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Forcing Posets with Large Dimension to Contain Large Standard Examples

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Abstract The dimension of a poset P, denoted $\dim(P)$, is the least positive integer d for which P is the intersection of d linear extensions of P. The maximum dimension of a poset P with $|P| \leq 2n+1$ is n, provided $n \geq 2$, and this inequality is tight when P contains the standard example S_n. However, there are posets with large dimension that do not contain the standard example S_2. Moreover, for each fixed $d \geq 2$, if P is a poset with $|P| \leq 2n+1$ and P does not contain the standard example S_d, then $\dim(P) = o(n)$. Also, for large n, there is a poset P with $|P| = 2n$ and $\dim(P) \geq (1 - o(1))n$ such that the largest d so that P contains the standard example S_d is $o(n)$. In this paper, we will show that for every integer $c \geq 1$, there is an integer $f(c) = O(c^2)$ so that for large enough n, if P is a poset with $|P| \leq 2n+1$ and $\dim(P) \geq n - c$, then P contains a standard example S_d with $d \geq n - f(c)$. From below, we show that $f(c) = \Omega(c^{4/3})$. On the other hand, we also prove an analogous result for fractional dimension, and in this setting $f(c)$ is linear in c. Here the result is best possible up to the value of the multiplicative constant.

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1 Introduction

When G is a graph, let $\chi(G)$ denote the *chromatic number* of G, and let $\omega(G)$ denote the *maximum clique size* of G. As is well known, there are triangle-free graphs (graphs with $\omega(G) \leq 2$) with large chromatic number. Moreover, just by analyzing the behavior of the Ramsey number $R(3, k) = \Theta(k^2/\log k)$, it follows that there are triangle-free graphs on n vertices with chromatic number $\Omega(\sqrt{n/\log n})$. However, when a graph on n vertices has chromatic number close to n, it must have a large clique. We state for emphasis the following self-evident proposition.[1]

Proposition 1 *Let c be a positive integer. If $n > 2c$ and G is a graph on n vertices with $\chi(G) \geq n - c$, then $\omega(G) \geq n - 2c$.*

This paper is concerned with analogous results for finite partially ordered sets (posets).

1.1 Posets and Dimension

We assume familiarity with basic notation and terminology for posets, including chains and antichains; comparable and incomparable elements; minimal and maximal elements; and linear extensions. For readers who seek additional background material on posets, Trotter's book [27] is a good reference.

We denote by $|P|$ the number of elements of P and we frequently refer to elements of P as points. Recall that the *height* of a poset P, denoted $\text{height}(P)$, is the maximum size of a chain in P, while the *width* of P, denoted $\text{width}(P)$, is the maximum size of an antichain in P. As there is no completely standard notation for this relation, here we will write $x \parallel y$ in P when x and y are distinct and incomparable points in a poset P.

A family $\mathscr{R} = \{L_1, L_2, \ldots, L_d\}$ of linear extensions of a poset P is called a *realizer* of P if $x < y$ in P if and only if $x < y$ in L_i for each $i = 1, 2, \ldots, d$. Equivalently, a family $\mathscr{R}$ of linear extensions of P is a realizer of P if and only if for every ordered pair (x, y) with $x \parallel y$ in P, there is some i with $1 \leq i \leq d$ with $x > y$ in L_i. Dushnik and Miller [9] defined the *dimension* of a poset P, denoted $\dim(P)$, as the least positive integer d for which P has a realizer of size d. When P is a poset, the *dual* of P is the poset Q with the same ground set as P with $x > y$ in Q if and only if $x < y$ in P. The following basic properties of dimension are self-evident.

1. $\dim(P) = 1$ if and only if P is a chain.

[1] Biró et al. [3] have studied the question of forcing large cliques in graphs in far greater detail than this elementary proposition. But this simple result suffices in establishing a parallel line of thought in graph theory.

2. If Q is a subposet of P, then $\dim(Q) \leq \dim(P)$.
3. If Q is the dual of P, then $\dim(P) = \dim(Q)$.
4. If P is an antichain of size at least 2, then $\dim(P) = 2$.

The following construction was first noted by Dushnik and Miller in [9]. For an integer $d \geq 2$, let S_d be the following height 2 poset: S_d has d minimal elements $\{a_1, a_2, \ldots, a_d\}$ and d maximal elements $\{b_1, b_2, \ldots, b_d\}$. The partial ordering on S_d is defined by setting $a_i < b_j$ in S_d if and only if $i \neq j$. Evidently, $\dim(S_d) \geq d$, since if L is any linear extension of S_d, there can be at most one value of i with $a_i > b_i$ in L. On the other hand, it is easy to see that $\dim(S_d) \leq d$. The poset S_d is called the *standard example* (of dimension d).

Hiraguchi [17] proved that if $n \geq 2$ and $|P| \leq 2n + 1$, then $\dim(P) \leq n$. The family of standard examples shows that this inequality is tight. Moreover,[2] we have the following theorem, with the even case due to Bogart and Trotter [6] and the odd case due to Kimble [22].

Theorem 1 *If* $n \geq 4$ *and* $|P| \leq 2n + 1$, *then* $\dim(P) < n$ *unless* P *contains the standard example* S_n.

If a poset P contains a large standard example, then the dimension of P is large, and the following result, which is both a generalization of Theorem 1 and a poset analogue of Proposition 1, is the first of the two principal results in this paper.

Theorem 2 *For every positive integer* c, *there is an integer* $f(c) = O(c^2)$ *so that if* $n > 10 f(c)$ *and* P *is a poset with* $|P| \leq 2n+1$ *and* $\dim(P) \geq n-c$, *then* P *contains a standard example* S_d *with* $d \geq n - f(c)$.

