

## TREE-WIDTH AND DIMENSION

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Over the last 30 years, researchers have investigated connections between dimension for posets and planarity for graphs. Here we extend this line of research to the structural graph theory parameter tree-width by proving that the dimension of a finite poset is bounded in terms of its height and the tree-width of its cover graph.

### 1. Introduction

In this paper, we investigate combinatorial problems involving finite graphs and partially ordered sets (posets), linking the well-studied concept of tree-width for graphs with the concept of dimension for posets. The following is our main result.

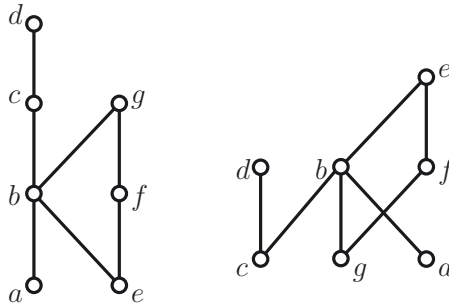
**Theorem 1.1.** *For every pair  $(t, h)$  of positive integers, there exists a least positive integer  $d = d(t, h)$  so that if  $P$  is a poset of height at most  $h$  and the tree-width of the cover graph of  $P$  is at most  $t$ , then the dimension of  $P$  is at most  $d$ . In particular, we have  $d(t, h) \leq 6 \cdot 2^{8t^{4h-2}}$ .*

The remainder of this paper is organized as follows. In the next section, we provide a brief summary of essential notation and terminology for posets and dimension. This is followed by an even more compact section on graphs and tree-width. These sections are included since we anticipate that many readers will be quite familiar with one of these topics but less so with the

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**Figure 1.** Two posets with the same cover graph

other. With these basics in hand, we discuss in Section 4 the background behind this line of research and the motivation for our principal theorem. The proof of our main theorem is given in Section 5, and we discuss some open problems in Section 6.

## 2. Posets and Dimension

A *partially ordered set* (here we use the short term *poset*) is a set  $P$  equipped with a reflexive, antisymmetric and transitive binary relation  $\leq$ . Elements of  $P$  are called points and here we will also call them vertices, since we will often consider graphs whose vertex set is the set of elements of  $P$ . When the poset  $P$  is fixed throughout the discussion, we abbreviate the statement  $x \leq y$  in  $P$  by just writing  $x \leq y$ . The notation  $x < y$  means of course  $x \leq y$  and  $x \neq y$ . These notations are reversible in the obvious manner, i.e.,  $x > y$  means the same as  $y < x$ .

We say  $x$  *covers*  $y$  (also  $y$  is *covered by*  $x$ ) when  $x > y$ , and there is no point  $z$  with  $x > z > y$ . Also, we associate with a poset  $P$  a *cover graph* having the same vertex set as  $P$ . The cover graph of  $P$  has an edge  $xy$  when one of  $x$  and  $y$  covers the other. A drawing (typically, we consider only drawings with straight line segments for the edges) of the cover graph of a poset  $P$  is called an *order diagram* (also, a *Hasse diagram*) if the point in the plane corresponding to the point  $x$  is higher than the point corresponding to the point  $y$  when  $x$  covers  $y$  in  $P$ . We show in Figure 1 order diagrams for two different posets, both with the same cover graph.

When  $x$  and  $y$  are distinct points in a poset  $P$ , and either  $x < y$  or  $y < x$ , we say  $x$  and  $y$  are *comparable*. When  $x$  and  $y$  are distinct points in  $P$ , and they are not comparable, we say they are *incomparable* and write  $x \parallel y$ . We use the notation  $\text{Inc}(P)$  for the set of all ordered pairs  $(x, y)$  with  $x \parallel y$ .

An element  $a$  in a poset  $P$  is *minimal*, respectively *maximal* when there is no point  $x$  with  $x < a$ , respectively  $x > a$ . When  $Q$  is a subset of a poset  $P$ , the restriction of the binary relation  $\leq$  to  $Q$  is a poset and we call this a *subposet* of  $P$ . A poset  $P$  is called a *linear order* (also a *total order*) if  $\text{Inc}(P) = \emptyset$ . When  $Q$  is a subposet of  $P$  and  $Q$  is a linear order, it is customary to call  $Q$  a *chain*. The largest positive integer  $h$  for which  $P$  has a subposet  $Q$  on  $h$  points which is a chain in  $P$  is called the *height* of  $P$ .

A poset  $P$  is called an *antichain* if it has height 1, i.e.,  $x \parallel y$  for all  $x$  and  $y$  with  $x \neq y$ . The largest integer  $w$  for which  $P$  contains a subposet on  $w$  points which is an antichain is called the *width* of  $P$ . The classic theorem of Dilworth [10] asserts that a poset of width  $w$  can be partitioned into  $w$  chains. Dually, Mirsky [24] proved that a poset of height  $h$  can be partitioned into  $h$  antichains.

Let  $P$  and  $L$  be posets. We call  $L$  a *linear extension* of  $P$  when  $L$  has the same ground set as  $P$ ,  $L$  is a linear order, and  $x < y$  in  $L$  whenever  $x < y$  in  $P$ . A family  $\mathcal{R} = \{L_1, L_2, \dots, L_t\}$  of linear extensions of  $P$  is called a *realizer* of  $P$  if  $x < y$  in  $P$  if and only if  $x < y$  in  $L_i$  for each  $i = 1, 2, \dots, t$ . Clearly, a family  $\mathcal{R} = \{L_1, L_2, \dots, L_t\}$  of linear extensions of  $P$  is a realizer if and only if for each  $(x, y) \in \text{Inc}(P)$ , there is some  $i$  with  $1 \leq i \leq t$  such that  $x > y$  in  $L_i$ .

Dushnik and Miller [11] defined the *dimension* of  $P$ , denoted  $\text{dim}(P)$ , as the least positive integer  $t$  for which  $P$  has a realizer  $\mathcal{R}$  with  $|\mathcal{R}| = t$ . Evidently,  $\text{dim}(P) = 1$  if and only if  $P$  is a linear order. Also, when  $P$  is a non-trivial antichain,  $\text{dim}(P) = 2$  as evidenced by the realizer  $\{L, L^d\}$  where  $L$  is an arbitrary linear order on the ground set of  $P$  and  $L^d$  is the *dual* of  $L$ , i.e.,  $x > y$  in  $L^d$  if and only if  $x < y$  in  $L$ .

In [20], Hiraguchi used Dilworth’s theorem to show that the dimension of a poset never exceeds its width. Hiraguchi also proved that if  $P$  is a poset on  $n$  points with  $n \geq 4$ , then  $\text{dim}(P) \leq \lfloor n/2 \rfloor$ . Both these inequalities are tight, as witnessed by a family of posets called *standard examples* and first studied in [11]. As these posets play an important role later in this paper, we include here some details on their structure and properties.

