) if $f(y)-f(x)>N(f(z)-f(y))$ whenever $f(x)<f(y)<f(z)$. Now AA(N) is an abbreviation for *advances aggressively* and again the reference to magnitude is dropped.

Dually, if f is an NC(N) order-reversing function, we say that f is RC(N) if $N(f(x)-f(y))<f(y)-f(z)$ whenever $f(x)>f(y)>f(z)$. We say that f is RA(N) if $f(x)-f(y)>N(f(y)-f(z))$ whenever $f(x)>f(y)>f(z)$.

Let $\mathscr{L}(N)=\{\mathrm{AC}(N),\mathrm{AA}(N),\mathrm{RC}(N),\mathrm{RA}(N),\mathrm{ACM}(N),\mathrm{RAM}(N)\}$. We call the elements of $\mathscr{L}(N)$ *change labels*. For a fixed value of $N>2$, at most one of these change labels applies to a function defined on a non-trivial poset — and for many functions, none of them is appropriate. The $6t$ elements of $\mathscr{L}(N)\times\{1,2,\ldots,t\}$ are called *change patterns*. A function $f:\boldsymbol{n}^t\to\mathbb{R}_0$ is said to be *N-uniform* if there exists a change pattern $(\boldsymbol{L},\alpha)$ so that f is $\boldsymbol{L}$ and is dominated by coordinate α. In this case, we say that *f satisfies the change pattern $(\boldsymbol{L},\alpha)$*.

With this background material in mind, we state a theorem which is only a gentle extension of Theorem 4.3. However, we will need an even stronger result, one for which the following theorem is an immediate corollary.

Theorem 4.6. *Given positive integers m, t and a real number N with $N>2$, there exists an integer n_0 so that if $n\geqslant n_0$ and $f:\boldsymbol{n}^t\to\mathbb{R}_0$ is any injective function, then there exist a subposet $\boldsymbol{Q}$ isomorphic to $\boldsymbol{m}^t$ and a change pattern $(\boldsymbol{L},\alpha)$, so that the restriction of f to $\boldsymbol{Q}$ is a N-uniform function satisfying $(\boldsymbol{L},\alpha)$.*

To prove our main theorem, we need to uniformize a large number of functions, a number which goes to infinity with n. The preceding result would allow us to handle only a bounded number of functions. Fortunately, the functions we need to uniformize have additional structure.

Let k and s be positive integers with $1\leqslant s\leqslant k$, and let A be a function which maps the $\boldsymbol{k}^t$ grids of $\boldsymbol{n}^t$ to $\mathbb{R}_0$. Then for each $(\boldsymbol{k}-\boldsymbol{1})^t$ grid g, we define a function $A_{g,s}$ on certain points in $\boldsymbol{n}^t$, namely on those points x (the set of such points may be empty) in $\boldsymbol{n}^t$ so that for each $i=1,2,\dots,t$, the coordinate $x(i)$ is larger than the smallest $s-1$ integers in the ith factor set of g and less than the largest $k-s$. Of course, when the ith coordinate of x is added to the ith factor set of g for $i=1,2,\dots,t$, we obtain a $\boldsymbol{k}^t$ grid g'. So we can define $A_{g,s}(x)=A(g')$. Note that the function $A_{g,s}$ has as its domain a poset which is a product of t chains — although in general the lengths of these chains is not constant. We call $A_{g,s}$ a (k,s)-*induced* function.

To make this more concrete, suppose we have an inclusion representation of $\boldsymbol{n}^3$ using spheres from $\mathbb{R}^d$. Then we can define a function A which maps the $\boldsymbol{3}^3$ grids from $\boldsymbol{n}^3$ to $\mathbb{R}_0$ as follows. With each $\boldsymbol{3}^3$ grid g', we associate a chain $x<y<z$, and then define $A(g')=\phi(x,y,z)$, the angle at x formed by $L(x,y)$ and $L(x,z)$. Now consider, for example, the value $s=2$. Then consider the $\boldsymbol{2}^3$ grid $g=\{10,23\}\times\{47,90\}\times\{18,45\}$. It follows that the $(3,2)$-induced function $A_{g,2}$ is defined on a subposet isomorphic to $\mathbf{12}\times\mathbf{42}\times\mathbf{26}$. Of course, the size of the subposet on which the function $A_{g,s}$ is defined depends both on g and s. However, if the set of points on which $A_{g,s}$ is defined is non-empty, we can discuss the issue of whether $A_{g,s}$ is N-uniform.

We are ready to present the main uniformizing theorem needed to prove Theorem 2.1.

Theorem 4.7. *Given positive integers m, t, k and a real number N with $N>2$, there exists an integer n_0 so that if $n\geqslant n_0$ and A is any injective function which maps the $\boldsymbol{k}^t$ grids of $\boldsymbol{n}^t$ to $\mathbb{R}_0$, then there exist k change patterns $(\boldsymbol{L}_1,\alpha_1),(\boldsymbol{L}_2,\alpha_2),\dots,(\boldsymbol{L}_k,\alpha_k)$ and a subposet $\boldsymbol{P}$ isomorphic to $\boldsymbol{m}^t$ so that for every $s=1,2,\dots,k$ and every $(\boldsymbol{k-1})^t$ grid g in $\boldsymbol{P}$, the (k,s)-induced function $A_{g,s}$ is N-uniform and satisfies change pattern $(\boldsymbol{L}_s,\alpha_s)$.*

Proof. Before beginning the proof, we comment that it is essential that the change pattern of an induced function $A_{g,s}$ depends only on s, not on g. There are only k choices for s, but the number of choices for g can be much larger than n. To help the reader keep track of sizes, we will always use g, g' and g' (with subscripts) to denote grids of size $(\boldsymbol{k-1})^t$, $\boldsymbol{k}^t$ and $(\boldsymbol{k+1})^t$, respectively.

Set $q=\lceil 100tkN\log Nm^t\rceil$ and $l=k(2^{4t}+3\cdot 2^{2t})$. Then set $r=2^l$. We now show that the value $n_0=\mathrm{PR}(q,k+1,r,t)$ satisfies the conclusion of our theorem. To accomplish this, we start with a poset $\boldsymbol{P}_0$ isomorphic to $\boldsymbol{n}_0^t$. We will then determine subposets $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ with $\boldsymbol{P}_{i+1}$ a subposet of $\boldsymbol{P}_i$ for $i=0,1$. For each $i=0,1,2$, $\boldsymbol{P}_i$ will be isomorphic to $\boldsymbol{n}_i^t$. The values of the other parameters are $n_1=q$ and $n_2=m$.

To show that the specified value of n_0 works, we first describe a coloring of the $(\boldsymbol{k+1})^t$ grids in $\boldsymbol{n}_0^t$.

Let A be any injective function which maps the $\boldsymbol{k}^t$ grids of $\boldsymbol{n}_0^t$ to $\mathbb{R}_0$. We use A to define a coloring of the $(\boldsymbol{k+1})^t$ grids of $\boldsymbol{n}_0^t$ using r colors.

Given a $(\boldsymbol{k+1})^t$ grid g'', we let $\{i_{j,1},i_{j,2},\dots,i_{j,k+1}\}$ denote the jth factor set of g'' for each $j=1,2,\dots,t$. For each $s=1,2,\dots,k$, we consider the set $G(g'',s)$ of all $\boldsymbol{k}^t$ grids having factor sets obtained from the factor sets of g'' by deleting exactly one of $i_{j,s}$ and $i_{j,s+1}$ for each $j=1,2,\dots,t$. For a fixed value of s, there is a natural correspondence which associates with each $\boldsymbol{k}^t$ grid $g'\in G(g'',s)$ a subset $S\subseteq\{1,2,\dots,t\}$

by taking $S=\{j:i_{j,s+1}$ belongs to the jth factor set of $g'\}$. So we can label the 2^s grids in $G(g'',s)$ as $g'(g'',s,S)$ where $S\subseteq\{1,2,\dots,t\}$. With this convention, $g'(g'',s,\emptyset)$ corresponds to the subgrid in which the jth factor set is $\{i_{j,1},i_{j,2},\dots,i_{j,s},i_{j,s+2},i_{j,s+3},\dots,i_{j,k+1}\}$. When the grid g'' and the value of s is fixed, we may just refer to a grid in $G(g'',s)$ as a subset of $\{1,2,\dots,t\}$.

Again, let g'' be a $(\boldsymbol{k}+\mathbf{1})^t$ grid and let s be an integer with $1\leqslant s\leqslant k$. Then consider all the images of the grids in $G(g'',s)$ under the map A, using the abbreviation $A(S)$ for $A(g'(g'',s,S))$. Some of the following statements will be true (T) and some will be false (F), for various subsets S_1,S_2,S_3,S_4 of $\{1,2,\dots,t\}$.

1. $A(S_1)<A(S_2)$.
2. $NA(S_1)<A(S_2)$.
3. $A(S_1)<A(S_2)(1+1/N)$.
4. $N(A(S_1)-A(S_2))<A(S_3)-A(S_4)$.

To emphasize that these statements actually depend on both g'' and s, we refer to them collectively as $\Sigma(g'',s)$.

In each of the first three patterns, there are 2^{2t} ordered pairs of variables for which the statement can be meaningfully expressed. In the last pattern, there are 2^{4t} ordered 4-tuples for which the statement makes sense. So summing over all s, there are $l=k(2^{4t}+3\cdot 2^{2t})$ statements altogether. It follows that we may associate with g'' a string of T's and F's of length l. There are $r=2^l$ such strings.