In our proof, the function $f(c)$ will be quadratic in c, but this may not be best possible. However, we are able to show that $f(c) = \Omega(c^{4/3})$. Of course, the restriction $n > 10 f(c)$ in the statement of Theorem 2 is just intended to make n sufficiently large in terms of c. This restriction also serves to keep us safely away from annoying small cases.

In the remainder of this introductory section, we briefly discuss results which are more substantive than Proposition 1 and serve to motivate our main theorem. In Sect. 2, we gather some essential preliminary material, and the proof of our main theorem is given in Sect. 3.

In Sect. 4, we prove an analogous theorem for fractional dimension, and in this setting, the function $f(c)$ is linear in c. Of course, this result is best possible up to the value of the multiplicative constant. We close with some open problems in Sect. 5.

[2] The inductive step in the proof of Theorem 1, as presented by Kimble in [22], is relatively compact, and some might even say that it is elegant. On the other hand, no entirely complete proof of the base case ($n = 4$) has ever been written down—nor is this likely to happen. The problem is to show that if $|P| = 9$ and $\dim(P) = 4$, then P contains S_4. The issue is that the analogous statement is not true when $n = 3$, as there are 20 posets of size 7 which have dimension 3 and do not contain a 3-dimensional subposet on 6 points.

1.2 Large Dimension Without Large Standard Examples

Dushnik and Miller [9] made the following observation: For an integer $n \geq 3$, let $P(1, 2; n)$ denote the poset consisting of the 1-element and 2-element subsets of $\{1, 2, \ldots, n\}$, and let $d(1, 2; n) = \dim(P(1, 2; n))$. Using the classic theorem of Erdős and Szekeres, they noted that $\dim(1, 2; n) = \Omega(\lg \lg n)$. While $P(1, 2; n)$ contains S_3, it does not contain S_d for any $d \geq 4$.

We pause to comment that much more can be said about the growth rate of $d(1, 2; n)$. By combining results of Hoşten and Morris [18] with estimates of Kleitman and Markovsky [23], the following theorem follows in a straightforward manner.

Theorem 3 *For every* $\epsilon > 0$, *there is an integer* n_0 *so that if* $n > n_0$ *and*

$$s = \lg \lg n + 1/2 \lg \lg \lg n + 1/2 \lg \pi + 1/2,$$

then $s - \epsilon < d(1, 2; n) < s + 1 + \epsilon$.

As a consequence, for almost all large values of n, we can compute the value of $d(1, 2; n)$ *exactly*; for the remaining small fraction of values, we are able to compute two consecutive integers and say that $d(1, 2; n)$ is one of the two.

But a poset can have large dimension without containing S_3. In [11], Felsner and Trotter observed that if d and g are positive integers, then there is a poset P of height 2 for which the girth of the comparability graph of P is at least g while the dimension of P is at least d. In fact, this observation is an immediate consequence of the well known fact that there is a graph G whose girth is at least g and whose chromatic number is at least d (see [11] and [12] for additional details on adjacency posets and related problems). Posets with large dimension and large girth will contain the standard example S_2. However, they do *not* contain the standard example S_3.

To close this story, we note that a poset can have large dimension without containing S_2. A poset P is called an *interval order* if there is a family $\mathscr{I} = \{[a_x, b_x] : x \in P\}$ of closed intervals of the real line $\mathbb{R}$ so that $x < y$ in P if and only if $b_x < a_y$ in $\mathbb{R}$. Fishburn [14] showed that a poset P is an interval order if and only if it does not contain the standard example S_2. We note that S_2 is isomorphic to $\mathbf{2} + \mathbf{2}$, the disjoint sum of two 2-element chains. In general posets of height 2 can have arbitrarily large dimension. However, Rabinovitch [24] showed that the dimension of an interval order is bounded in terms of its height. Specifically, he showed that the maximum dimension d_n of an interval order of height n is $O(\log n)$.

In [5], Bogart et al. considered the *canonical* interval order $I(n)$ consisting of all intervals with distinct integer end points in $\{1, 2, \ldots, n\}$ and showed that $\dim(I(n))$ goes to infinity with n. However, by combining the results of Füredi et al. [15] with the same analysis used for Theorem 3, it is easy to verify that $|\dim(I(n)) - d(1, 2 : n)| \leq 5$ and $|d_n - d(1, 2; n)| \leq 5$, for all $n \geq 2$. So the values of $\dim(I(n))$ and d_n can be computed "almost exactly."

1.3 Forcing Large Standard Examples

With this background in mind, it is natural to ask whether there are conditions which force a poset of large dimension to contain a large standard example. A partial answer is provided by the following theorem of Biró et al. [1].

Theorem 4 *For every integer $d \geq 2$ and every $\epsilon > 0$, there is an integer n_0 so that if $n > n_0$ and P is a poset with $|P| \leq 2n + 1$ and P does not contain the standard example S_d, then* $\dim(P) < \epsilon n$.

Paralleling our earlier discussion on chromatic number, it was natural for Biró, Füredi and Jahanbekam to conjecture[3] in [3] that when n is large and P is a poset with $|P| \leq 2n + 1$, if the dimension of P is close to n, then P must contain a standard example S_d with d also close to n.

Here are two examples to show that when $|P| \leq 2n + 1$, the dimension must be quite close to n in order to force P to contain a large standard example. The first example was studied, for a quite different purpose, by Howard and Trotter [19]. In dual form, this poset was also studied by Füredi and Kahn [16].

Example 1 Consider a finite projective plane of order q. We associate with this geometry a poset P of height 2 with $|P| = 2n = 2(q^2 + q + 1)$. The minimal elements of P are the points of the geometry and the maximal elements of P are the lines. In P, point x is less than line y when x is *not* on y. It can be derived from results in [4] that if X is a set of points and Y is a set of lines in a finite projective plane of order q, and no point of X belongs to any line in Y, then $|X||Y| \leq q^3$. It follows that if S_d is contained in P, then $d \leq 2q^{3/2}$. In fact, Illés et al. [20] tightened this elementary bound and showed that $d \leq q\sqrt{q} + 1$.