For  $d \geq 2$ , the standard example  $S_d$  is a height 2 poset with minimal elements  $\{a_1, a_2, \dots, a_d\}$  and maximal elements  $\{b_1, b_2, \dots, b_d\}$ . The relation  $\leq$  is defined on  $S_d$  by setting  $a_i < b_j$  if and only if  $i \neq j$ , for all  $i, j = 1, 2, \dots, d$ . For each  $d \geq 2$ , the width of  $S_d$  is  $d$  so  $\text{dim}(S_d) \leq d$ . On the other hand,  $\text{dim}(S_d) \geq d$ . This follows from the observation that if  $L$  is a linear extension of  $S_d$ , there can only be one integer  $i$  with  $1 \leq i \leq d$  and  $a_i > b_i$  in  $L$ . Moreover, when  $d \geq 3$ , it is easy to see that  $S_d$  is *d-irreducible*, i.e., removing any point from  $S_d$  lowers the dimension to  $d - 1$ .

There is a natural notion of *isomorphism* for posets, and it obvious that isomorphic posets have the same dimension. So it is natural to say that a poset  $P$  *contains* a poset  $Q$  when there is a subposet of  $P$  which is isomorphic to  $Q$ . In this vein, a poset  $P$  has large dimension when it contains a large standard example. But this is far from necessary.

A poset  $P$  is called an *interval order* when there is a family  $\{[a_x, b_x]: x \in P\}$  of closed intervals of the real line  $\mathbb{R}$  so that  $x < y$  in  $P$  if and only if  $b_x < a_y$  in  $\mathbb{R}$ . Fishburn [16] showed that a poset  $P$  is an interval order if and only if it does not contain the standard example  $S_2$ . In [17], Füredi, Hajnal, Rödl and Trotter show that the maximum dimension of an interval order  $P$  of height  $h$  is  $\lg \lg h + (1/2 + o(1)) \lg \lg \lg h$ . In particular, note that in order for an interval order to have large dimension, it must have very large height.

The standard examples show that in general, large height is not necessary for large dimension, and in [14], Felsner, Li and Trotter show that for every pair  $(g, d)$  of positive integers, there is a height 2 poset  $P$  with  $\dim(P) \geq d$  so that the girth of the cover graph of  $P$  is at least  $g$ . The posets resulting from this construction contain  $S_2$  but they do not contain  $S_3$ , when  $g > 6$ .

Although cover graphs are useful in providing diagrams of posets, they do not seem to tell us much about the combinatorial properties of the posets associated with them. For example, the two posets shown in Figure 1 have the same cover graph. However, the poset on the left has height 4, width 2 and 21 linear extensions, while the poset on the right has height 3, width 3 and 84 linear extensions. Both posets have dimension 2.

At the extreme, a linear order on  $n$  points has height  $n$ , width 1 and of course, a unique linear extension. However, when  $n \geq 2$ , the associated cover graph is bipartite, and the height 2 poset with this same cover graph is called a *fence*. Now the width is  $\lceil n/2 \rceil$  and the number of linear extensions is exponentially large in  $n$ . On the other hand, the dimension of a fence is 2 when  $n \geq 3$ , so based only on these observations, one might conjecture that posets with the same cover graph have approximately the same dimension. But even this is not true. Later in the paper, we will show that for each  $d \geq 1$ , there are two posets having the same cover graph, one having dimension 2 and the other having dimension at least  $d$ .

However, there is another natural way to associate a graph with a poset. Like the cover graph, the *comparability graph* of  $P$  has the same vertex set as  $P$  but now we make  $xy$  an edge if  $x$  and  $y$  are comparable. The comparability graph of a poset contains the cover graph as a subgraph. Furthermore, if  $P$  and  $Q$  are posets with isomorphic comparability graphs, then they have the same height, width, number of linear extensions and dimension. The

fact that they have the same height and width is immediate. The fact that they have the same number of linear extensions and the same dimension follows in a straightforward manner from the pioneering work of Gallai [18] on comparability graphs.

With these remarks in mind, and with no additional background information to suggest otherwise, the principal result of this paper would then have to be viewed as a surprise.

### 3. Graphs and Tree-Width

In this paper, we consider only finite graphs without loops or multiple edges, and we assume that readers are familiar with basic concepts such as trees, paths, cycles, complete graphs, subgraphs, induced subgraphs, components, chromatic number, girth, genus, distance and diameter. Given a graph  $G$ , an induced subgraph of  $G$  is determined entirely by its vertex set. In particular, when  $T$  is a tree, we will identify subtrees of  $T$  just by specifying their vertex sets. So when  $T'$  and  $T''$  are subtrees of a tree  $T$ , the statement  $T' \cap T'' \neq \emptyset$  just means that  $T'$  and  $T''$  have one or more vertices of  $T$  in common.

Let  $G$  be a graph with vertex set  $V$  and edge set  $E$ . The *tree-width*<sup>1</sup> of  $G$  is the least positive integer  $t$  for which there is a tree  $T$  and a family  $\{T(x) : x \in V\}$  of non-empty subtrees of  $T$  so that

- (1) for all vertices  $u$  in  $T$ ,  $|\{x \in V : u \in T(x)\}| \leq t + 1$ ,
- (2)  $T(x) \cap T(y) \neq \emptyset$  for all  $xy \in E$ .

Trivially, a graph has tree-width 0 if and only if it has no edges, while the tree-width of the complete graph  $K_n$  on  $n$  vertices is  $n - 1$  for all  $n \geq 1$ . Furthermore, if  $G = (V, E)$  is a tree with at least one edge, then the tree-width of  $G$  is 1. To see this, simply subdivide each edge  $e = xy$  in  $E$  by inserting a new vertex  $m_{xy}$  in the interior of  $e$ . Let  $T$  denote the resulting tree. Then for each  $x \in V$ , take  $T(x)$  as the subtree of  $T$  with vertex set  $\{x\} \cup \{m_{xy} : xy \in E\}$  (each  $T(x)$  is a star). Conversely, it is easy to see that a graph  $G$  has tree-width at most 1 if and only if it is acyclic.

Consider the following three basic operations on a graph: (1) delete an edge; (2) delete a vertex; (3) contract an edge. Given a graph  $G$ , any graph  $H$  that can be obtained from  $G$  by applying a sequence of these basic operations

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<sup>1</sup> We refer the reader to the text by Diestel [9] for a concise exposition of some of the key concepts behind this parameter. Diestel also provides interesting details on its history and the twenty year time period spanned by Robertson and Seymour's proof of the Graph Minor Theorem. Also our notation for tree-width and some of our examples are taken from exercises in this text.

is called a *minor* of  $G$ . The following fundamentally important theorem, called the Graph Minor Theorem, is due to Robertson and Seymour [27]<sup>2</sup>.