So we have described a coloring of the $(\boldsymbol{k}+\mathbf{1})^t$ grids of $\boldsymbol{n}_0^t$ using r colors. Since $n_0=\mathrm{PR}(q,k+1,r,t)$, there is a subposet $\boldsymbol{P}_1$ isomorphic to $\boldsymbol{q}^t$ so that all $(\boldsymbol{k}+\mathbf{1})^t$ grids in $\boldsymbol{P}_1$ receive the same color. This uniform color is then a string of T's and F's which tells which statements in $\Sigma(g'',s)$ are true and which are false. Furthermore, the string depends only on s and not on g''. Accordingly, for the subposet $\boldsymbol{P}_1$ in which all grids receive the same color, we can refer to statements in the family $\Sigma(s)$, deleting g'' from our earlier notation.

Since $n_1=q=\lceil 100tkN\log Nm^t\rceil$, we may use the spacing proposition to choose a subposet $\boldsymbol{P}_2$ of $\boldsymbol{P}_1$, with $\boldsymbol{P}_2$ isomorphic to $\boldsymbol{m}^t$, so that $\boldsymbol{P}_2$ is embedded by I in $\boldsymbol{P}_1$ with gap size at least $\lceil 50tkN\log N\rceil$. This value is chosen so that it is comfortably larger than $\max\{t,k,N\log N\}$.

In the remainder of the proof, when we discuss coordinates of points from $\boldsymbol{P}_2$, we use the coordinates of their images in $\boldsymbol{P}_1$ — via the embedding I.

Now fix a value of s. We show that there exists a change pattern $(\boldsymbol{L},\alpha)$ so that if g is any $(\boldsymbol{k}-\mathbf{1})^t$ grid in $\boldsymbol{P}_2$, the induced (k,s) function $A_{g,s}$ is N-uniform and satisfies the change pattern $(\boldsymbol{L},\alpha)$. Once we have accomplished this goal, the proof of our theorem is complete.

Let g be any $(\boldsymbol{k}-\mathbf{1})^t$ grid in $\boldsymbol{P}_2$. We may assume without loss of generality that the subposet $\boldsymbol{Q}_2$ of points in $\boldsymbol{P}_2$ on which $A_{g,s}$ is defined is non-trivial, else there is nothing to prove. In the subposet $\boldsymbol{P}_1$, we let $\boldsymbol{Q}_1$ denote the domain of the (k,s)-induced function determined by the grid g. Of course, $\boldsymbol{Q}_2$ is a subposet of $\boldsymbol{Q}_1$. Since $\boldsymbol{Q}_2$ is non-trivial and the gap size of the embedding I is $4tk$, $\boldsymbol{Q}_1$ is isomorphic to a product of chains each having $4tk+2$ or more points.

If x and y are distinct points from $\boldsymbol{Q}_2$, then the coordinates of x and y — in $\boldsymbol{P}_1$ under the embedding I — together with the grid g forms a $(\boldsymbol{k}+\mathbf{1})^t$ grid g''. In the grid g'', we label the jth factor set $\{i_{j,1}<i_{j,2}<\cdots<i_{j,k+1}\}$. Note that $\{x(j),y(j)\}=\{i_{j,s},i_{j,s+1}\}$ for all $j=1,2,\ldots,t$.

As before, we associate x and y with subsets of $\{1,2,\ldots,t\}$. If $x<y$, then $x=\emptyset$ and $y=\{1,2,\ldots,t\}$, so $A_{g,s}$ is order-preserving if the statement

$$A(\emptyset)<A(\{1,2,\ldots,t\})$$

from $\Sigma(s)$ is true. Furthermore, $A_{g,s}$ is order-reversing if this statement is false.

In the remainder of the argument, we assume that $A_{g,s}$ is order-preserving. The argument when it is order-reversing is dual.

Next, we show that $A_{g,s}$ is dominated by a coordinate α which depends only on s and not on g. Consider the $(\boldsymbol{k}+\mathbf{1})^t$ grid g_0'' in $\boldsymbol{Q}_1$ with all factor sets $\{1,2,\ldots,k+1\}$.

For our fixed value of s, consider the grids in $G(g_0'',s)$ which correspond to singleton subsets. These are the grids of the form $g(g_0'',s,\{i\})$ where $i\in\{1,2,\ldots,t\}$. Using the abbreviation $\{i\}$ for $g(g_0'',s,\{i\})$, we choose the unique element $\alpha\in\{1,2,\ldots,t\}$ so that $A(\{\alpha\})>A(\{i\})$ for all $i\in\{1,2,\ldots,t\}$ with $\alpha\neq i$. We now show that $A_{g,s}$ is dominated by coordinate α. Without loss of generality, we may assume that $\alpha=1$.

We now turn our attention to the suposet $\boldsymbol{Q}_1$. For element $u\in\boldsymbol{Q}_1$, we will write $A(u)$ rather than $A_{g,s}(u)$.

Consider the following points $v_1,v_2,\ldots,v_t$ in $\boldsymbol{Q}_1$:

$$v_i(1)=i;$$

$$v_i(j)=t \quad \text{for } j=2,3,\ldots,t-i+1;$$

$$v_i(j)=1 \quad \text{for } j=t-i+2,\ldots,t.$$

Now let $i\in\{1,2,\ldots,t-1\}$. Then consider the $(\boldsymbol{k}+\mathbf{1})^t$ grid g_i'' in $\boldsymbol{P}_1$ whose jth factor set is the union of the jth factor set of g and the following coordinate values in $\boldsymbol{Q}_1$:

$$\{i,i+1\} \quad \text{if } j=1;$$

$$\{t,t+1\} \quad \text{if } j\leqslant 2\leqslant t-i;$$

$$\{1,t\} \quad \text{if } j=t-i+1;$$

$$\{0,1\} \quad \text{if } t-i+1<j\leqslant t.$$

Observe that in the grid g_i'', the point v_i corresponds to the singleton set $\{t-i+1\}$ while v_{i+1} corresponds to $\{1\}$. As a consequence, we see that $A(v_i)<A(v_{i+1})$ for all $i=1,2,\ldots,t-1$. By transitivity, $A(v_1)<A(v_t)$. Now observe that in the grid g_{t+1}'' formed by g and the coordinates of v_1 and v_t, v_t corresponds to the singleton set $\{1\}$ while v_1 corresponds to the complementary set $\{2,3,\ldots,t\}$. It follows that the statement $A(\{2,3,\ldots,t\})<A(\{1\})$ from $\Sigma(s)$ is true.

Now let x and y be distinct points from $\boldsymbol{Q}_2$ with $x(1)<y(1)$. We show that $A_{g,s}(x)<A_{g,s}(y)$. This is certainly true if $x<y$, so we assume that x and y are

incomparable. Again, consider the grid g'' in $\boldsymbol{P}_1$ formed by g and the coordinates of x and y. For this grid, let u be the element of $G(g'',s)$ corresponding to the singleton set $\{1\}$ and let v be the element corresponding to the complementary set $\{2,3,\dots,t\}$. Note that $x \leqslant v$ and $y \leqslant u$. Since the coloring of grids is uniform, we know that $A_{g,s}(u)=A(u)<A(v)=A_{g,s}(v)$. Since $A_{g,s}$ is order-preserving, we then conclude that $A_{g,s}(x) \leqslant A_{g,s}(u)<A_{g,s}(v) \leqslant A_{g,s}(v)$, so that $A_{g,s}(x)<A_{g,s}(y)$ as claimed.

We now show that the restriction of $A_{g,s}$ to $\boldsymbol{Q}_2$ is N-uniform and has a change pattern which depends only on s.

Consider the following statement from $\Sigma(s)$:

$$A(\{1,2,\dots,t\})<A(\emptyset)(1+1/N).$$

Suppose first that this statement is false. Then we know that $A_{g,s}(y) \geqslant (1+1/N)A_{g,s}(x)$ for every 2-element chain from $\boldsymbol{Q}$.

Let $q_0=\lceil 25N\log N\rceil$. Then each of the chains whose product forms the poset $\boldsymbol{Q}_1$ is at least $2q_0$ in length. Consider the chain $u_0<u_1<\cdots<u_{q_0}$ in $\boldsymbol{Q}_1$, where $u_i=(i,i,\dots,i)$. Observe that $A(u_{i+1}) \geqslant (1+1/N)A(u_i)$ for $i=1,2,\dots,q_0-1$. Since $q_0 \geqslant 25N\log N$, it follows that $NA(u_1)<A(u_{q_0})$. Therefore the statement

$$NA(\emptyset)<A(\{1,2,\dots,t\})$$

from $\Sigma(s)$ is true, and $NA_{g,s}(x)<A_{g,s}(y)$ for every 2-element chain $x<y$.

Now suppose that x and y are any two points from $\boldsymbol{Q}_2$ and that $A_{g,s}(x)<A_{g,s}(y)$. Since $A_{g,s}$ is dominated by coordinate α, we know that $x(\alpha)<y(\alpha)$. Since the gap size is at least 3, we may choose an integer β so that $x(\alpha)<\beta<\beta+1<y(\alpha)$. Now let u and v be any two points in $\boldsymbol{Q}_1$ so that $u<v$, $u(\alpha)=\beta$ and $v(\alpha)=\beta+1$. Then $A_{g,s}(x)<A_{g,s}(u)$, $NA_{g,s}(u)<A_{g,s}(v)$ and $A_{g,s}(v)<A_{g,s}(y)$. It follows that $NA_{g,s}(x)<A_{g,s}(y)$, so that $A_{g,s}$ is ACM(N).