Now suppose that $t = \dim(P)$ and let $\mathscr{R} = \{L_1, L_2, \ldots, L_t\}$ be a realizer of P. Suppose that $t < q^2 + q + 1 - q^{3/2}$. Then there is a set X of points, with $|X| > q^{3/2}$, so that no point of X is the top point in any linear extension in $\mathscr{R}$. (Here "point" refers to a point in the geometry.) Dually, there is a set Y of lines, with $|Y| > q^{3/2}$, so that no line in Y is the lowest line in any linear extension in $\mathscr{R}$. Using the basic property of a finite geometry that two points determine a unique line, it follows easily that $x < y$ in L_i for all $x \in X$, $y \in Y$ and for all $i = 1, 2, \ldots, t$. Indeed, if $x \parallel y$, then there is a linear extension in which $y < x$; but there are also a point x' and a line y' such that in the same linear extension $y' < y < x < x'$, which would imply $\{y, y'\} \parallel \{x, x'\}$, a contradiction.

We have shown that no point of X is on any of the lines in Y. This is a contradiction, since $|X||Y| > q^3$. It follows that $\dim(P) \geq q^2 + q + 1 - q^{3/2}$, so when q is large, $\dim(P) = (1 - o(1))n$, yet if S_d is a standard example contained in P, then $d = O(n^{3/4}) = o(n)$. As we will detail later, this example provides the lower bound $f(c) = \Omega(c^{4/3})$ for the function $f(c)$ in our main theorem.

[3] To be precise, in [3], Biró et al. conjectured that for a fixed small $\epsilon > 0$, a poset P with at most $2n + 1$ points and dimension at least $(1 - \epsilon)n$ must contain a standard example S_d with d a positive fraction of n. Examples 1 and 2 show that this conjecture is too strong. Nevertheless, our Theorem 2 confirms the basic intuition behind their conjecture.

Example 2 This example uses results from [10] where Erdös et al. study the behavior of a random poset P of height 2. In this setting $P = A \cup B$, where A and B are n-element antichains. Fix a probability p (in general p is a function of n). Then for each of the n^2 pairs $(a, b) \in A \times B$, we set $a < b$ in P with probability p. Events corresponding to distinct pairs are independent.

The following statement is simply an extraction of more comprehensive results proved in [10]: If $p = 1/2$, and n is very large, then almost surely, the following two statements hold: (1) $\dim(P) > n - 1000n/\log n$, and (2) P does not contain any standard example S_d with $d \geq 100 \log n$.

2 Essential Preliminary Material

The proof of our main theorem will require a number of well-known inequalities in dimension theory. These results use the following notation and terminology.

When P is a poset, $\mathrm{Min}(P)$ and $\mathrm{Max}(P)$ denote, respectively, the set of minimal elements and the set of maximal elements of P. A subposet D of P is called a *down set* if $y \in D$ whenever $x \in D$ and $x > y$ in P. Dually, a subposet U is called an *up set* in P if $y \in U$ whenever $x \in U$ and $y > x$ in P. Of course, D is a down set in P if and only if $U = P - D$ is an up set in P. When x is a point in P and Q is a subposet of P, we let;

1. $D(x, Q) = \{u \in Q : u < x \text{ in } P\}$;
2. $U(x, Q) = \{v \in P : v > x \text{ in } P\}$; and
3. $I(x, Q) = \{y \in Q - \{x\} : y \parallel x \text{ in } P\}$.

When A is a maximal antichain in P, the subposet $P - A$ is naturally partitioned into a subposet $D_P(A) = \{x \in P - A : x < a \text{ for some } a \in A\}$ and a subposet $U_P(A) = \{y \in P - A : y > a \text{ for some } a \in A\}$. When no confusion may arise, we simply write $D(A)$ and $U(A)$ rather than $D_P(A)$ and $U_P(A)$. In discussing subposets of P, we use the natural convention that when Q is empty, $\mathrm{width}(Q) = \dim(Q) = 0$.

With this notation in hand, we list in the following theorem the essential results we will need.

Theorem 5 *Let P be poset. Then the following inequalities hold.*

1. $\dim(P) \leq \mathrm{width}(P)$.
2. *If* $|P| \geq 2$ *and* $x \in P$, *then* $\dim(P) \leq 1 + \dim(P - \{x\})$.
3. $\dim(P) \leq \max\{2, 1 + \mathrm{width}(P - \mathrm{Min}(P))\}$.
4. *If A is a maximal antichain in P, then* $\dim(P) \leq \max\{2, |P - A|\}$.
5. *If A is a maximal antichain in P, then* $\dim(P) \leq \max\{2, 1 + 2\,\mathrm{width}(P - A)\}$.
6. *If D is a down set in P, and $U = P - D$, then* $\dim(P) \leq \dim(D) + \mathrm{width}(U)$.
7. *If* $a \in \mathrm{Min}(P)$, $b \in \mathrm{Max}(P)$ *and* $a \parallel b$ *in* P, *then* $\dim(P) \leq 1 + \dim(P - \{a, b\})$.

The first inequality is due to Dilworth [8]. The second and seventh are due to Hiraguchi [17]. The third, fourth and fifth are due to Trotter [26] (the fourth was discovered independently by Kimble [22]). The sixth is due to Trotter and Wang [31].

These inequalities have forms which can be applied to the dual of a poset, and we will use these dual forms without comment.