**Theorem 3.1.** *If  $\{G_n : n \geq 1\}$  is an infinite sequence of graphs, then there are integers  $i$  and  $j$  with  $1 \leq i < j$  so that  $G_i$  is isomorphic to a minor of  $G_j$ .*

A class  $\mathcal{G}$  of graphs is *minor-closed* if  $H$  is in  $\mathcal{G}$  whenever  $G$  is in  $\mathcal{G}$  and  $H$  is isomorphic to a minor of  $G$ . Examples of minor closed classes of graphs include the family of all planar graphs and, more generally, for fixed  $g \geq 0$ , the family of all graphs having genus at most  $g$ . Also, it is easy to see that for each  $t \geq 1$ , the class of all graphs having tree-width at most  $t$  is minor-closed.

Any proper minor-closed class of graphs admits a characterization by “forbidden minors”, i.e., a minimum family  $\mathcal{F}$  of graphs such that a graph  $G$  belongs to  $\mathcal{G}$  if and only if it does not contain a minor isomorphic to a graph in  $\mathcal{F}$ . By the Graph Minor Theorem, the class  $\mathcal{F}$  is finite. The classic theorem of Wagner [32] asserts that the list of forbidden minors for the class of planar graphs consists of the complete graph  $K_5$  and the complete bipartite graph  $K_{3,3}$ .

Planar graphs can have large tree-width. Note that any bipartite graph is both the cover graph and the comparability graph of a height 2 poset. In particular, the  $n \times n$  planar grid is bipartite and has tree-width  $n$  (see Diestel [9], Exercises 14 and 21 on page 369). However, the tree-width of a planar graph is bounded in terms of its diameter<sup>3</sup>. Classes of graphs where tree-width is bounded in terms of diameter are said to satisfy the *diameter tree-width property* (also called the *bounded local tree-width property*).

The concept of *path-width* for graphs is defined just like tree-width except that it is required that the tree  $T$  be a path, and of course the subtrees of  $T$  are then just subpaths of  $T$ . Trivially, the tree-width of a graph is at most its path-width. However, the tree-width of an outerplanar graph is at most 2 (this follows from the observation that in a maximal outerplanar graph, there is always a vertex  $x$  of degree two such that the neighbors of  $x$  are adjacent to each other). On the other hand, outerplanar graphs can have arbitrarily large path-width. In fact, trees can have arbitrarily large path-width (see Diestel [9], Exercise 31 on page 370).

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<sup>2</sup> The proof given by Robertson and Seymour for the Graph Minor Theorem appears in a series of papers published over the time span 1983 through 2004, and we cite here the culminating paper in that series.

<sup>3</sup> This result is implicit in the work of Baker [1] and made explicit by Bodlaender in [5].

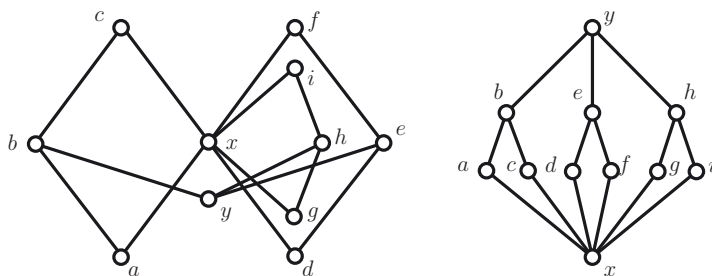


Figure 2. A non-planar poset with planar cover graph

### 4. Background and Motivation

A poset  $P$  is *planar* if its order diagram can be drawn without edge crossings in the plane. In Figure 2, we show on the left the order diagram of a height 3 nonplanar poset. However, the cover graph of this poset is planar as witnessed by the drawing on the right.

We note that if  $P$  is a height 2 poset, then  $P$  is planar if and only if its cover graph is planar [25,3]. We also note that it is NP-complete to test whether a poset is planar [19], while there are linear-time algorithms for testing whether a graph is planar [21]. Also, it is NP-complete to test whether a graph is a cover graph [6,26].

When  $P$  is a poset with only one minimal element, this single element is usually called a *zero*. Similarly, in a poset with only one maximal element, this element is called a *one*. The first result linking planarity and dimension is the following theorem of Baker, Fishburn and Roberts [2].

**Theorem 4.1.** *If  $P$  is a planar poset with a zero and a one, then  $\dim(P) \leq 2$ .*

Subsequently, Trotter and Moore [30] proved the following extension.

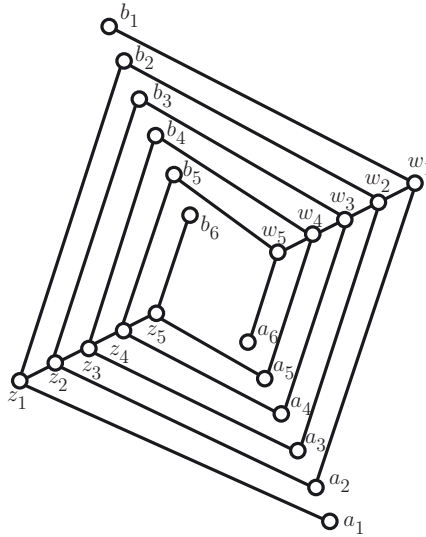
**Theorem 4.2.** *If  $P$  is a planar poset with a zero or a one, then  $\dim(P) \leq 3$ .*

Trotter and Moore [30] also obtained the following result as an immediate corollary to the preceding theorem.

**Corollary 4.3.** *If  $P$  is a poset whose cover graph is a tree, then  $\dim(P) \leq 3$ .*

With the benefit of hindsight, one can argue that the line of research carried out in this paper might reasonably have been triggered 35 years ago, based solely on possible extensions to Corollary 4.3.

It is an easy exercise to show that the standard example  $S_d$  is planar when  $d \leq 4$ , and as a consequence, there are 4-dimensional planar posets.



**Figure 3.** Kelly’s construction

On the other hand,  $S_d$  is non-planar when  $d \geq 5$ . For a brief time in the late 1970’s, it was thought that it might be the case that  $\dim(P) \leq 4$  whenever  $P$  is a planar poset.

However, in 1981, Kelly [23] showed that for each  $d \geq 5$ , the standard example  $S_d$  is a subposet of a planar poset  $P_d$ . We illustrate Kelly’s construction in Figure 3 for the specific value  $d=6$ .

We pause here to answer a question raised earlier concerning the dimension of posets with the same cover graph. Specifically, we show that for each  $d \geq 2$ , there are posets  $Q_d$  and  $Q'_d$  with the same cover graph with  $\dim(Q'_d)=2$  and  $\dim(Q_d) \geq d$ . First, we consider a poset  $Q_d$  formed by modifying Kelly’s example as follows. For each  $i=1, 2, \dots, d-1$ , we add two new minimal points  $u_i$  and  $v_i$  with  $u_i$  covered by  $b_i$  and  $b_{i+1}$ , while  $v_i$  is covered by  $a_i$  and  $a_{i+1}$ . Clearly,  $P_d$  is a subposet of  $Q_d$  so that  $\dim(Q_d) \geq d$ .