Now suppose that the statement

$$A(\{1,2,\dots,t\})<A(\emptyset)(1+1/N)$$

from $\Sigma(s)$ is true. Then $A_{g,s}(y)<A_{g,s}(x)(1+1/N)$ for every 2-element chain $x<y$ from $\boldsymbol{Q}_2$. Let B be the bottom element of $\boldsymbol{Q}_2$ and let T be the top element. Then $A_{g,s}(B)<A_{g,s}(x)<A_{g,s}(T)$ for every other point x from $\boldsymbol{P}_1$. This shows that $A_{g,s}$ is NC(N).

We now show that $A_{g,s}$ is either AC(N) or AA(N). Suppose first that the statement

$$N(A(\{1\})-A(\emptyset))<A(\{1,2,\dots,t\})-A(\{1\})$$

from $\Sigma(s)$ is true. Then it follows that for every 3-element chain $x<y<z$ in $\boldsymbol{Q}$, $N(A_{g,s}(y)-A_{g,s}(x))<A_{g,s}(z)-A_{g,s}(y)$. Now let x, y and z be any three points from $\boldsymbol{P}$ with $A_{g,s}(x)<A_{g,s}(y)<A_{g,s}(z)$. Then, since the gap size of the embedding I is more than 3 and $A_{g,s}$ is dominated by coordinate α, we may find a 3-element chain $w_1<w_2<w_3$ so that $w_1(\alpha)<x(\alpha)<y(\alpha)<w_2(\alpha)<w_3(\alpha)<z(\alpha)$. Since $A_{g,s}(y)-A_{g,s}(x) <A_{g,s}(w_2)-A_{g,s}(w_1)$ and $A_{g,s}(w_3)-A_{g,s}(w_2)<A_{g,s}(z)-A_{g,s}(y)$, it follows that $N(A_{g,s}(y)-A_{g,s}(x))<A_{g,s}(z)-A_{g,s}(y)$. We conclude that $A_{g,s}$ is AC(N).

Dually, if the statement

$$N(A(\{1,2,\dots,t\}) - A(\{1\})) < A(\{1\}) - A(\emptyset)$$

from $\Sigma(s)$ is true, then $A_{g,s}$ is AA(N).

Now suppose that both statements from $\Sigma(s)$ are false. Then, referring to the chain $u_0 < u_1 < \dots, u_{q_0}$ discussed earlier in the proof, we note that if $0 \leqslant i < j < k < l \leqslant q_0$, we have

$$(A(u_j) - A(u_i))/N \leqslant A(u_l) - A(u_k) \leqslant N\big(A(u_j) - A(u_i)\big).$$

Observe that the interval $[A(u_0), A(u_{q_0})]$ is divided up into q_0 disjoint subintervals of the form $[A(u_j), A(u_{j+1})]$ where $0 \leqslant j < q$. Choose an integer j with $1 \leqslant j \leqslant q_0 - 2$ so that the length of the interval $[A(u_j), A(u_{j+1})]$ is as small as possible. Then set $i = 0$, $k = j+1$ and $l = q-1$ to conclude that the length of $[A(u_0), A(u_i)]$ is at most N times the length of $[A(u_j), A(u_k)]$. Similarly, the length of $[A(u_l), A(u_{q-1})]$ is at most N times the length of $[A(u_j), A(u_k)]$. Being generous, we can conclude that $j \leqslant N$ and $q - j \leqslant N$, so that $q \leqslant 2N$. This contradicts the fact that $q_0 = \lceil 25 N \log N \rceil$.

A dual argument shows that when $A_{g,s}$ is order-reversing, it is either RAM(N) or NC(N). When it is NC(N), it is either RC(N) or RA(N). □

Note that Theorem 4.6 is just the special case of Theorem 4.7 obtained when $k = 1$.

Although we stated Theorem 4.7 in terms of a single function A, it is clear that we can apply it to a bounded number of functions. In fact, this result — and for that matter, all the Ramsey theoretic material discussed here — can be treated in much greater generality.

5. Some technical preliminaries

Recall that we have defined a sequence of 'large constants' by setting $N_0 = 10^6$ and $N_{i+1} = 10^6 N_i$ for $i \geqslant 0$. One important theme which runs through our argument will be applications of the triangle inequality which we state in a 'weak' form.

The following elementary proposition is immediate.

Proposition 5.1. *Let $i \geqslant 1$ and let e_1, e_2 and e_3 be positive real numbers which satisfy the weak triangle inequality that the sum of any two is larger than the third divided by $1 + 1/N_{i+1}$. If $e_1 > N_{i+1} e_2$, then*

$$e_1/(1 + 1/N_i) < e_3 < e_1(1 + 1/N_i) \quad \text{so that } e_1 \approx e_3.$$

The situation described in Proposition 5.1 will arise when e_1, e_2 and e_3 are the lengths of the three sides of a triangle. It will also arise when e_1, e_2 and e_3 are angles formed by three rays intersecting in a common point. In these situations, the quantities will actually satisfy the ordinary triangle inequality, i.e., the sum of any two is larger than the third.

However, we will also use Proposition 5.1 when e_1, e_2 and e_3 are the heights of three triangles which share a common point, and in this case, the weak form is needed.

In our proof, we will refer to Proposition 5.1 as the 'triangle proposition'. We comment that the condition $i \geqslant 1$ in the triangle proposition is necessary to insure that the tolerance discussed in Section 2 for using the $\approx$ notation is respected.

We will make extensive use of 'small angle' approximations and other elementary trigonometric inequalities as summarized in the following proposition. More accurate approximations are available, but we do not need such precision here.

Proposition 5.2. *The following inequalities hold*:
1. $\sin\theta < \theta$ *when* $0 < \theta < \pi/2$;
2. $1 - \cos\theta < \theta^2/2$ *when* $0 < \theta < \pi/2$;
3. $\theta/2 < \sin\theta$ *when* $0 < \theta < 0.01$;
4. $\theta^2/10 < 1 - \cos\theta$ *when* $0 < \theta < 0.01$;
5. $\theta/(1 + 1/N_i) < \sin\theta$ *when* $i \geqslant 0$ *and* $0 < \theta < 1/N_i$;
6. $(\theta^2/2)/(1 + 1/N_i)) < 1 - \cos\theta$ *when* $i \geqslant 0$ *and* $0 < \theta < 1/N_i$;
7. *For every angle* θ *with* $0 < \theta < \pi/2$, $\sin^2\theta < 2(1 - \cos\theta)$.

6. Part 1: uniformizing the representation

This section begins the proof of Theorem 1.1. As discussed in Section 2, we prove Theorem 1.1 by showing that if n is sufficiently large, the finite 3-dimensional poset $\boldsymbol{n}^3$ is not a sphere order. We start with the assumption that we have an inclusion representation F of $\boldsymbol{n}^3$ using spheres from $\mathbb{R}^d$ and then argue to a contradiction — provided n is sufficiently large. The issue of how large n must be is decided in six steps. We begin by setting $n = n_0$ and $\boldsymbol{P} = \boldsymbol{P}_0 = \boldsymbol{n}_0^3$. Then, for each $i = 1, 2, \ldots, 6$, we will choose an appropriate subposet $\boldsymbol{P}_i$ of $\boldsymbol{P}_{i-1}$, with $\boldsymbol{P}_i$ isomorphic to $\boldsymbol{n}_i^3$. At each step, we increase the uniformity of the inclusion representation for the remaining points. At the final step, we will halt with $n_6 = 11$ and $\boldsymbol{P}_6$ isomorphic to $\mathbf{11}^3$. The relative sizes between $n_0, n_1, \ldots, n_6$ will be clear from the material to follow.

To begin, we assume that the spheres used in our representation are in 'general position', i.e.:
1. no two spheres are tangent;
2. all centers are distinct;
3. no three centers are collinear;
4. no four centers are coplanar;
5. all radii are distinct and positive;
6. the angles determined by any three centers are distinct;
7. the distances from any center to the line passing through two other centers are all distinct.

This assumption is allowed by the fact that we may add (in an order preserving manner) a small quantity to each radius without disturbing the inclusion relation. We may then make small perturbations in the center locations. Note that the fourth condition requires $d \geqslant 3$, and it is clear that we may make this assumption without loss of generality.

Assuming that n_0 is sufficiently large in terms of n_1 and the large constant N_{10}, we may apply Theorem 4.6 to find a subposet $\boldsymbol{P}_1$ isomorphic to $\boldsymbol{n}_1^3$ on which the radius function r is N_{10}-uniform. For the remainder of the paper, all discussions of uniformity of functions will be in terms of the parameter N_{10}, so for example, we will just write that a function is ACM rather than ACM(N_{10}).

When $x<y$, we know that $r(x)<r(y)$, so the function r must be order-preserving on $\boldsymbol{P}_1$. Without loss of generality, we assume that it is dominated by coordinate 1. So r satisfies one of the following three change patterns: (ACM, 1), (AC, 1), or (AA, 1).

Claim 1. *We may assume without loss of generality that r is* ACM.

Proof. Should r be AA, we explain how to modify our representation so that r is AC. We then show how to transform a representation in which r is AC into one where r is ACM.