2.1 Reversible Sets and Alternating Cycles

Let P be a poset and let $\mathrm{Inc}(P) = \{(x, y) : x \parallel y \text{ in } P\}$. When $(x, y) \in \mathrm{Inc}(P)$ and L is a linear extension of P, we say L reverses (x, y) when $x > y$ in L. More generally, we say a family $\mathscr{F}$ of linear extensions *reverses* a set $S \subseteq \mathrm{Inc}(P)$ when for every $(x, y) \in S$, there is some $L \in \mathscr{F}$ which reverses (x, y). The dimension of P is then the least positive integer d for which there is a family $\mathscr{F}$ of d linear extensions of P which reverses $\mathrm{Inc}(P)$. This reformulation of dimension in terms of a partition of the set of incomparable pairs was first stated explicitly by Rabinovitch and Rival in [25].

A set $S \subseteq \mathrm{Inc}(P)$ is said to be *reversible* when there is a single linear extension L of P which reverses every pair in S. So the dimension of a poset P which is not a chain is then the least d for which $\mathrm{Inc}(P)$ can be partitioned into d reversible sets.

When $k \geq 2$, a sequence $\{(a_i, b_i) : 1 \leq i \leq k\}$ of pairs from $\mathrm{Inc}(P)$ is called an *alternating cycle* when $a_i \leq b_{i+1}$ in P for every $i = 1, 2, \ldots, k$ (this requirement is interpreted cyclically, i.e., we intend that $a_k \leq b_1$ in P). An alternating cycle is *strict* when $a_i \leq b_j$ in P if and only if $j = i + 1$. The following elementary lemma is proved (with different notation) in [29].

Lemma 1 *Let P be a poset and let $S \subseteq \mathrm{Inc}(P)$. Then the following statements are equivalent.*

1. *S is reversible.*
2. *S does not contain an alternating cycle.*
3. *S does not contain a strict alternating cycle.*

In many instances, the last statement of Lemma 1 is particularly useful, since if $\{(a_i, b_i) : 1 \leq i \leq k\}$ is a strict alternating cycle, then $\{a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_k\}$ are k-element antichains in P.

The following elementary proposition was first exploited by Hiraguchi in proving the last inequality listed in Theorem 5.

Proposition 2 *Let P be a poset and let $(x, y) \in \mathrm{Inc}(P)$. Then the set*

$$S = \{(x, u) : (x, u) \in \mathrm{Inc}(P)\} \cup \{(v, y) : (v, y) \in \mathrm{Inc}(P)\}$$

is reversible.

Proof Since $x \parallel y$ in P, the set S cannot contain a strict alternating cycle.

Framing dimension problems in terms of reversible sets of incomparable pairs has been a very useful approach, and we point to the recent papers [13] and [21] as examples.

2.2 Bipartite Posets

In [28], Trotter and Bogart defined the *interval dimension* of a poset P, denoted $\mathrm{Idim}(P)$, as the least positive integer d for which there are d interval orders P_1,

$P_2, \ldots, P_d$ so that $x < y$ in P if and only if $x < y$ in P_i for $i = 1, 2, \ldots, d$. Since a linear order is an interval order, we always have $\mathrm{Idim}(P) \leq \dim(P)$. At one extreme, it is easy to see that $\mathrm{Idim}(S_d) = \dim(S_d) = d$, for every $d \geq 2$. At the other extreme, the family of canonical interval orders discussed previously show that every $d \geq 1$, there is a poset with $\mathrm{Idim}(P) = 1$ and $\dim(P) = d$.

In the arguments to follow, we will quickly reduce the problem to the case where P is a poset of height 2, and there the real work begins. In fact, it will be useful to consider the notion of a *bipartite poset*. This is a poset P with a partition $P = A \cup B$ where $A \subseteq \mathrm{Min}(P)$ and $B \subseteq \mathrm{Max}(P)$. So a bipartite poset has height at most 2, but elements of P which are both minimal and maximal (some authors call these elements "loose" points) can belong to A or B. In fact, an antichain can be made into a bipartite poset. When we refer to a subposet Q of bipartite poset $P = A \cup B$, then we automatically consider $Q = (Q \cap A) \cup (Q \cap B)$ in the bipartite form it inherits from P. Also, in the bipartite poset setting, it makes sense to speak of the standard example $S_1 = \{a_1\} \cup \{b_1\}$ with $a_1 \parallel b_1$ in S_1.

For the balance of this subsection, we restrict our attention to bipartite posets. The following two results are extracted from [28], and we caution the reader to remember that they hold only in this special setting.

Proposition 3 *When $P = A \cup B$ is a bipartite poset*, $\mathrm{Idim}(P)$ *is the least positive integer d for which there is a family $\mathscr{F}$ of linear extensions of P so that for every pair* $(a, b) \in \mathrm{Inc}(P) \cap (A \times B)$, *there is some $L \in \mathscr{F}$ with $a > b$ in P.*

Proposition 4 *When $P = A \cup B$ is a bipartite poset*, $\mathrm{Idim}(P) \leq \dim(P) \leq 1 + \mathrm{Idim}(P)$.

In view of the two preceding propositions, when $P = A \cup B$ is a bipartite poset, we let $\mathrm{Inc}_0(P) = \mathrm{Inc}(P) \cap (A \times B)$, so that when $\mathrm{Inc}_0(P) \neq \emptyset$, $\mathrm{Idim}(P)$ is the least positive integer for which $\mathrm{Inc}_0(P)$ can be partitioned into d reversible sets. Also, borrowing from the terminology discussed earlier, when $A' \subseteq A$ and $B' \subseteq Y$, we will say that a linear extension L *reverses* A' with B' when L reverses all pairs $(a, b) \in \mathrm{Inc}_0(P) \cap (A' \times B')$. When $A' = \{a\}$ and L reverses A' with B', we will just say that L reverses a with B'. Similarly, when $B' = \{b\}$ and L reverses A' with B', we will just say that L reverses A' with b. More generally, we will say that a family $\mathscr{F}$ of linear extensions of P *reverses* A' with B' when for every pair $(a, b) \in \mathrm{Inc}_0(P) \cap (A' \times B')$, there is some $L \in \mathscr{F}$ with $a > b$ in L. For convenience, we will just say that a family $\mathscr{F}$ which reverses A with B is a *reversing family* for P.