On the other hand, there are exponentially many posets having the same cover graph as  $Q_d$ . One of them, which we denote  $Q'_d$ , has  $b_i > u_i > b_{i+1}$  and  $a_{i+1} > v_i > a_i$ , for each  $i=1, 2, \dots, d-1$ . Obviously, both  $Q_d$  and  $Q'_d$  are planar poset as witnessed by trivial modifications to the diagram for  $P_d$  given in Figure 3. Moreover, in  $Q'_d$ , the point  $a_1$  is now a zero and the point  $b_d$  is now a one. So by Theorem 4.1,  $\dim(Q'_d)=2$ .

Returning to the general subject of the dimension of posets with planar cover graphs, Felsner, Li and Trotter [14] proved the following result in 2010:



**Theorem 4.4.** *Let  $P$  be poset of height 2. If the cover graph of  $P$  is planar, then  $\dim(P) \leq 4$ .*

Actually, this was obtained as an easy corollary to the following theorems of Brightwell and Trotter [8,7], published in 1997 and 1993, respectively (a new and quite elegant proof of this result has just been obtained by Felsner [13]).

**Theorem 4.5.** *Let  $G$  be a planar multi-graph and let  $P$  be the vertex-edge-face poset determined by a drawing without edge crossings of  $G$  in the plane. Then  $\dim(P) \leq 4$ . Furthermore, if  $G$  is a simple, 3-connected planar graph, then the subposet determined by the vertices and faces is 4-irreducible.*

The inequality in Theorem 4.4 is best possible as evidenced by the standard example  $S_4$ . Noting that the poset  $P_d$  in Kelly’s construction has height  $d + 1$ , Felsner, Li and Trotter [14] conjectured the following generalization, which was proved by Streib and Trotter [29] in 2012.

**Theorem 4.6.** *For every positive integer  $h$ , there is a least positive integer  $c_h$  so that if  $P$  is a poset with a planar cover graph and the height of  $P$  is at most  $h$ , then  $\dim(P) \leq c_h$ .*

We have  $c_1 = 2$  and  $c_2 = 4$ . For  $h \geq 3$ , the upper bound on the constant  $c_h$  produced in the proof of Theorem 4.6 is very large, as several iterations of Ramsey theory are used. From below, it is straightforward to modify Kelly’s original construction and decrease the height to  $d - 1$ . This can be accomplished by deleting  $a_1, a_d, b_1$  and  $b_d$  and relabelling  $z_1, z_{d-1}, w_1$  and  $w_{d-1}$  as  $a_1, b_d, b_1$  and  $a_d$ , respectively. Wiechert [34] constructed a planar poset  $Q$  of height 3 with  $\dim(Q) = 5$ ; however, this construction does not seem to generalize for larger values of  $d$ . Accordingly, when  $d \geq 6$ , we do not know whether there is a planar poset  $P$  of height  $d - 2$  with  $\dim(P) = d$ . On the other hand, Streib and Trotter [29] showed that for each  $d \geq 5$ , there is a poset  $P$  of height  $d - 2$  with  $\dim(P) = d$  so that the cover graph of  $P$  is planar.

Theorems 4.1 and 4.2, as well as Corollary 4.3 all provide conditions where the dimension of a planar poset can be bounded independent of its height. In [15], Felsner, Trotter and Wiechert gave the following additional results of this nature.

**Theorem 4.7.** *Let  $P$  be a poset.*

- (1) *If the cover graph of  $P$  is outerplanar, then  $\dim(P) \leq 4$ .*
- (2) *If the comparability graph of  $P$  is planar, then  $\dim(P) \leq 4$ .*

Both inequalities in Theorem 4.7 are best possible. The proof of the first inequality in Theorem 4.7 is relatively straightforward, but it takes a bit of work to show that it is best possible. However, the second inequality in Theorem 4.7 is quite different, and now the argument depends on the full strength of the Brightwell-Trotter inequality for the dimension of the vertex-edge-face poset determined by a drawing of a planar multi-graph, with the edges now playing a key role.

To the best of our knowledge, the following observation concerning Kelly's 1981 construction was not made until 2012: The cover graphs of the posets in this construction have bounded tree-width. In fact, they have bounded path-width. We leave the following elementary observations as an exercise.

**Exercise 4.8.** *Let  $d \geq 2$ , let  $P_d$  be the poset illustrated in Kelly's construction, and let  $G_d$  be the cover graph of  $P_d$ . Then the height of  $P_d$  is  $d+1$ , and the path-width of  $G_d$  is at most 3. In fact, when  $d \geq 5$ ,  $G_d$  contains  $K_4$  as a minor, so its path-width is exactly 3.*

We made some effort to construct large dimension posets with bounded height and cover graphs having bounded tree-width and were unable to do so. So consider the following additional information:

- (1) A poset whose cover graph has tree-width 1 has dimension at most 3.
- (2) A poset whose cover graph is outerplanar has dimension at most 4. As noted previously, outerplanar graphs can have arbitrarily large path-width, but they have tree-width at most 2.
- (3) On the one hand, the tree-width of the cover graph of a planar poset can be arbitrarily large, even when the height of  $P$  is 2. As an example, just take a height 2 poset whose cover graph is an  $n \times n$  grid. On the other hand, the proof given by Streib and Trotter [29] to show that the dimension of a poset with a planar cover graph can be bounded in terms of its height used a reduction to the case where the cover graph of the poset is both planar and has diameter bounded in terms of the height of the poset. Again, as noted previously, a planar graph of bounded diameter has bounded tree-width.

Taking into consideration this body of evidence together with our inability to prove otherwise, it is natural to conjecture that the dimension of a poset is bounded in terms of its height and the tree-width of its cover graph, and this is what we now prove.

## 5. Proof of the Main Theorem

### 5.1. Preliminaries

A subset  $I$  of  $\text{Inc}(P)$  is said to be *reversible* if there is a linear extension  $L$  of  $P$  with  $x > y$  in  $L$  for every  $(x, y) \in I$ . It is then immediate that  $\dim(P)$  is the least positive integer  $d$  so that there is a partition  $\text{Inc}(P) = I_1 \cup I_2 \cup \dots \cup I_d$  with each  $I_i$  reversible. In view of this formulation, it is handy to have a simple test to determine whether a given subset  $I$  of  $\text{Inc}(P)$  is reversible.