Now suppose that r is AA. Choose a large positive number R_0, large enough so that $2N_{10}r(x)<R_0$ for every $x \in \boldsymbol{P}_1$. We then consider the function $\hat{r} : \boldsymbol{P}_1^d \rightarrow \mathbb{R}_0$ defined by $\hat{r}(x)=R_0 - r(x)$ for every $x \in \boldsymbol{P}_1^d$. Also let $\hat{F}$ be the inclusion representation which assigns to each $x \in \boldsymbol{P}_1^d$ the sphere with center at $c(x)$ and radius $\hat{r}(x)$. Note that $\hat{F}$ is an inclusion representation of $\boldsymbol{P}_1^d$. Furthermore, if $\hat{r}(x)<\hat{r}(y)$, then $(1 + 1/N_{10})N_{10}r(x)<2N_{10}r(x)<R_0<R_0 + N_{10}r(y)$. Thus $\hat{r}(y)=R_0 - r(y)<(1+1/N_{10})(R_0 - r(x))=\hat{r}(x)$, so $\hat{r}$ is NC.

Also, if $\hat{r}(x)<\hat{r}(y)<\hat{r}(z)$, then $r(x)>r(y)>r(z)$. Since r is AA, it follows that $r(y)-r(z)>N_{10}((r(x)-r(y))$. Thus, $N_{10}(\hat{r}(y)-\hat{r}(x))<\hat{r}(z)-\hat{r}(y)$. This shows that $\hat{r}$ is AC on of $\boldsymbol{P}_1^d$. So in this case, noting that $\boldsymbol{P}_1^d$ is isomorphic to $\boldsymbol{P}_1$, we drop the hats from the notation and replace $\boldsymbol{P}_1$ by $\boldsymbol{P}_1^d$. We now have a representation where the radius function is AC.

Now suppose we have a representation of $\boldsymbol{P}_1$ where the radius function is AC. Now let $B_1=(0,0,0)$ and $r_0=r(B_1)$. We then define a new radius function $\hat{r}(x)$ by setting $\hat{r}(x)=r(x)-r_0$ for every $x \in \boldsymbol{P}_1$. Since $\hat{r}(y)-\hat{r}(x)=r(y)-r(x)$ for every x and y in $\boldsymbol{P}_1$, we could equally well use $\hat{r}$ as our radius function.

Now let $x<y$ be any two elements of $\boldsymbol{P}_1$ with $\hat{r}(x)<\hat{r}(y)$ and $x>B_1$. Then $r(B_1)<r(x)<r(y)$. It follows that $N_{10}\hat{r}(x)=N_{10}(r(x) - r_0)=N_{10}(r(x) - r(B_1))<r(y) - r(x) <r(y) - r(B_1)=\hat{r}(y)$. It follows that $\hat{r}$ is ACM. Again, we drop the hats and use r to denote the new radius function. However, we now have a representation where the least element has a circle of radius zero. Since the criteria for uniformity and for inclusion are expressed in terms of strict inequalities, we add a small quantity to the radius of the bottom element.

These remarks complete the proof of our claim that we may assume that r is ACM. □

We next describe three functions A, B and C to which we will apply Theorem 4.7. In each case, we take the value $k=3$. As discussed in the previous section, with each $\mathbf{3}^3$ grid g in $\boldsymbol{P}_1$, we associate a 3-element chain $x<y<z$ and then set $A(g)=\phi(x,y,z)$, $B(g)=h(x,y,z)$ and $C(g)=h(x,y,z)\phi(x,y,z)/2$.

After applying Theorem 4.7 three times, once for each of these functions, we may assume that we have a subposet $\boldsymbol{P}_2$ isomorphic to $\boldsymbol{n}_2^3$ so that we have nine change patterns, one for each ordered pair from $\{A,B,C\}\times\{1,2,3\}$, so that the nine classes of $(3,s)$-induced functions they produce are N_{10}-uniform and have a change pattern depending only on the class.

We are only concerned with five of these nine classes:

1. The $(3,2)$ and $(3,3)$ functions induced by A.
2. The $(3,1)$ and $(3,2)$ functions induced by B.
3. The $(3,2)$ function induced by C.

We find it convenient to use the symbols Φ, Θ, K, H, and G to denote these functions, so that:

1. For each 2-element chain $x<z$, the $(3,2)$-induced function $\Phi(x,y,z)$ is defined on those y with $x<y<z$ by setting $\Phi(x,y,z)=\phi(x,y,z)$.
2. For each 2-element chain $x<y$, the $(3,3)$-induced function $\Theta(x,y,z)$ is defined on those z with $x<y<z$ by setting $\Theta(x,y,z)=\phi(x,y,z)$.
3. For each 2-element chain $y<z$, the $(3,1)$-induced function $K(x,y,z)$ is defined on those x with $x<y<z$ by setting $K(x,y,z)=h(x,y,z)$.
4. For each 2-element chain $x<z$, the $(3,2)$-induced function $H(x,y,z)$ is defined on those y with $x<y<z$ by setting $H(x,y,z)=h(x,y,z)$.
5. For each 2-element chain $x<z$, the $(3,2)$-induced function $G(x,y,z)$ is defined on those y with $x<y<z$ by setting $G(x,y,z)=h(x,y,z)\phi(x,y,z)/2$.

We will return to the discussion of these induced functions after we develop some geometric implications among the remaining spheres — those which provide a representation of $\boldsymbol{P}_2$.

7. Part 2: geometric implications

In the subposet $\boldsymbol{P}_2$, let $u_0=(0,n_2-1,n_2-1)$, and let $B_2=(0,0,0)$ denote the bottom (least) element of $\boldsymbol{P}_2$. Setting $n_3=n_2-2$, and letting $\boldsymbol{P}_3$ consist of all $x\in\boldsymbol{P}_2$ whose coordinates satisfy $0<x(i)<n_2-1$ for $i=1,2,3$, it follows that $\boldsymbol{P}_3$ is isomorphic to $\boldsymbol{n}_3^3$. Now let $x\in\boldsymbol{P}_3$. Since $u_0(1)=0<x(1)$, and r is ACM and dominated by coordinate 1, we know that $N_{10}r(u_1)<r(x)$.

The next two claims show that we are able to approximate the distance between points in $\boldsymbol{P}_3$ by the larger of the two radii.

Claim 2. *For all* $x\in\boldsymbol{P}_3$,

$$r(x)/(1+1/N_9)<\rho(x,B_2)<r(x) \quad \textit{so that } r(x)\approx\rho(x,B_2).$$

Proof. Let $x \in \boldsymbol{P}_3$. We first establish the upper bound. To accomplish this, consider a point $u \in \boldsymbol{P}_2$ with $u \neq B_2$. Then $B_2 < u$, so that $r(u) - r(B_2) > \rho(u, B_2)$. Thus $\rho(u, B_2) < r(u)$. In particular, $\rho(x, B_2) < r(x)$. Also, $\rho(u_0, B_2) < r(u_0)$.

For the lower bound,observe that x is incomparable to u_0, so $r(x) - r(u_0) < \rho(x, u_0) < \rho(x, B_2) + \rho(u_0, B_2) < \rho(x, B_2) + r(u_0)$. It follows that $r(x) < \rho(x, B_2) + 2r(u_0) < \rho(x, B_2) + 2r(x)/N_{10}$, and thus $r(x)/(1 + 1/N_9) < r(x)(1 - 2/N_{10}) < \rho(x, B_2)$. □

The next claim is our first application of the triangle proposition.

Claim 3. *For all $x, y \in \boldsymbol{P}_3$ with $x(1) < y(1)$,*

$$r(y)(1 - 1/N_8) < \rho(x, y) < r(y)(1 + 1/N_8) \quad \text{so that } \rho(x, y) \approx r(y).$$

Proof. Consider the triangle formed by the points $c(B_2), c(x)$ and $c(y)$ and the lengths of the three sides of this triangle: $e_1 = \rho(y, B_2)$, $e_2 = \rho(x, B_2)$ and $e_3 = \rho(x, y)$. The sum of any two of these quantities is larger than the third.

From Claim 2, we know that

$$r(x)/(1 + 1/N_9) < \rho(x, B_2) < r(x),$$

$$r(y)/(1 + 1/N_9) < \rho(y, B_2) < r(y).$$

Since r is ACM, dominated by coordinate 1 and $x(1) < y(1)$, we know that $N_{10} r(x) < r(y)$. Therefore, $e_1 > N_9 e_2$ and the claim follows from the triangle proposition. □

When x, y and z are distinct points in $\boldsymbol{P}_3$ and $x(1) < y(1) < z(1)$, we know that $\Delta(x, y, z) = \rho(x, y) + \rho(y, z) - \rho(x, z) > 0$. However, we can actually write the following elementary identity:

$$\Delta(x, y, z) = \rho(x, y)(1 - \cos\phi(x, y, z)) + \rho(y, z)(1 - \cos\gamma(x, y, z)). \tag{2}$$

The next claim shows that it is only the first term on the right-hand side of Eq. (2) which matters.

Claim 4. *If x, y and z are distinct points in $\boldsymbol{P}_3$ and $x(1) < y(1) < z(1)$, then*

$$\rho(x, y)(1 - \cos\phi(x, y, z)) > N_9 \rho(y, z)(1 - \cos\gamma(x, y, z)).$$

Proof. Note first that

$$\rho(x, y)\sin\phi(x, y, z) = h(x, y, z) = \rho(y, z)\sin\gamma(x, y, z).$$

Using only the fact that $\sin\phi(x, y, z) < 1$, we see that

$$\gamma(x, y, z) < \rho(x, y)/\rho(y, z) < 2r(y)/r(z) < 1/N_9.$$

Using the estimates developed in Section 5 and the bounds in Claim 3, we see that

$$\begin{aligned}\rho(y,z)(1-\cos\gamma(x,y,z)) &< r(z)\gamma^2(x,y,z)\\ &< 2r(z)\sin^2\gamma(x,y,z)\\ &= 2r(z)\sin^2\phi(x,y,z)\rho^2(x,y)/\rho^2(y,z)\\ &< 10\rho(x,y)\sin^2\phi(x,y,z)/N_{10}\\ &< 20\rho(x,y)(1-\cos\phi(x,y,z))/N_{10}\\ &< \rho(x,y)(1-\cos\phi(x,y,z))/N_9. \qquad \square\end{aligned}$$

The next claim follows immediately from Eq. (2) and Claim 4.