Our arguments for bipartite posets will make extensive use of the following special case of Proposition 2.

Proposition 5 *Let $P = A \cup B$ be a bipartite poset. If* $(a, b) \in \mathrm{Inc}_0(P)$, *then there is a linear extension $L = L(a, b, P)$ of P which reverses a with B and A with b.*

In the remainder of the paper, when $P = A \cup B$ is a bipartite poset, $(a, b) \in \mathrm{Inc}_0(P)$, then $L(a, b, P)$ will always denote a linear extension of P which reverses a with B and A with b. There may be many linear extensions which satisfying these two requirements, and in most settings, it will not matter which one is chosen. However, later in this section, we will discuss a special case where we will attempt to find a linear extension $L(a, b, P)$ which also satisfies a third requirement.

We say a subposet Q of a bipartite poset $P = A \cup B$ is *balanced* when $|Q \cap A| = |Q \cap B|$. When $m \geq 1$ and Q is a balanced subposet of P with $|Q| = 2m$, a labelling $Q = \{u_1, u_2, \ldots, u_m\} \cup \{v_1, v_2, \ldots, v_m\}$ of the elements of Q will be called a *matching* of Q when $u_i \in A$, $v_i \in B$ and $u_i \parallel v_i$ in P for all $i = 1, 2, \ldots, m$.

Here are two essential—although straightforward—lemmas concerning matchings.

Lemma 2 *Let $P = A \cup B$ be a bipartite poset, let Q be a non-empty balanced subposet of P and let $Q = \{u_1, u_2, \ldots, u_m\} \cup \{v_1, v_2, \ldots, v_m\}$ be a matching of Q. Then* $\mathrm{Idim}(P) \leq m + \mathrm{Idim}(P - Q)$. *Furthermore, if Q is maximal, then* $\mathrm{Idim}(P) \leq m$.

Proof Let $t = \mathrm{Idim}(P - Q)$ and let $\{M_1, M_2, \ldots, M_t\}$ be a family of linear extensions of $P - Q$ which forms a reversing family for $P - Q$. For each $i = 1, 2, \ldots, t$, let L_i be a linear extension of P so that the restriction of L_i to $P - Q$ is M_i. Then for each $j = 1, 2, \ldots, m$, let $L_{t+j} = L(u_j, v_j, P)$. Then $\mathscr{F} = \{L_1, L_2, \ldots, L_t, L_{t+1}, \ldots, L_{t+m}\}$ shows that $\mathrm{Idim}(P) \leq m + \mathrm{Idim}(P - Q)$.

Furthermore, if Q is maximal, then we can take $t = 0$ and the initial family empty; then $\{L_1, L_2, \ldots, L_t\}$ is a realizer.

Lemma 3 *Let $P = A \cup B$ be a bipartite poset, and let $m = \min\{|A|, |B|\}$. If $m \geq 2$, then* $\mathrm{Idim}(P) < m$ *unless P contains the standard example S_m.*

Proof Without loss of generality, we assume $|A| \leq |B|$ and let $A = \{a_1, a_2, \ldots, a_m\}$. Clearly, we may assume that this labelling has been done so that there is no element $b \in B$ with $b \parallel a_m$ in P and $b > a_i$ in P for each $i = 1, 2, \ldots, m - 1$. For each $i = 1, 2, \ldots, m - 1$, let L_i be a linear extension of P with the following block structure:

$$A - \{a_i, a_m\} < I(a_i, B) \cap I(a_m, B) < a_m < I(a_i, B) \cap U(a_m, B) < a_i < U(a_i, B).$$

Clearly, the family $\{L_1, L_2, \ldots, L_{m-1}\}$ shows that $\mathrm{Idim}(P) \leq m - 1$.

The next lemma is more substantive and more technical in nature. However, this result will prove to be a key detail in our proof. Also, it is one of two new inequalities prompted by the general problem investigated in this paper—the second such result will be presented in the next section. We alert the reader that we will be discussing linear extensions of the form $L(a, b, P)$ where we search for such an extension which also reverses a pair $(a', b') \in \mathrm{Inc}_0(P)$ with $a \neq a'$ and $b \neq b'$.

Lemma 4 *Let $s \geq 1$ and $t = 5s$. Then let $P = A \cup B$ be a balanced bipartite poset with $|P| = 4t$. Suppose that P can be partitioned into two disjoint balanced subposets T and T', each of which is a copy of the standard example S_t. Also, let $T = \{a_1, a_2, \ldots, a_t\} \cup \{b_1, b_2, \ldots, b_t\}$ and $T' = \{w_1, w_2, \ldots, w_t\} \cup \{z_1, z_2, \ldots, z_t\}$ be matchings. If the subposet $\{a_i, b_i, w_i, z_i\}$ is not S_2, for each $i = 1, 2, \ldots, t$, then* $\mathrm{Idim}(P) \leq 9s$.

Proof Set $A_0 = \{a_1, a_2, \ldots, a_t\}$ and $W_0 = \{w_1, w_2, \ldots, w_t\}$ so that $A = A_0 \cup W_0$. Also, set $B_0 = \{b_1, b_2, \ldots, b_t\}$ and $Z_0 = \{z_1, z_2, \ldots, z_t\}$ so that $B = B_0 \cup Z_0$.