Let  $k \geq 2$ . An indexed subset  $I = \{(x_i, y_i) : 1 \leq i \leq k\}$  of  $\text{Inc}(P)$  is called an *alternating cycle* when  $x_i \leq y_{i+1}$  in  $P$  for each  $i \in \{1, 2, \dots, k\}$ , where we interpret the subscripts cyclically (i.e., we require  $x_k \leq y_1$  in  $P$ ). Reversing an alternating cycle  $I$  would require a linear extension in which the cyclic arrangement  $y_1, x_1, \dots, y_k, x_k$  alternates between strict inequalities of the form  $y_i < x_i$  (needed to reverse  $I$ ) and inequalities of the form  $x_i \leq y_{i+1}$  (forced by  $P$ ). Consequently, alternating cycles are not reversible. The following elementary lemma, proved by Trotter and Moore in [30] using slightly different terminology, states that alternating cycles are the only obstruction to being reversible.

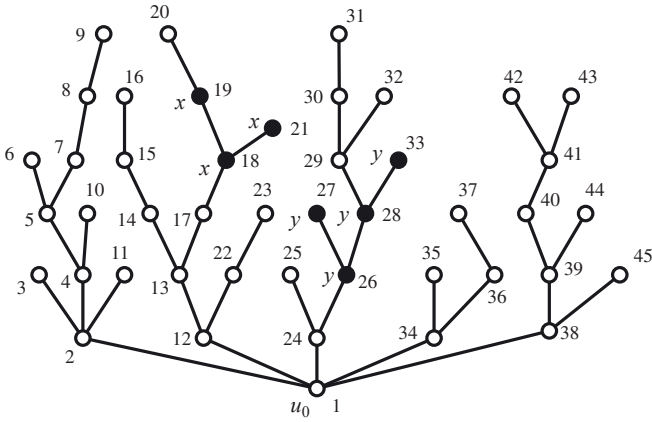
**Lemma 5.1.** *If  $P$  is a poset and  $I \subseteq \text{Inc}(P)$ , then  $I$  is reversible if and only if  $I$  contains no alternating cycle.*

For the remainder of this section, we fix integers  $t$  and  $h$ , assume that  $P$  is a poset with height  $h$  and cover graph  $G$ , and assume that the tree-width of  $G$  is  $t$ . Of course, we may also assume that  $\text{Inc}(P) \neq \emptyset$ . The remainder of the argument is organized to show that we can partition the set  $\text{Inc}(P)$  into  $d$  reversible sets, where  $d$  is bounded in terms of  $t$  and  $h$ .

Let  $X$  denote the ground set of  $P$ , so that  $X$  is also the vertex set of the cover graph  $G$ . Since the tree-width of  $G$  is  $t$ , there is a tree  $T$  and a family  $\mathcal{F} = \{T(x) : x \in X\}$  of subtrees of  $T$  such that (1) for each vertex  $u$  of  $T$ , the number of elements  $x$  of  $X$  with  $u \in T(x)$  is at most  $t + 1$ , and (2) for each edge  $xy$  of  $G$ , we have  $T(x) \cap T(y) \neq \emptyset$ .

Let  $H$  be the intersection graph determined by the family  $\mathcal{F}$  of subtrees of  $T$  (some researchers refer to  $H$  as the *chordal completion* of  $G$ ). Evidently, the tree-width of  $H$  is  $t$ , and every edge of  $G$  is an edge of  $H$ . Of course, the set  $X$  is also the vertex set of  $H$ . In the discussion to follow, we will go back and forth, without further comment, between referring to members of  $X$  as elements of the poset  $P$  and as vertices in the cover graph  $G$  and the intersection graph  $H$ .

To help distinguish between vertices of  $T$  and elements of  $X$ , we will use the letters  $r, u, v$  and  $w$  (possibly with subscripts) to denote vertices of the



**Figure 4.** A tree with root  $u_0$  and vertices labelled using number  $1, 2, \dots, 45$  according to the depth-first, left-to-right search order  $L_1$ . Two subtrees  $T(x) = \{18, 19, 21\}$  and  $T(y) = \{26, 27, 28, 33\}$  are marked with the darkened points. The root  $r(x)$  of  $T(x)$  is 18, and the root  $r(y)$  of  $T(y)$  is 26.

tree  $T$ , while the letters  $x, y$  and  $z$  (again with subscripts) will be used to denote members of  $X$ . The letters  $i, j, k, \ell, m$  and  $n$  will denote non-negative integers with the meaning of  $n$  fixed by setting  $n = |X|$ . The Greek letters  $\phi$  and  $\tau$  will denote proper colorings of the graph  $H$ . The colors assigned by  $\phi$  will be positive integers, while the colors assigned by  $\tau$  will be sets of triples. Later, we will define a function  $\sigma$  which assigns to each incomparable pair  $(x, y)$  a *signature*, to be denoted  $\sigma(x, y)$ . We will use the Greek letter  $\Sigma$  to denote a signature. The number of signatures will be the value  $d$ , and we will use Lemma 5.1 to show that any set of incomparable pairs having the same signature is reversible. Of course, we must be careful to insure that  $d$  is bounded in terms of  $t$  and  $h$ .

We consider the tree  $T$  as a *rooted* tree by taking an arbitrary vertex  $u_0$  of  $T$  as root. Draw the tree without edge crossings in the canonical manner. The root is at the bottom, and each vertex that is not the root has a unique neighbor below—its parent (equipped with such a drawing,  $T$  is called a *planted tree*). We suggest such a drawing in Figure 4.

For each  $x \in X$ , let  $r(x)$  denote the root of the subtree  $T(x)$ , i.e., the unique vertex of  $T(x)$  that is closest to the root  $u_0$  of  $T$ . Expanding vertices of  $T$  if necessary, we may assume that  $r(x) \neq r(y)$  whenever  $x$  and  $y$  are distinct elements of  $X$ .

The tree  $T$  may be considered as a poset by setting  $u \leq v$  in  $T$  when  $u$  lies on the path from  $v$  to  $u_0$  in  $T$ . Let  $L_1$  denote the depth-first, left-to-right search order of  $T$ . Let  $L_2$  denote the depth-first, right-to-left search order

of  $T$ . It follows that  $u \leq v$  in  $T$  if and only if  $u \leq v$  in  $L_1$  and  $u \leq v$  in  $L_2$ <sup>4</sup>. This shows  $\dim(T) \leq 2$  with  $\dim(T) = 2$  unless  $L_1 = L_2$ . It is natural to say that  $u$  is *left* of  $v$  in  $T$ , when  $u < v$  in  $L_1$  and  $v < u$  in  $L_2$ . Also, we say that  $u$  is *below*  $v$  in  $T$  when  $u < v$  in  $T$ . When  $u$  and  $v$  are distinct elements of  $T$ , exactly one of the following four statements holds: (1)  $u$  is below  $v$  in  $T$ ; (2)  $v$  is below  $u$  in  $T$ ; (3)  $u$  is left of  $v$  in  $T$ ; and (4)  $v$  is left of  $u$  in  $T$ .

The *lowest common ancestor* of two vertices  $u$  and  $v$  of  $T$ , denoted  $u \wedge v$ , is the greatest vertex  $w$  with  $w \leq u$  and  $w \leq v$  in  $T$ .