Claim 5. *If x, y and z are distinct points in $\boldsymbol{P}_3$ and $x(1)<y(1)<z(1)$, then*

$$\begin{aligned}r(y)(1-\cos\phi(x,y,z))/(1+1/N_8) &< \Delta(x,y,z)\\ &< r(y)(1-\cos\phi(x,y,z))(1+1/N_8),\end{aligned}$$

so that

$$\Delta(x,y,z)\approx r(y)(1-\cos\phi(x,y,z)).$$

When $x(1)<y(1)<z(1)$, we have already noted that $\gamma(x,y,z)<1/N_9$. But at this point, we cannot make such a claim for $\phi(x,y,z)$. However, we now show that we may assume that all $\phi(x,u,z)$ are very small, provided $x<u<z$.

To accomplish this, we use the Product Ramsey Theorem. With each $\mathbf{3}^3$ grid g, we associate a chain $x<u<z$ as described in the preceding section. Color the grid red if $\phi(x,u,z)<1/N_{10}$; otherwise, color it blue. Setting $n_3=\mathrm{PR}(n_4,3,2,3)$, we may find a subposet $\boldsymbol{P}_4$ isomorphic to $\boldsymbol{n}_4$ so that all $\mathbf{3}^3$ grids in $\boldsymbol{P}_4$ receive the same color. Now set $n_4=n_5^3$ and $n_5=17$. We may then choose a subposet $\boldsymbol{P}_5$ isomorphic to $\boldsymbol{n}_5^3$ via the embedding I (with gap size 1) as defined in the spacing proposition.

Claim 6. *For every 3-element chain $x<u<z$ in $\boldsymbol{P}_5$, $\phi(x,u,z)<1/N_{10}$.*

Proof. Suppose to the contrary that $\phi(x,u,z)\geqslant 1/N_{10}$ for some 3-element chain in $\boldsymbol{P}_5$. Considering coordinates in $\boldsymbol{P}_4$, we see that $\boldsymbol{P}_4$ contains a blue $\mathbf{3}^t$ grid. Thus all $\mathbf{3}^3$ grids in $\boldsymbol{P}_4$ are blue.

Then consider the 6-element chain $u_1<u_2<\ldots<u_6$ in $\boldsymbol{P}_4$, where $u_i=(i,i,i)$ for $i=1,2,\ldots,6$. Then let $x=u_1=(1,1,1)$, $v=(2,0,7)$, $u=u_5=(5,5,5)$ and $z=u_6=(6,6,6)$. Because r is dominated by coordinate 1, we know that $N_{10}r(v)<r(u_3)$, $N_{10}r(u_3)<r(u_4)$ and $N_{10}r(u_4)<r(u_5)=r(u)$. Thus $N_{10}^3r(v)<r(u)$.

Since

$$\Delta(x,u,z)>r(u)(1-\cos\phi(x,u,z))/(1+1/N_8)$$

and $\phi(x,u,z)\geqslant 1/N_{10}$, we conclude that $\Delta(x,u,z)>r(u)/(10N_{10}^2)$.

On the other hand, since $(1 - \cos\phi(x,v,z)) \leqslant 1$, we know that $\Delta(x,v,z) < 10r(v)$. Thus, $r(u)/(10N_{10}^2) < 10r(v)$ so that $r(u) < 100N_{10}^2 r(v)$, which is a contradiction. $\square$

For the remainder of the proof, we will use the symbols $B=(0,0,0)$ and $T=(16,16,16)$ to denote the bottom and top elements of $\boldsymbol{P}_5$. Also, we let $B'=(1,1,1)$, $B''=(2,2,2)$, $T''=(14,14,14)$, $T'=(15,15,15)$. We then let $\boldsymbol{P}_6$ consist of those points x in $\boldsymbol{P}_5$ with $2<x(i)<14$ for $i=1,2,3$. Then $B<B'<x<T'<T$ for every x in $\boldsymbol{P}_6$. As anticipated, $n_6=n_5-6=11$.

Also, for the remainder of the proof, we will let $C=\{u_1<u_2<\ldots<u_9\}$ be the 9-element chain in $\boldsymbol{P}_6$ defined by setting $u(i)=(i,i,i)$ for each $i=1,2,\ldots,9$. Of course, we intend that the coordinates of the points in C are given in $\boldsymbol{P}_6$ rather than in $\boldsymbol{P}_5$.

For emphasis, we point out that the triangle inequality holds for angles in $\mathbb{R}^d$.

Proposition 7.1. *If s_1,s_2,s_3 and s_4 are distinct, noncoplanar points in $\mathbb{R}^d$, $e_1=\phi(s_1,s_2,s_3)$, $e_2=\phi(s_1,s_3,s_4)$ and $e_3=\phi(s_1,s_2,s_4)$, then e_1,e_2 and e_3 satisfy the triangle inequality, i.e., the sum of any two is larger than the third.*

When we apply Proposition 7.1, each s_i will be the center of one of the spheres in our representation. The resulting proposition is stated for clarity.

Proposition 7.2. *Let x_1,x_2,x_3 and x_4 be distinct points from $\boldsymbol{P}_5$. Then let*

1. *$e_1=\phi(x_1,x_2,x_3)$, $e_2=\phi(x_1,x_3,x_4)$ and $e_3=\phi(x_1,x_2,x_4)$, and*
2. *$e_1'=\gamma(x_1,x_2,x_4)$, $e_2'=\gamma(x_1,x_3,x_4)$ and $e_3'=\gamma(x_2,x_3,x_4)$.*

Then the two sets $\{e_1,e_2,e_3\}$ and $\{e_1',e_2',e_3'\}$ each satisfy the triangle inequality, i.e., the sum of any two quantities in the set is larger than the third.

If s_1, s_2 and s_3 are distinct points from $\mathbb{R}^d$, then $\phi(s_1,s_2,s_3)=\phi(s_1,s_3,s_2)$ and $\gamma(s_1,s_2,s_3)=\gamma(s_2,s_1,s_3)$. On the other hand, note that $h(s_1,s_2,s_3)\neq h(s_2,s_1,s_3)$ in general. In fact, the two quantities can be far apart. However, due to the uniform behavior of the radius function, we do have approximate symmetry in the first two coordinates for centers.

Proposition 7.3. *Let x_1,x_2 and x_3 be distinct points from $\boldsymbol{P}_5$ with $x_1(1)<x_2(1)<x_3(1)$. Then*

$$r(x_3)\gamma(x_1,x_2,x_3)/(1+1/N_8) < h(x_1,x_2,x_3) < r(x_3)\gamma(x_1,x_2,x_3)(1+1/N_8),$$
$$r(x_3)\gamma(x_1,x_2,x_3)/(1+1/N_8) < h(x_2,x_1,x_3) < r(x_3)\gamma(x_1,x_2,x_3)(1+1/N_8),$$

so that

$$h(x_1,x_2,x_3) \approx h(x_2,x_1,x_3) \approx r(x_3)\gamma(x_1,x_2,x_3).$$

Proof. Observe that

$$h(x_1,x_2,x_3) = \rho(x_2,x_3)\sin\gamma(x_1,x_2,x_3) = \rho(x_1,x_2)\sin\phi(x_1,x_2,x_3).$$

Since $\sin\phi(x_1,x_2,x_3)<1$, it follows that $\gamma(x_1,x_2,x_3)<1/N_9$.

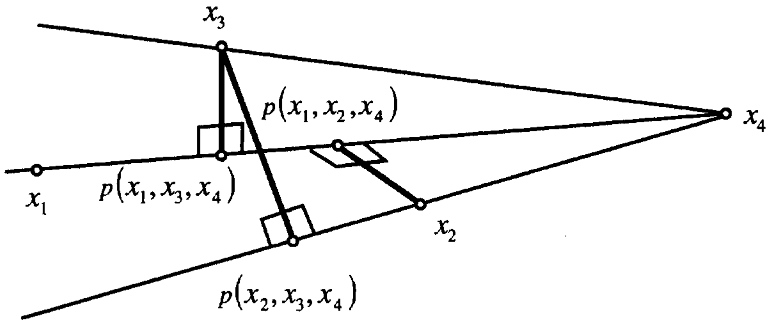

Fig. 2.

We also observe that

$$h(x_2,x_1,x_3)=\rho(x_1,x_3)\sin\gamma(x_2,x_1,x_3).$$

The conclusions of the proposition then follow from the fact that $\gamma(x_1,x_2,x_3)=\gamma(x_2,x_1,x_3)$ and the inequalities:

$$r(x_3)/(1+1/N_9)<\rho(x_2,x_3)<r(x_3)(1+1/N_9),$$
$$r(x_3)/(1+1/N_9)<\rho(x_1,x_3)<r(x_3)(1+1/N_9). \qquad \square$$

Taking advantage of the properties of our radius function, we will now derive a weak triangle inequality involving heights (see Fig. 2). From an intuitive standpoint, we consider this the 'view back from infinity'.