Before proceeding with the proof, we pause to make two comments. First, we have the trivial upper bound $\text{Idim}(P) \leq |A| = 10s$, so the purpose of the lemma is just to lower this upper bound down to $9s$. Second, when a family $\mathscr{F}$ of linear extensions of P is reversing, it must reverse the "vertical" matched pairs $\{(a_i, b_i) : 1 \leq i \leq t\} \cup \{(w_i, z_i) : 1 \leq i \leq t\}$ as well as the "diagonal" pairs in

$$\big(\text{Inc}_0(P) \cap (A_0 \times Z_0)\big) \cup \big(\text{Inc}_0(P) \cap (W_0 \times B_0)\big).$$

Let s_1 be the largest non-negative integer for which there are sets $C = \{i_1, \ldots, i_{s_1}\}$ and $D = \{j_1, \ldots, j_{s_1}\}$, each of which are s_1-element subsets of $\{1, 2, \ldots, t\}$ so that for each $k = 1, 2, \ldots, s_1$, $a_{i_k} \parallel z_{j_k}$ and $b_{i_k} \parallel w_{j_k}$ in P. Note that we may have $s_1 = 0$. Regardless, we show that $\text{Idim}(P) \leq 10s - s_1$. There is nothing to prove if $s_1 = 0$, so we consider the case where $s_1 > 0$. Let Q_1 be the following subposet of P:

$$Q_1 = \cup\big\{\{a_{i_k}, b_{i_k}, w_{j_k}, z_{j_k}\} : 1 \leq k \leq s_1\big\}.$$

By Lemma 2, we have $\text{Idim}(P) \leq 10s - 2s_1 + \text{Idim}(Q_1)$. To show that $\text{Idim}(P) \leq 10s - s_1$, we need only show that $\text{Idim}(Q_1) \leq s_1$. However, this follows from the fact that for every $k = 1, 2, \ldots, s_1$, there is a linear extension $L(a_{i_1}, b_{i_1}, Q_1)$ which also reverses (w_{i_1}, z_{i_1}).

If $s_1 \geq s$, then it follows that $\text{Idim}(P) \leq 9s$. So for the balance of the argument, we will assume that $s_1 < s$.

Let $E = C \cup D$. Then $|E| < 2s$, so we may assume that after a relabelling of subscripts that $E \cap \{1, 2, \ldots, 3s\} = \emptyset$. In particular, this implies that for all $i, j = 1, 2, \ldots, 3s$, if $a_i \parallel z_j$ in P, then $w_j < b_i$ in P. Also, if $w_j \parallel b_i$ in P, then $a_i < z_j$ in P. In particular, it implies that for each $i = 1, 2, \ldots, 3s$, exactly one of the following two statements applies:

1. $a_i \parallel z_i$ and $w_i < b_i$ in P.
2. $w_i \parallel b_i$ and $a_i < z_i$ in P.

Let Q_2 denote the subposet determined by all elements with subscripts (after the relabelling) in the range $1 \leq i \leq 3s$. To complete the proof of the lemma, we need only show that $\text{Idim}(Q_2) \leq 5s$.

We define an auxiliary directed graph G whose vertex set is $\{1, 2, \ldots, 3s\}$. In G we have a directed edge (i, j) when i and j are distinct integers in $\{1, 2, \ldots, 3s\}$ and one of the following two conditions applies:

1. $a_i \parallel z_j$ and $b_i \parallel w_i$ in P.
2. $b_i \parallel w_j$ and $a_i \parallel z_i$ in P.

We note that these conditions are mutually exclusive. We also note that we may have edges (i, j) and (j, i) simultaneously.

Now let M be a maximal matching in the auxiliary graph G and suppose that M consists of r edges, where $0 \leq r \leq 3s/2$. Let Q_3 be the subposet of Q_2 determined by the elements whose subscripts are in the maximal matching M. Using Lemma 2, and that $|Q_3| = 4r$, we have $\text{Idim}(Q_2) \leq 6s - 4r + \text{Idim}(Q_3)$.

We show that $\mathrm{Idim}(Q_3) \leq 3r$. There is nothing to prove if $r = 0$ and $Q_3 = \emptyset$, so we assume $r > 0$. We construct a family $\mathscr{F}_3$ of linear extensions of Q_3 by the following rule: For each edge (i, j) in the matching M, we add to $\mathscr{F}_3$ three linear extensions determined as follows:

1. If the first condition above applies, i.e. $a_i \parallel z_j$ and $b_i \parallel w_i$ in Q_2, then we add $L(a_i, z_j, Q_2)$, $L(w_i, b_i, Q_2)$ and $L(a_j, b_j, Q_2)$ to $\mathscr{F}_3$.
2. If the second condition above applies, i.e. $b_i \parallel w_j$ and $a_i \parallel z_i$ in Q_2, then we add $L(w_j, b_i, Q_2)$, $L(a_i, z_i, Q_2)$ and $L(a_j, b_j, Q_2)$ to $\mathscr{F}_3$.

We claim that $\mathscr{F}_3$ is reversing for Q_3. Indeed, if (i, j) is in M, then any incomparable pair involving a_i, a_j, b_i, or b_j are reversed, and depending on which condition is satisfied, either z_i and w_j, or z_j and w_i are reversed with all elements they are incomparable with. Now it is elementary to verify that all vertical and diagonal pairs are reversed in $\mathscr{F}_3$.

Furthermore, $|\mathscr{F}_3| = 3r$. If $r \geq s$, this implies that

$$\mathrm{Idim}(Q_2) \leq 6s - 4r + \mathrm{Idim}(Q_3) \leq (6s - 4r) + 3r = 6s - r \leq 5s.$$

Accordingly, we may assume that $r < s$.

Now let $Q_4 = Q_2 - Q_3$, i.e., Q_4 is the subposet of Q_2 determined by elements whose subscripts are *not* in the maximal matching M.