### 5.2. Induced Paths in the Intersection Graph

Observe that  $xy$  is an edge of the graph  $H$  if and only if one of the following statements is true: (1)  $r(x) < r(y)$  in  $T$  and  $r(y) \in T(x)$ , (2)  $r(y) < r(x)$  in  $T$  and  $r(x) \in T(y)$ .

We write  $x \triangleright_k y$  when there is a sequence  $(z_0, z_1, \dots, z_m)$  of elements of  $X$  such that  $0 \leq m \leq k$ ,  $z_0 = x$ ,  $z_m = y$ , and  $r(z_i) \in T(z_{i+1})$  for each  $i \in \{0, 1, \dots, m-1\}$ . Note that a shortest such sequence is an induced path in the graph  $H$ . Therefore, we could alternatively have written this definition as follows:  $x \triangleright_k y$  when there is an induced path  $(z_0, z_1, \dots, z_m)$  in  $H$  with  $0 \leq m \leq k$ ,  $z_0 = x$ ,  $z_m = y$ , and  $r(z_0) > r(z_1) > \dots > r(z_m)$  in  $T$ . As it will turn out, our proof will use the relation  $\triangleright_k$  for  $k \leq 2h - 2$ .

**Lemma 5.2.** *The relation  $\triangleright_k$  has the following properties:*

- (1) if  $x \triangleright_k y$  and  $k \leq \ell$ , then  $x \triangleright_\ell y$ ,
- (2)  $x \triangleright_0 y$  if and only if  $x = y$ ,
- (3)  $x \triangleright_{k+\ell} z$  if and only if there exists  $y \in X$  with  $x \triangleright_k y$  and  $y \triangleright_\ell z$ ,
- (4) if  $x \triangleright_k y$ , then  $r(y) \leq r(x)$  in  $T$ ,
- (5) if  $x \triangleright_k z$  and  $r(z) \leq r(y) \leq r(x)$  in  $T$ , then  $y \triangleright_k z$ ,
- (6) if  $x \triangleright_k y$  and  $x \triangleright_k z$ , then  $y \triangleright_k z$  or  $z \triangleright_k y$ ,
- (7) for each  $x \in X$ ,  $|\{y \in X : x \triangleright_k y\}| \leq 1 + t + t^2 + \dots + t^k$ .

**Proof.** Properties (1)–(4) follow directly from the definition of  $\triangleright_k$ . To see (5), let  $(z_0, z_1, \dots, z_m)$  be a sequence of elements of  $X$  such that  $0 \leq m \leq k$ ,  $z_0 = x$ ,  $z_m = z$ , and  $r(z_i) \in T(z_{i+1})$  for each  $i \in \{0, 1, \dots, m-1\}$ . Now note that since  $(z_0, \dots, z_m)$  is a path in  $H$  and  $r(x) = r(z_0) \in T(z_1)$ , the union  $\bigcup_{i=1}^m T(z_i)$  is a subtree of  $T$  containing the path from  $r(x)$  to  $r(z)$ . In particular,  $r(y) \in \bigcup_{i=1}^m T(z_i)$ , so there must be a positive  $i$  with  $r(y) \in T(z_i)$ , and  $(y, z_i, \dots, z_m)$  witnesses  $y \triangleright_k z$ . To see (6), observe that  $x \triangleright_k y$

<sup>4</sup> Note that the poset obtained by adding a one to  $T$  is planar. Now the argument given in [2] implies that  $\dim(T) \leq 2$ , as evidenced by these two linear extensions.

and  $x \triangleright_k z$  imply  $r(y) \leq r(z) \leq r(x)$  or  $r(z) \leq r(y) \leq r(x)$  in  $T$ , and the conclusion follows from (5). Finally, the fact that  $t$  is the tree-width of  $H$  yields  $|\{z' \in X - \{z\} : r(z) \in T(z')\}| \leq t$  for each  $z \in X$ , whence (7) follows. ■

We will use the properties listed in Lemma 5.2 implicitly, without further reference.

**Lemma 5.3.** *If  $x \leq y$  in  $P$ , then there exists  $z \in X$  such that:*

- (1)  $x \leq z \leq y$ ,
- (2)  $x \triangleright_{h-1} z$  and  $y \triangleright_{h-1} z$ .

**Proof.** Since  $H$  contains the cover graph  $G$  of  $P$ , there is a path  $(z_0, z_1, \dots, z_k)$  in  $H$  with  $z_0 = x$ ,  $z_k = y$ , and  $z_0 < z_1 < \dots < z_k$  in  $P$ . Take the shortest such path. Since  $h$  is the height of  $P$ , we have  $k \leq h - 1$ . For each  $i$  with  $0 \leq i < k$ , since  $z_i z_{i+1}$  is an edge of  $H$ , we have  $r(z_{i+1}) \in T(z_i)$  when  $r(z_i) < r(z_{i+1})$  in  $T$  or  $r(z_i) \in T(z_{i+1})$  when  $r(z_i) > r(z_{i+1})$  in  $T$ . If there is an index  $i$  with  $0 < i < k$  and  $r(z_{i-1}) < r(z_i) > r(z_{i+1})$  in  $T$ , then we have  $r(z_i) \in T(z_{i-1}) \cap T(z_{i+1})$ , so  $z_{i-1} z_{i+1}$  is an edge of  $H$  and we can obtain a shorter path by removing  $z_i$ . Therefore, there is a unique index  $i \in \{0, 1, \dots, k\}$  with  $r(z_0) > r(z_1) > \dots > r(z_i) < r(z_{i+1}) < \dots < r(z_k)$  in  $T$ . The definition of  $\triangleright_k$  yields  $z_0 \triangleright_k z_i$  and  $z_k \triangleright_k z_i$ . Since  $k \leq h - 1$ , the conclusion follows for  $z = z_i$ . ■

### 5.3. Colorings of the Ground Set

Order the elements of  $X$  as  $x_0, x_1, \dots, x_{n-1}$  so that the following holds: if  $r(x_j) \leq r(x_i)$  in  $T$ , then  $j \leq i$ . In particular, we have  $j \leq i$  whenever  $x_i \triangleright_k x_j$ . Define a coloring  $\phi$  of  $X$  with positive integers using the following inductive procedure. Start by setting  $\phi(x_0) = 1$ . Thereafter, for  $1 \leq i < n$ , let  $\phi(x_i)$  be the least positive integer that does not belong to  $\{\phi(x_j) : 0 \leq j < i \text{ and } x_i \triangleright_{2h-2} x_j\}$ . The reason why we take  $\triangleright_{2h-2}$  in this definition will become clear at the very end of the proof. The number of colors used by  $\phi$  is at most  $1 + t + t^2 + \dots + t^{2h-2}$ . Actually, we are not that interested in how many colors  $\phi$  will use exactly, except that this number must be bounded in terms of  $t$  and  $h$ , which it is.