Proposition 7.4. *Let x_1,x_2,x_3 and x_4 be points from $\boldsymbol{P}_5$ with $x_i(1)<x_4(1)$ for $i=1,2,3$. Then let $e_1=h(x_1,x_2,x_4)$, $e_2=h(x_2,x_3,x_4)$ and $e_3=h(x_1,x_3,x_4)$. It follows that the sum of any two of e_1,e_2 and e_3 is larger than the third divided by $(1+1/N_7)$.*

Proof. Let $e_1'=\gamma(x_1,x_2,x_4)$, $e_2'=\gamma(x_2,x_3,x_4)$ and $e_3'=\gamma(x_1,x_3,x_4)$. Then e_1', e_2' and e_3' satisfy the triangle inequality.

From Proposition 7.3, we note that

$$r(x_4)\gamma(x_1,x_2,x_4)/(1+1/N_8)<h(x_1,x_2,x_4)<r(x_4)\gamma(x_1,x_2,x_4)(1+1/N_8),$$
$$r(x_4)\gamma(x_2,x_3,x_4)/(1+1/N_8)<h(x_2,x_3,x_4)<r(x_4)\gamma(x_2,x_3,x_4)(1+1/N_8),$$

and

$$r(x_4)\gamma(x_1,x_3,x_4)/(1+1/N_8)<h(x_1,x_3,x_4)<r(x_4)\gamma(x_1,x_3,x_4)(1+1/N_8).$$

Clearly, these statements imply the conclusion of the proposition. $\square$

Next we revisit the issue of the size of the angle $\phi(x,y,z)$ when $x(1)<y(1)<z(1)$.

Claim 7. *For all x,y and z in $\boldsymbol{P}_6$ with $x(1)<y(1)<z(1)$, $\phi(x,y,z)<1/N_9$.*

Proof. From Proposition 7.4, we know that $h(x,y,z)/(1+1/N_7)<h(B,x,z)+h(B,y,z)$. Also, we know that $h(x,y,z)=\rho(x,y)\sin\phi(x,y,z)$. Thus (being generous)

$$\sin\phi(x,y,z)<2h(x,y,z)/r(y).$$

Now

$$\begin{aligned} h(x,y,z) &< 2(h(B,x,z)+h(B,y,z)) \\ &= 2(\rho(B,x)\sin\phi(B,x,z)+\rho(B,y)\sin\phi(B,y,z)) \\ &< 4(r(x)+r(y)\sin\phi(B,y,z)). \end{aligned}$$

Thus $\sin\phi(x,y,z)<8(r(x)/r(y)+\sin\phi(B,y,z))$.

We note that $B<y<T$ and $B<z<T$ in $\boldsymbol{P}_6$, so that $\phi(B,y,T)<1/N_{10}$ and $\phi(B,z,T)<1/N_{10}$. It follows that $\phi(B,y,z)<\phi(B,y,T)+\phi(B,z,T)<2/N_{10}$. Since $r(x)/r(y)<1/N_{10}$ and $\sin\phi(B,y,z)<\phi(B,y,z)<2/N_{10}$, it follows that $\sin\phi(x,y,z)<3/N_{10}$ so that $\phi(x,y,z)<1/N_9$ as claimed. □

We may now use the following estimates for any three points x,y and z with $x(1)<y(1)<z(1)$:

$$r(y)\phi^2(x,y,z)/[2(1+1/N_8)]<\Delta(x,y,z)<(1+1/N_8)r(y)\phi^2(x,y,z)/2,$$

$$r(y)\phi(x,y,z)/(1+1/N_8)<h(x,y,z)<r(y)\phi(x,y,z)(1+1/N_8).$$

Of course, we may also write:

$$\Delta(x,y,z)\approx r(y)\phi^2(x,y,z)/2,$$

$$h(x,y,z)\approx r(y)\phi(x,y,z).$$

Although it will not be used in the proof, we note that when $x(1)<y(1)<z(1)$, our previous upper bound $\gamma(x,y,z)<1/N_9$ can now be improved to $\gamma(x,y,z)<1/N_9^2$.

8. Part 3: applications of uniformity

This section develops properties of the various functions involving angles and distances. Already, we know that the radius function r is ACM and dominated by coordinate 1.

Let $\mathscr{L}=\mathscr{L}(N_{10})$. Then there exist change patterns $(\boldsymbol{L}_1,\alpha_1),\dots,(\boldsymbol{L}_5,\alpha_5)$ so that:

1. There is a coordinate α_1 and a change label $\boldsymbol{L}_1\in\mathscr{L}$ so that for every 2-element chain $x<z$ in $\boldsymbol{P}_5$, the map $\Phi(x,y,z)$, defined on those y with $x<y<z$ is N_{10}-uniform and satisfies change pattern $(\boldsymbol{L}_1,\alpha_1)$.

2. There is a coordinate α_2 and a change label $L_2 \in \mathscr{L}$ so that for every 2-element chain $x<y$ in P_5, the map $\Theta(x,y,z)$, defined on those z with $x<y<z$ is N_{10}-uniform and satisfies change pattern (L_2,α_2).
3. There is a coordinate α_3 and a change label $L_3 \in \mathscr{L}$ so that for every 2-element chain $y<z$ in P_5, the map $K(x,y,z)$, defined on those x with $x<y<z$ is N_{10}-uniform and satisfies change pattern (L_3,α_3).
4. There is a coordinate α_4 and a change label $L_4 \in \mathscr{L}$ so that for every 2-element chain $x<z$ in P_5, the map $H(x,y,z)$, defined on those y with $x<y<z$ is N_{10}-uniform and satisfies change pattern (L_4,α_4).
5. There is a coordinate α_5 and a change label $L_5 \in \mathscr{L}$ so that for every 2-element chain $x<z$ in P_5, the map $G(x,y,z)$, defined on those y with $x<y<z$ is N_{10}-uniform and satisfies change pattern (L_5,α_5).

When $x \in P_6$, we use the shorthand notations: $\Phi(x)=\Phi(B,x,T)$, $\Theta(x)=\Theta(B,B',x)$, $K(x)=K(x,T',T)$, $H(x)=H(B,x,T)$ and $G(x)=G(B,x,T)$. Also, for example, when we say that Φ is dominated by coordinate α_1, we mean that $\Phi(x)=\Phi(B,x,T)$ is dominated by α_1. It is important to remember that, for example, for all $x<z$, the function $\Phi(x,y,z)$, defined on y with $x<y<z$, satisfies the same change pattern as $\Phi(x)$.

Note. For the remainder of the argument, we will use the shorthand notaions $e_1 \ll e_2$, $e_1 \lesssim e_2$ and $e_1 \approx e_2$ as discussed in Section 2. In all cases, the shorthand notation will remind us of a more precise inequality which we have obtained at an earlier point in the argument.

We now begin to gather some information about other patterns present in P_5. For reasons which will become clear, we concentrate on the (3,2)-induced functions Φ and H.

Claim 8. *The function Φ cannot be* ACM.

Proof. Suppose to the contrary that Φ is ACM. Then for every 2-element chain $x<z$, the map $\Phi(x,y,z)$ defined on those y with $x<y<z$ is ACM.

Consider the three 3-element sets $\{e_1,e_2,e_3\}$, $\{e_1',e_2',e_3'\}$ and $\{e_1'',e_2'',e_3''\}$ where
1. $e_1=\phi(u_1,u_4,u_5)$, $e_2=\phi(u_1,u_3,u_5)$ and $e_3=\phi(u_1,u_3,u_4)$;
2. $e_1'=\phi(u_1,u_3,u_4)$, $e_2'=\phi(u_1,u_2,u_4)$ and $e_3'=\phi(u_1,u_2,u_3)$; and
3. $e_1''=\phi(u_1,u_3,u_5)$, $e_2''=\phi(u_1,u_2,u_5)$ and $e_3''=\phi(u_1,u_2,u_3)$.

Each of these 3-element sets satisfies the triangle inequality. Furthermore, $e_1>N_{10}e_2$, $e_1'>N_{10}e_2'$ and $e_1''>e_2''$, so that $e_1\approx e_3$, $e_1'\approx e_3'$ and $e_1''\approx e_3''$. Since $e_3=e_1'$ and $e_3'=e_3''$, we conclude that $e_1\approx e_3=e_1'\approx e_3'=e_3''\approx e_1''$, i.e., $e_1\approx e_1''$. But $e_1''=e_2$ and therefore $e_1>N_{10}e_1''$. The contradiction completes the proof. □

The next claim is dual to the preceding one — except for the fact that it uses the weak version of the triangle inequality.

Claim 9. *The function H cannot be* RAM.

Proof. Suppose to the contrary that H is RAM. Then for every 2-element chain $x<z$, the map $H(x,y,z)$ defined on those y with $x<y<z$ is RAM.

Consider the three 3-element sets $\{e_1,e_2,e_3\}$, $\{e_1',e_2',e_3'\}$ and $\{e_1'',e_2'',e_3''\}$ where

1. $e_1=h(u_1,u_2,u_5)$, $e_2=h(u_1,u_3,u_5)$ and $e_3=h(u_2,u_3,u_5)$;
2. $e_1'=h(u_2,u_3,u_5)$, $e_2'=h(u_2,u_4,u_5)$ and $e_3'=h(u_3,u_4,u_5)$; and
3. $e_1''=h(u_1,u_3,u_5)$, $e_2''=h(u_1,u_4,u_5)$ and $e_3''=h(u_3,u_4,u_5)$.