Then $\mathrm{Idim}(Q_2) \leq 4r + \mathrm{Idim}(Q_4)$. Again, we note that this inequality holds even if $r = 0$. We build a family $\mathscr{F}_4$ of linear extensions of Q_4 as follows. For each vertex i which is not covered by the maximal matching M, we add $L(a_i, z_i, Q_4)$ to $\mathscr{F}_4$ if $a_i \parallel z_i$ in Q_4. On the other hand, if $a_i < z_i$ in Q_4, then $w_i \parallel b_i$ in Q_4 and in this case, we add $L(w_i, b_i, Q_4)$ to $\mathscr{F}_4$.

We now show that $\mathscr{F}_4$ is a reversing family for Q_4. First note that the vertical pair (a_i, b_i) gets reversed, because either $L(a_i, z_i, Q_4) \in \mathscr{F}_4$ or $L(w_i, b_i, Q_4) \in \mathscr{F}_4$. Similarly the vertical pairs (w_i, b_i) are reversed. Now consider a diagonal pair $(u, v) \in \mathrm{Inc}_0(Q_4)$. There are integers i and j with $1 \leq i, j \leq 3s$, for which $u \in \{a_i, w_i\}$ and $v \in \{z_j, b_j\}$. We may assume $i \neq j$, for otherwise $L(u, v, Q_4) \in \mathscr{F}_4$. If $u = a_i$ and $v = z_j$, then the maximality of M implies that G has neither an (i, j) nor a (j, i) edge, so $a_i \parallel z_j$ implies $w_i < b_i$, and if also $a_i < z_i$, then $\{w_i, b_i, a_i, z_i\}$ would form an S_2. Hence $a_i \parallel z_i$ in Q_4. (Symmetric argument shows $a_j \parallel z_j$ in Q_4.) So (u, v) is reversed in $L(a_i, z_i, Q_4)$ and in $L(a_j, z_j, Q_4)$, both of which belong to $\mathscr{F}_4$. The case, when $u = w_i$ and $v = b_j$ is handled similarly.

This completes the proof that $\mathscr{F}_4$ is a reversing family for Q_4. Furthermore, it is clear that $|\mathscr{F}_4| = 3s - 2r$, so that

$$\mathrm{Idim}(Q_2) \leq 4r + \mathrm{Idim}(Q_4) \leq 4r + (3s - 2r) = 3s + 2r < 5s.$$

This completes the proof.

As we bring this subsection to a close, we remind the reader that we will no longer be restricting our attention to bipartite posets.

2.3 A New Inequality

The following lemma will be an essential tool for reducing the problem to posets of height 2. Its formulation was motivated entirely by the problem at hand; however, the ideas behind the proof are a relatively straightforward extension of techniques first introduced by Trotter and Monroe in [30].

Lemma 5 *Let A be a maximal antichain in a poset P which is not an antichain. If $X = D(A)$ and $Y = U(A)$ are both antichains, $|X| = s$ and $|Y| = s + t$ where $s, t \geq 0$, then* $\dim(P) \leq 1 + t + \lceil 4s/3 \rceil$.

Proof We argue by contradiction. First, we assume the lemma is false, and let P be a counterexample with $|P|$ as small as possible. Suppose first that $s = |X| = 0$. Then $A = \mathrm{Min}(P)$. Furthermore, since P is not an antichain, $U(A) \neq \emptyset$. Then, by Theorem 5(3), $\dim(P) \leq 1 + \mathrm{width}(Y) \leq 1 + t$. The contradiction shows that $s > 0$.

Now suppose that $t > 0$. Let $y \in Y$ and consider the poset $Q = P - \{y\}$. Since P is a minimum size counterexample, we know that $\dim(Q) \leq t + \lceil 4s/3 \rceil$, but this implies that $\dim(P) \leq 1 + t + \lceil 4s/3 \rceil$. The contradiction forces $t = 0$.

Now suppose $s = 1$. Then $|P - A| = 2$ so by Theorem 5(4), $\dim(P) \leq 2 < 1 + \lceil 4/3 \rceil = 3$. The contradiction shows $s > 1$. Now suppose $s = 2$. Then $|P - A| = 4$, so $\dim(P) \leq 4 = 1 + \lceil 8/3 \rceil$. The contradiction shows $s \geq 3$.

Next, we observe that we must have $x < y$ in P for all $x \in X$ and $y \in Y$. For if there was an incomparable pair $(x, y) \in X \times Y$, we could remove x and y and decrease the dimension by at most 1. This would again produce a counterexample of smaller size.

We now attempt to construct a realizer $\mathscr{R}$ of P with $|\mathscr{R}| = 1 + \lceil 4s/3 \rceil$. We first consider the case where $s \equiv 0 \mod 3$, as the other two residue classes are easy modifications of this base case. Furthermore, the first non-trivial case is $s = 3$.

Label the elements of X as $\{x_1, \ldots, x_s\}$ and the elements of Y as $\{y_1, \ldots, y_s\}$. Set $r = s/3$. For each $j = 1, 2, \ldots, r$, we construct four linear extensions L_{4j-3}, L_{4j-2}, L_{4j-1} and L_{4j}. These four extensions will focus on the elements of

$$\{x_{3j-2}, x_{3j-1}, x_{3j}\} \cup \{y_{3j-2}, y_{3j-1}, y_{3j}\}.$$

In each of the four extensions, x_{3j-2}, x_{3j-1} and x_{3j} will be the three highest elements of X. Also, y_{3j-2}, y_{3j-1} and y_{3j} will be the three lowest elements of Y. Furthermore, the restriction of the four extensions to these six elements will be:

$$\begin{aligned}
&x_{3j-1} < x_{3j} < x_{3j-2} < y_{3j-2} < y_{3j-1} < y_{3j} \quad \text{in} \quad L_{4j-3}.\\
&x_{3j} < x_{3j-1} < x_{3j-2} < y_{3j-1} < y_{3j-2} < y_{3j} \quad \text{in} \quad L_{4j-2}.\\
&x_{3j-2} < x_{3j} < x_{3j-1} < y_{3j} < y_{3j-2} < y_{3j-1} \quad \text{in} \quad L_{4j-1}.\\
&x_{3j-2} < x_{3j-1} < x_{3j} < y_{3j} < y_{3j-1} < y_{3j-2} \quad \text{in} \quad L_{4j}.
\end{aligned}$$