**Lemma 5.4.** *If  $x \triangleright_{2h-2} z$ ,  $x \triangleright_{2h-2} z'$  and  $\phi(z) = \phi(z')$ , then  $z = z'$ .*

**Proof.** Suppose  $z \neq z'$ . Since  $x \triangleright_{2h-2} z$  and  $x \triangleright_{2h-2} z'$ , we have  $z \triangleright_{2h-2} z'$  or  $z' \triangleright_{2h-2} z$ . Whichever of these holds, the definition of  $\phi$  yields  $\phi(z) \neq \phi(z')$ . ■

Let  $(x, z)$  be a pair of elements of  $X$  with  $x \triangleright_{2h-2} z$ . There are four cases of how  $x$  and  $z$  are related in  $P$ : (1)  $x = z$ , (2)  $x < z$ , (3)  $x > z$ , or (4)  $x \parallel z$ .

We associate with  $(x, z)$  a triple  $\text{val}(x, z) = (\phi(x), \phi(z), t(x, z))$ , where  $t(x, z)$  is the number in  $\{1, 2, 3, 4\}$  denoting which of the above four cases holds. Since the number of distinct colors used by  $\phi$  is bounded in terms of  $t$  and  $h$ , so is the number of distinct triples of the form  $\text{val}(x, z)$  for all pairs  $(x, z)$  considered.

We define a new coloring  $\tau$  of  $X$  by assigning to each element  $x$  of  $X$ , the family  $\tau(x) = \{\text{val}(x, z) : z \in X \text{ and } x \triangleright_{2h-2} z\}$ . Thus the colors used by  $\tau$  are sets of triples, and the number of distinct colors used by  $\tau$  is bounded in terms of  $t$  and  $h$ . Note that the color classes of  $\tau$  refine the color classes of  $\phi$ , as the first element of each triple in  $\tau(x)$  is  $\phi(x)$ , and  $\tau(x)$  is non-empty since  $x \triangleright_0 x$ .

**Lemma 5.5.** *If  $x \triangleright_{2h-2} z$ ,  $x' \triangleright_{2h-2} z$  and  $\tau(x) = \tau(x')$ , then:*

- (1)  $x \leq z$  in  $P$  if and only if  $x' \leq z$  in  $P$ ,
- (2)  $x \geq z$  in  $P$  if and only if  $x' \geq z$  in  $P$ .

**Proof.** Since  $x' \triangleright_{2h-2} z$  and  $\tau(x) = \tau(x')$ , there is  $z' \in X$  with  $x \triangleright_{2h-2} z'$  and  $\text{val}(x, z') = \text{val}(x', z)$ . In particular, we have  $\phi(z) = \phi(z')$ , which implies  $z = z'$  in view of Lemma 5.4. The conclusion now follows from  $\text{val}(x, z) = \text{val}(x', z)$ . ■

### 5.4. Signatures for Incomparable Pairs

Each incomparable pair  $(x, y)$  in  $P$  satisfies exactly one of the following six conditions:

- (1)  $r(x)$  is below  $r(y)$  in  $T$ ,
- (2)  $r(y)$  is below  $r(x)$  in  $T$ ,
- (3)  $r(x)$  is left of  $r(y)$  in  $T$  and  $r(y')$  is left of  $r(y)$  in  $T$  for each  $y' \in X$  with  $\tau(y') = \tau(y)$  and  $x \leq y'$  in  $P$ ,
- (4)  $r(x)$  is left of  $r(y)$  in  $T$  and there exists  $y' \in X$  with  $\tau(y') = \tau(y)$ ,  $x \leq y'$  in  $P$  and  $r(y')$  not left of  $r(y)$  in  $T$ ,
- (5)  $r(y)$  is left of  $r(x)$  in  $T$  and  $r(x')$  is left of  $r(x)$  in  $T$  for each  $x' \in X$  with  $\tau(x') = \tau(x)$  and  $x' \leq y$  in  $P$ ,
- (6)  $r(y)$  is left of  $r(x)$  in  $T$  and there exists  $x' \in X$  with  $\tau(x') = \tau(x)$ ,  $x' \leq y$  in  $P$  and  $r(x')$  not left of  $r(x)$  in  $T$ .

We define the *signature* of  $(x, y)$  to be the triple  $\sigma(x, y) = (\tau(x), \tau(y), s(x, y))$ , where  $s(x, y)$  is the number in  $\{1, 2, \dots, 6\}$  denoting which of the above six cases holds for  $(x, y)$ . Since the number of distinct colors used by  $\tau$  is bounded in terms of  $t$  and  $h$ , so is the number of distinct signatures.

Let  $\text{Inc}(P, \Sigma) = \{(x, y) \in \text{Inc}(P) : \sigma(x, y) = \Sigma\}$ . To finish the proof of our main theorem, we show that  $\text{Inc}(P, \Sigma)$  is reversible for each signature  $\Sigma$ .

We argue by contradiction. Fix a signature  $\Sigma$ , and suppose that  $\text{Inc}(P, \Sigma)$  is not reversible. In view of Lemma 5.1,  $\text{Inc}(P, \Sigma)$  contains an alternating cycle  $\{(x_i, y_i) : 1 \leq i \leq k\}$ . Since all the signatures  $\sigma(x_i, y_i)$  are equal, we have all the  $\tau(x_i)$  equal and all the  $\tau(y_i)$  equal. Moreover, all the pairs  $(x_i, y_i)$  satisfy the same one of the conditions (1)–(6) above. This gives us six cases to consider. Case (2) is dual to (1), (5) is dual to (3), and (6) is dual to (4). Therefore, it is enough that we show that each of the cases (1), (3) and (4) leads to a contradiction. In the arguments below, we always interpret the index  $i$  cyclically in  $\{1, 2, \dots, k\}$ .

Suppose that (1) holds for all  $(x_i, y_i)$ . There must be an index  $i$  such that  $r(x_i)$  is not below  $r(x_{i-1})$  in  $T$ . We have  $x_{i-1} \leq y_i$  in  $P$ , so let  $z$  be an element of  $X$  claimed by Lemma 5.3 for  $(x_{i-1}, y_i)$ . Thus  $x_{i-1} \leq z \leq y_i$  in  $P$ ,  $x_{i-1} \triangleright_{h-1} z$ , and  $y_i \triangleright_{h-1} z$ . Since  $r(x_i)$  is below  $r(y_i)$  and not below  $r(x_{i-1})$  in  $T$ , we have  $r(z) \leq r(x_i) < r(y_i)$  in  $T$  and thus  $x_i \triangleright_{h-1} z$ . We also have  $\tau(x_{i-1}) = \tau(x_i)$ . Consequently, by Lemma 5.5, we have  $x_i \leq z \leq y_i$  in  $P$ , which is a contradiction.