Each of these 3-element sets satisfies the property that the sum of any two quantities from the set is larger than the third divided by $1+1/N_7$. Furthermore, $e_1>N_{10}e_2$, $e_1'>N_{10}e_2'$ and $e_1''>e_2''$, so that $e_1\approx e_3$, $e_1'\approx e_3'$ and $e_1''\approx e_3''$. Since $e_3=e_1'$ and $e_3'=e_3''$, we conclude that $e_1\approx e_3=e_1'\approx e_3'=e_3''\approx e_1''$, i.e., $e_1\approx e_1''$. But $e_1''=e_2$ and therefore $e_1>N_{10}e_1''$. The contradiction completes the proof. □

Next we begin to consider the issue of coordinate domination. The next two claims are again dual.

Claim 10. *If Φ is* NC, *then H is* ACM *and dominated by coordinate* 1.

Proof. Let $s_1=(1,2,2)$ and $s_2=(2,1,1)$. Then $\Phi(s_1)\approx\Phi(s_2)$. Since $s_1(1)<s_2(1)$, $r(s_1)\ll r(s_2)$. Noting that $r(x)\Phi(x)\approx H(x)$ for all x, we conclude that $H(s_1)\ll H(s_2)$. From the preceding claim, we know that H cannot be RAM. Evidently, it is not NC, so it must be ACM. Furthermore, it must be dominated by coordinate 1, since $s_1(i)>s_2(i)$ for $i=2,3$. □

Claim 11. *If H is* NC, *then Φ is* RAM *and dominated by coordinate* 1.

Proof. Again, let $s_1=(1,2,2)$ and $s_2=(2,1,1)$. Then $H(s_1)\approx H(s_2)$. Since $s_1(1)<s_2(1)$, $r(s_1)\ll r(s_2)$. Noting that $\Phi(x)\approx H(x)/r(x)$ for all x, we conclude that $\Phi(s_1)\gg\Phi(s_2)$. From above, we know that Φ cannot be ACM. Evidently, it is not NC, so it must be RAM. Furthermore, it must be dominated by coordinate 1, since $s_1(i)>s_2(i)$ for $i=2,3$. □

Here is another useful property.

Claim 12. *If Φ is dominated by coordinate* 2, *then H is* ACM *and dominated by coordinate* 1.

Proof. Once again, consider $s_1=(1,2,2)$ and $s_2=(2,1,1)$. The inequalities $N_{10}r(s_1)<r(s_2)$ and $\phi(s_2)>\phi(s_1)$ imply $N_{10}H(s_1)<H(s_2)$, so H is ACM and dominated by coordinate 1. □

The remainder of the argument is by cases which depend on the change patterns of Φ and H. Originally, this would have resulted in $324=18^2$ cases, which would have been unbearable even for the most patient of readers. But in view of the results of the claims in this section, we only have 3 cases:

Case 1: Φ is RAM; H is ACM.

Case 2: Φ is NC; H is ACM.

Case 3: H is NC; Φ is RAM.

Moreover, in Case 2, we know that H is dominated by coordinate 1, while in Case 3, we know that Φ is dominated by coordinate 1. Also, following the pattern evidenced in this section, Cases 2 and 3 will be dual.

Since we are arguing by contradiction, we will show that each of the cases is impossible. When this is accomplished, our proof will be complete.

9. Part 4: case 1 of 3

In this section, we assume Φ is RAM and H is ACM. We assume without loss of generality that α_1, the coordinate which dominates Φ, is either 1 or 2.

Let y be a point with $x(1)<y(1)<z(1)$ and $x<z$. We obtain some estimates on $\phi(x,y,z)$ and $\Delta(x,y,z)$. These estimates assume that the coordinates of all three points are distinct.

First, consider the quantities

$$e_1=\phi(B,x,T),\quad e_2=\phi(B,z,T)\quad \text{and}\quad e_3=\phi(B,x,z).$$

These three quantities satisfy the triangle inequality. Furthermore, since Φ is RAM, we know that $\phi(B,x,T)>N_{10}\phi(B,z,T)$. Applying the triangle proposition, we conclude that

$$\Phi(x)=\phi(B,x,T)\approx\phi(B,x,z).$$

Now consider the quantities $e_1'=\phi(B,y,T)$, $e_2'=\phi(B,z,T)$ and $e_3'=\phi(B,y,z)$. These three quantities satsify the triangle inequality. Furthermore, one of $\phi(B,y,T)$ and $\phi(B,z,T)$ is much larger (by a factor of N_{10}) than the other. This depends on which is larger, $y(\alpha_1)$ or $z(\alpha_1)$. From the triangle proposition, we conclude that

$$\phi(B,y,z)\approx\max\{\phi(B,y,T),\phi(B,z,T)\}.$$

Alternatively, we may write

$$\phi(B,y,z)\approx\max\{\Phi(y),\Phi(z)\}.$$

Now suppose that $x<u<z$ is a chain. We know that $N_{10}\Phi(z)<\Phi(u)$, so that $\Phi(B,u,z)\approx\Phi(u)$. Since H is ACM, we know that $N_{10}h(B,x,z)<h(B,u,z)$. Applying the triangle proposition, we conclude that $h(B,u,z)\approx h(x,u,z)$. Therefore, $\Phi(u)\approx h(B,u,z)/r(u)\approx h(x,u,z)/r(u)\approx\phi(x,u,z)$, i.e., $\Phi(u)\approx\phi(x,u,z)$.

Recall that $G(x)=H(x)\Phi(x)/2$. It follows that $\Delta(x,u,z)\approx G(u)$. The important fact here is that this estimate is *independent* of both x and z.

For the remainder of this case, we will fix notation for the following points in $\boldsymbol{P}_6$: $x=(1,1,1)$, $z=(9,9,9)$, $v=(5,0,10)$ and $w=(5,10,0)$. Note that x and z are just the bottom and top elements of the chain $C=\{u_1<u_2<\cdots<u_9\}$.

As outlined in Section 3, we have the following lower bound on $\mathrm{gap}(x,z)$.

$$\mathrm{gap}(x,z)>\Delta(x,C,z)=\sum_{i=1}^{4}\Delta(u_{2i-1},u_{2i},u_{2i+1}).$$

Since $\Delta(u_{2i-1},u_{2i},u_{2i+1})\gtrsim G(u_{2i})$, we can write

$$\Delta(x,C,z)\gtrsim G(u_2)+G(u_4)+G(u_6)+G(u_8).$$

We now turn our attention to the problem of finding relatively tight upper bounds on $\mathrm{gap}(x,z)$.

To do this, we consider the points v and w, but we need to consider subcases depending on the coordinate that dominates Φ.

Subcase 1a. Φ is dominated by coordinate 1.

In this subcase, we note that $v(1)=w(1)=5<9=z(1)$, so that $\Phi(v)\gg\Phi(z)$ and $\Phi(w)\gg\Phi(z)$. It follows that $\phi(B,v,z)\approx\Phi(v)$ and $\phi(B,w,z)\approx\Phi(w)$. Recall that $\phi(B,x,z)\approx\Phi(x)$, so that $H(x)\approx r(x)\Phi(x)\approx r(x)\phi(B,x,z)\approx h(B,x,z)$. Also, $h(B,v,z)\approx r(v)\phi(x,v,z)\approx r(v)\Phi(v)\approx H(v)$.

Using the property that H is ACM, we know that exactly one of the following statements is true:

1. $H(x)\gg H(v)$.
2. $H(v)\gg H(x)$.

If $H(x)\gg H(v)$, we consider the quantities $H(x)\approx h(B,x,z)$, $H(v)\approx h(B,v,z)$ and $h(x,v,z)$ and use the triangle property to conclude that $h(B,x,z)\approx h(x,v,z)$. In this case, we see that $\phi(x,v,z)\approx r(x)\Phi(x)/r(v)$.

On the other hand, if $H(v)\gg H(x)$, then $h(B,v,z)\approx h(x,v,z)$. In this case, we conclude that $\phi(x,v,z)\approx\Phi(v)$. So we may then write

$$\phi(x,v,z)\approx\max\{r(x)\Phi(x)/r(v),\Phi(v)\}.$$

Applying the same argument to w, we can write

$$\phi(x,w,z)\approx\max\{r(x)\Phi(x)/r(w),\Phi(w)\}.$$

Therefore,

$$\Delta(x,v,z)\approx\max\{r(x)G(x)/r(v),G(v)\},$$

$$\Delta(x,w,z)\approx\max\{r(x)G(x)/r(w),G(w)\}.$$

Now we consider the implications of the following inequality discussed first in Section 3.

$$\Delta(x,C,z)<\min\{\Delta(x,v,z),\Delta(x,w,z)\}.$$

At this point, the argument depends on the coordinate dominating G. Suppose first that G is dominated by coordinate 1. If G is order-preserving, then

$$\begin{aligned} G(v) &= \max\{r(x)G(x)/r(v), G(v)\} \\ &\approx \Delta(x,v,z) \\ &\gtrsim \Delta(x,C,z) \\ &\gtrsim G(u_6)+G(u_8) \\ &\geqslant 2G(v), \end{aligned}$$

which is a contradiction.