If (3) holds for all  $(x_i, y_i)$ , then we have  $r(y_{i+1})$  left of  $r(y_i)$  in  $T$  for each  $i$ , which is clearly a contradiction.

Finally, suppose that (4) holds for all  $(x_i, y_i)$ . There must be an index  $i$  such that  $r(x_i)$  is not left of  $r(x_{i-1})$  in  $T$ . To simplify the notation, let  $x = x_i$ ,  $y = y_i$  and  $x' = x_{i-1}$ . Thus we have  $x \parallel y$  and  $x' \leq y$  in  $P$ ,  $\tau(x') = \tau(x)$ ,  $r(x)$  left of  $r(y)$  in  $T$ , and  $r(x)$  not left of  $r(x')$  in  $T$ . Furthermore, since  $(x, y)$  satisfies condition (4), there is  $y' \in X$  with  $\tau(y') = \tau(y)$ ,  $x \leq y'$  in  $P$  and  $r(y')$  not left of  $r(y)$  in  $T$ . All this implies that the paths in  $T$  connecting  $r(x')$  to  $r(y)$  and  $r(x)$  to  $r(y')$  both pass through  $r(x) \wedge r(y)$ . Now, let  $z$  be an element of  $X$  claimed by Lemma 5.3 for  $(x', y)$ , and  $z'$  be an element of  $X$  claimed by Lemma 5.3 for  $(x, y')$ . Thus we have  $x' \leq z \leq y$  and  $x \leq z' \leq y'$  in  $P$ ,  $x' \triangleright_{h-1} z$ ,  $y \triangleright_{h-1} z$ ,  $x \triangleright_{h-1} z'$ , and  $y' \triangleright_{h-1} z'$ . Since  $r(z) \leq r(x')$  and  $r(z) \leq r(y)$  in  $T$ , it follows that  $r(z)$  is below every vertex in the path from  $r(x')$  to  $r(y)$ , and in particular,  $r(z) \leq r(x) \wedge r(y)$ . Similarly,  $r(z') \leq r(x) \wedge r(y)$  in  $T$ . Thus  $r(z) \leq r(z')$  or  $r(z') \leq r(z)$  in  $T$ . If  $r(z) \leq r(z')$ , then  $z' \triangleright_{h-1} z$  and thus  $x \triangleright_{2h-2} z$ . This, by Lemma 5.5, implies  $x \leq z \leq y$  in  $P$ , which is a contradiction. If  $r(z') \leq r(z)$ , then we get a similar contradiction  $x \leq z' \leq y$ . This completes the proof of Theorem 1.1.

### 6. Questions and Problems

Our main result establishes the existence of the function  $d(t, h)$  without emphasis on optimizing our bound. Let  $p$  be the number of colors used in  $\phi$ . The number of signatures of incomparable pairs is at most  $6 \cdot 2^{8p^2}$ . We



compute  $p \leq 1 + t + \dots + t^{2h-2} \leq t^{2h-1}$ , and it follows that  $d(t, h) \leq 6 \cdot 2^{8t^{4h-2}}$ . One immediate challenge is to tighten the bounds on this function. It may even be true that for each  $t$ , there is a constant  $c_t$  so that  $d(t, h) \leq c_t h$ . It is conceivable that better techniques may prove an exact formula for  $d(t, h)$ , for all  $t$  and  $h$ .

As noted in the introductory section, when the tree-width of the cover graph of  $P$  is 1,  $\dim(P) \leq 3$ , independent of the height of  $P$ . Also, when the cover graph of  $P$  is outerplanar (so it has tree-width at most 2),  $\dim(P) \leq 4$  independent of the height of  $P$ . On the other hand, the posets in Kelly's construction have path-width 3. Accordingly, it is natural to raise the following questions.

**Question 6.1.** Does there exist a constant  $d_0$  so that if  $P$  is a poset and the path-width of the cover graph of  $P$  is at most 2, then  $\dim(P) \leq d_0$ ?

**Question 6.2.** Does there exist a constant  $d_1$  so that if  $P$  is a poset and the tree-width of the cover graph of  $P$  is at most 2, then  $\dim(P) \leq d_1$ ?

The first of these two questions was recently settled in the affirmative by Biró, Keller and Young [4], and we firmly believe that the second one has an affirmative answer as well.

Kelly's construction actually raises two other questions. First, is it true that a planar poset with large dimension contains a large standard example? We believe the answer is yes and make the following conjecture.

**Conjecture 6.3.** *For every integer  $d \geq 2$ , there is an integer  $D = D(d)$  so that if  $P$  is a planar poset with  $\dim(P) \geq D$ , then  $P$  contains the standard example  $S_d$ .*

Second (and this specific question was posed to us by Stanley [28]), is it true that a planar poset with large dimension has many minimal elements? The answer is yes. Recently, Trotter and Wang [31] proved the following result.

**Theorem 6.4.** *If  $P$  is a planar poset with  $t$  minimal elements, then  $\dim(P) \leq 2t + 1$ .*

This inequality is best possible for  $t = 1$  and  $t = 2$ , but for larger values of  $t$ , a lower bound of  $t + 3$  is proved in [31].

The first of these two questions has a natural extension to tree-width, so we would also make the following conjecture.

**Conjecture 6.5.** *For every pair  $(d, t)$  of positive integers with  $d \geq 2$ , there is an integer  $D = D(d, t)$  so that if  $P$  is a poset such that the tree-width of the cover graph of  $P$  is at most  $t$  and  $\dim(P) \geq D$ , then  $P$  contains the standard example  $S_d$ .*

While the second question concerning the number of minimal elements makes sense, it is easily answered in the negative, since adding a zero to a poset can increase the tree-width of the cover graph by at most one.

Finally, we close with what we believe is a very ambitious conjecture.

**Conjecture 6.6.** *Let  $\mathcal{G}$  be a proper minor-closed class of graphs. Then for every integer  $h \geq 1$ , there is a least positive integer  $d = d(\mathcal{G}, h)$  so that if  $P$  is a poset of height  $h$  and the cover graph of  $P$  belongs to  $\mathcal{G}$ , then  $\dim(P) \leq d$ .*

Our main theorem shows that the conjecture is true when  $\mathcal{G}$  is the class of graphs of tree-width at most  $t$ . In [29], a general reduction is described which allows one to restrict to the case where the cover graph has bounded diameter (as a function of the height). It follows as an immediate corollary that the conjecture holds whenever  $\mathcal{G}$  has the diameter tree-width property. For this reason, we have an alternative proof of Theorem 4.6. Graphs of bounded genus, and more generally graphs excluding an apex graph as a minor also have the diameter tree-width property (see [12]). Therefore, the above conjecture also holds in these special cases.

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**Note added in proof.** Conjecture 6.6 has been settled in the affirmative by B. Walczak in [33]. Also, Question 6.2 has been answered in the positive by G. Joret, P. Micek, W. T. Trotter, R. Wang and V. Wiechert in [22].

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