Now suppose G is order-reversing. Then $\Delta(x,C,z)\gtrsim 2G(w)$ and $\Delta(x,C,z)\gtrsim 2G(v)$, which implies that $r(x)G(x)/r(v)\gg G(v)$ and $r(x)G(x)/r(w)\gg G(w)$. Thus $H(x)\gg H(v)$ and $H(x)\gg H(w)$. However, there is no coordinate $i\in\{1,2,3\}$ for which $x(i)>v(i)$ and $x(i)>w(i)$. We conclude that G is not dominated by coordinate 1.

Because the definitions of v and w are symmetric between coordinates 2 and 3, we can assume without loss of generality that G is dominated by coordinate 2. If G is order-preserving, then $\max\{r(x)G(x)/r(v), G(v)\}\leq G(x)$, but $\Delta(x,C,z)\gtrsim 2G(x)$.

So G must be order-reversing. Now $\Delta(x,C,z)\gtrsim 2G(w)$, so $r(x)G(x)/r(w)>G(w)$. This implies that $H(x)>H(w)$, so that H must be dominated by coordinate 3. This is impossible, because Φ is dominated by coordinate 1, G by coordinate 2 and $G\approx H\phi/2$. The contradiction completes the proof of this subcase.

Subcase 1b. Φ is dominated by coordinate 2.

In this subcase, we know from Claim 12 that H is dominated by coordinate 1. It follows without loss of generality that we may assume G is dominated by coordinate 1 or 2.

Now it is straightforward to verify that

1. $\phi(B,v,z)\approx\Phi(v)$;
2. $h(B,v,z)\approx H(v)$;
3. $h(B,x,z)\approx H(x)$.

Since H is ACM and dominated by coordinate 1, we know that $H(v)\gg H(x)$. Therefore, $h(B,v,z)\approx h(x,v,z)$, $\Phi(v)\approx\phi(x,v,z)$ and $\Delta(x,v,z)\approx G(v)$.

We now consider the implications of $\Delta(x,C,z)\lesssim\Delta(x,v,z)\approx G(v)$. Regardless of whether G is order-preserving or order-reversing, since G is dominated by coordinate 1 or 2, we see that $\Delta(x,C,z)\gtrsim 2G(v)$. The contradiction completes both the proof of the subcase as well as Case 1.

10. Part 5: case 2 of 3

In this case, we assume Φ is NC. By Claim 10, H is ACM and dominated by coordinate 1. Without loss of generality, we may assume that α_2, the coordinate which dominates Θ, is either 1 or 2.

Claim 13. *The function Θ is* ACM.

Proof. Suppose to the contrary that Θ is not ACM. Let $x<y<z<w$ be a 4-element chain in $\boldsymbol{P}_5$. Since Φ is NC, we know $\phi(x,y,w)\approx\phi(x,z,w)$. Since Θ is not ACM, we know $\phi(x,y,w)\lesssim\phi(x,y,z)$, and thus $\phi(x,z,w)\lesssim\phi(x,y,z)$.

Since H is ACM, we know that $h(x,z,w)\gg h(x,y,w)$, so that $h(x,z,w)\approx h(y,z,w)$. Thus $\phi(x,z,w)\approx\phi(y,z,w)$. It follows that

$$\phi(x,y,z)\gtrsim\phi(x,z,w)\approx\phi(x,y,w)\approx\phi(y,z,w).$$

In particular, $\phi(x,y,z)\gtrsim\phi(y,z,w)$.

On the other hand, $\phi(x,y,z)<\phi(x,y,w)+\phi(x,z,w)\lesssim 2\phi(y,z,w)$. It follows that $\phi(x,y,z)\approx\phi(y,z,w)$.

Now let $w_1<w_2<\cdots<w_6$ be a chain in $\boldsymbol{P}_5$. It follows that

$$\phi(w_1,w_2,w_3)\approx\phi(w_2,w_3,w_4)\approx\phi(w_3,w_4,w_5)\approx\phi(w_4,w_5,w_6),$$

and therefore

$$\phi(x_1,y_1,z_1)\approx\phi(x_2,y_2,z_2)$$

for any two 3-element chains $x_1<y_1<z_1$ and $x_2<y_2<z_2$ from $\boldsymbol{P}_5$.

Now consider the following points in $\boldsymbol{P}_6$: $x=(1,1,1)$, $v=(2,0,4)$, $u=(3,3,3)$ and $z=(4,4,4)$. Since $x<u<z$ is a 3-element chain, we know that $\phi(x,u,z)\approx\phi(B,B',B'')$ so that $\Delta(x,u,z)\approx r(u)\phi^2(B,B',B'')/2$.

On the other hand, $\phi(x,v,z)<\phi(x,z,T)+\phi(x,v,T)$. Also, $h(B,v,T)\gg h(B,x,T)$ implies $h(x,v,T)\approx h(B,v,T)$ so that $\phi(x,v,T)\approx\phi(B,v,T)$. Thus $\phi(x,v,z)\lesssim 2\phi(B,B',B'')$.

In turn, this implies that $\Delta(x,v,z)\lesssim 2r(v)\phi^2(B,B',B'')$, and thus $r(u)\lesssim 2r(v)$. However, $r(u)\gg r(v)$. The contradiction completes the proof. □

Claim 14. *For all 4-element chains $x<y<z<w$,*

$$\phi(x,y,w)\approx\phi(y,z,w)\approx\Theta(w).$$

Proof. Since Φ is NC, we know that $\phi(x,y,w)\approx\phi(x,z,w)$. Thus $h(x,y,w)\ll h(x,z,w)$. This implies that $h(x,z,w)\approx h(y,z,w)$ and $\phi(x,z,w)\approx\phi(y,z,w)$. It follows that $\phi(x,y,w)\approx\phi(y,z,w)$.

Observing that this pattern holds for any 4-element chain, we may also conclude that

$$\Theta(w)=\phi(B,B',w)\approx\phi(B',x,w)\approx\phi(x,y,w). \qquad \square$$

So for chains, the behavior of Θ depends only on the last coordinate. The next claim extends this to certain triples which are not chains.

Claim 15. *If $x(1)<y(1)<z(1)$, $x(2)<z(2)$ and $y(2)<z(2)$, then*

$$\phi(x,y,z)\approx\Theta(z).$$

Proof. Since Θ is ACM and dominated by coordinate 1 or 2, we know that

$$\phi(B,B',z)=\Theta(z)\gg\Theta(y)=\phi(B,B',y).$$

Thus $\Theta(z)\approx\phi(B,y,z)$.

Similarly, we know that $\Theta(z)\approx\phi(B,x,z)$. Now H is dominated by coordinate 1, so $h(B,y,z)\gg h(B,x,z)$. Thus $h(B,y,z)\approx h(x,y,z)$ and $\Theta(z)\approx\phi(B,y,z)\approx\phi(x,y,z)$. □

Now we consider the following points in $\boldsymbol{P}_6$: $x=(1,1,1)$, $v=(2,0,5)$, $u=(3,3,3)$ and $z=(4,4,4)$.

From Claim 15, it follows that $\phi(x,u,z)\approx\Theta(z)\approx\phi(x,v,z)$. Thus $\Delta(x,u,z)\approx r(u)\Theta^2(z)/2$ and $\Delta(x,v,z)\approx r(v)\Theta^2(z)/2$. This requires $r(u)\lesssim r(v)$. Since $u(1)>v(1)$, we know that $r(u)\gg r(v)$. The contradiction completes the proof of Case 2.

11. Part 4: case 3 of 3

In this case, we assume that H is NC and Φ is RAM. Because this case is dual to Case 2, we outline only the statements necessary to complete the proof. Of course, the key idea here is to focus on the function K.

From Claim 11, we know that Φ is dominated by coordinate 1. So first, we prove the following claim.

Claim 16. *The function K is* RAM.

The reader should note that the proof will hinge on the situation where $h(x,y,z)$ is nearly constant for all 3-element chains $x<y<z$. But this will lead to a contradiction by considering the same four points as in the proof of Claim 13.

Next, the following claims are established.

Claim 17. *For all 4-element chains $x<y<z<w$,*

$$h(x,y,w)\approx h(x,z,w)\approx H(x).$$

Claim 18. *If $x(1)<y(1)<z(1)$, $x(2)<z(2)$ and $x(2)<y(2)$, then*

$$h(x,y,z)\approx H(x).$$

To complete the argument, we consider the following points: $x=(1,1,1)$, $u=(2,2,2)$, $w=(3,0,5)$ and $z=(4,4,4)$. In this case, we conclude that

$$\Delta(x,u,z)\approx h^2(x,u,z)/2r(u)\approx H^2(x)/2r(u),$$

while

$$\Delta(x,w,z)\approx h^2(x,w,z)/2r(w)\approx H^2(x)/2r(w).$$

Thus, we must have $r(w) \lesssim r(u)$. Instead, we know $r(w) \gg r(u)$. With this remark, the proof of Case 3 and our principal theorem is complete. □

12. Concluding remarks

Not surprisingly, our original proof was quite different from the one given here. It was specific to the plane and showed only that there was a finite 3-dimensional poset that was not a circle order. Many details of this approach did not extend to the general problem, and some new techniques were necessary to work around the apparent obstacles. In the end, the proof of the general result is simpler.

It is tempting to conjecture that there is a poset of modest size, say at most 100 points, which is not a sphere order. Certainly, new ideas will be required to prove the existence of such a poset.

One interesting open problem remains.

Question 3. *Does there exist a finite poset $\boldsymbol{P}$ so that $\boldsymbol{P} \times \boldsymbol{n}$ is a circle order for all $n \geqslant 1$ but $\boldsymbol{P} \times \mathbb{N}$ is not a circle order?*

Of course, this question can also be stated for sphere orders in general.

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