In each of these four extensions, we have seven places (blocks) into which elements of A can be placed. But recall that our goal is to reverse pairs of the form (a, y) where $a \in A$ and $y \in \{y_{3j-2}, y_{3j-1}, y_{3j}\}$ as well as pairs of the form (x, a) where $x \in$

$\{x_{3j-2}, x_{3j-1}, x_{3j}\}$. So in the discussion to follow, when we refer to an incomparable pair (a, y), we intend that $a \in A$ and $y \in \{y_{3j-2}, y_{3j-1}, y_{3j}\}$. An analogous remark applies to pairs (x, a).

First, let $a \in A$ and suppose that by placing a in the highest possible block in L_{4j-3}, we have succeeded in reversing all pairs (if any) of the form (a, y). Then a can be pushed down into the lowest possible blocks in L_{4j-2}, L_{4j-1} and L_{4j} and we will certainly reverse all pairs of the form (x, a). An analogous statement holds if we could put a in the highest possible block in L_{4j-2} and reverse all pairs (if any) of the form (a, y).

Dually, suppose that by placing a as low as possible in L_{4j}, we have succeeded in reversing all pairs (x, a). Then we could push a up in L_{4j-3}, L_{4j-2} and L_{4j-1} and we will have certainly reversed all pairs of the form (a, y). An analogous statement holds if we could put a in the lowest possible block in L_{4j-1} and reverse all pairs (if any) of the form (x, a).

There are three cases left to consider:

1. $a \parallel y_{3j-2}$ in P, $a \parallel y_{3j}$ in P and $a < y_{3j-1}$ in P.
2. $a \parallel y_{3j-1}$ in P, $a \parallel y_{3j}$ in P and $a < y_{3j-2}$ in P.
3. $a < y_{3j-2}$ in P, $a < y_{3j-1}$ in P and $a \parallel y_{3j}$ in P.

In the first case, we push a up in L_{4j-1} and down in the other three linear extensions in our group. (Note that we can not have $a \parallel x_{3j-1}$ and $a > x_{3j-2}$, because otherwise we would have pushed down a in L_{4j-1} or L_{4j} as explained above. So $a \parallel x_{3j-1}$ implies $a \parallel x_{3j-2}$ and so it will be pushed under x_{3j-1} in L_{4j-2}.) In the second, we push a up in L_{4j} and down in the other three. Finally, in the third case, we push a up in L_{4j} and down in the other three.

These remarks complete the proof in the case when $s \equiv 0 \mod 3$. When $s \equiv 1 \mod 3$ and $s = 3r + 1$, we note that $\lceil 4s/3 \rceil = 4r + 2$. So to the family constructed above, we add two additional linear extensions each having x_s as the highest element of X and y_s as the lowest element of Y. Now, elements of A are pushed down in the first of the two new linear extensions and pushed up in the second.

When $s \equiv 2 \mod 3$ and $s = 3r + 2$, we note that $\lceil 4s/3 \rceil = 4r + 3$. So to the family of size $3r$ constructed above, we add three additional linear extensions. Each has x_{s-1} and x_s as the highest elements of X and y_{s-1} and y_s as the lowest elements of Y. The first two have $x_{s-1} < x_s$ while the third has $x_s < x_{s-1}$. However, only the first has $y_{s-1} < y_s$ with $y_s < y_{s-1}$ in both the second and the third. It is easy to see how to appropriately position elements of A in these three new linear extensions, and these observations complete the proof of the lemma.

We comment that when $s \geq 1$, we can actually prove that $\dim(P) \leq t + \lceil 4s/3 \rceil$. We do not include the proof as the technical details are formidable, and the minor improvement is not central to the results of this paper. However, when $t = 0$, the resulting inequality $\dim(P) \leq \lceil 4s/3 \rceil$ is tight, as evidenced by examples constructed in [30].

3 Proof of the Main Theorem

For the readers convenience, we restate here the theorem we are about to prove:

Theorem *For every positive integer c, there is an integer $f(c) = O(c^2)$ so that if $n > 10f(c)$ and P is a poset with $|P| \leq 2n+1$ and $\dim(P) \geq n-c$, then P contains a standard example S_d with $d \geq n - f(c)$.*

Proof Let c be a positive integer. Then set $s = 41c$, $t = 5s$, and $f(c) = 17ct$. We note that $f(c) = 3485c^2$.

Let $n > 10f(c)$ (this bound is generous) and let P be a poset with $|P| \leq 2n + 1$ and $\dim(P) \geq n - c$. We will show that P contains a standard example S_d with $d \geq n - f(c)$. Clearly, we may assume that $|P| = 2n + 1$, as otherwise we can just add loose points.

Let A be a maximum antichain in P. Since $n-c \leq \dim(P) \leq \text{width}(P)$, we know $|A| \geq n - c$. Since $\dim(P) \leq |P - A|$, we also know that $|A| \leq n + c + 1$. Let $D = P - U(A)$. By Theorem 5,

$$\dim(P) \leq \dim(D) + \text{width}(U(A)) \leq 1 + \text{width}(D(A)) + \text{width}(U(A)),$$

so we may choose antichains $X \subseteq D(A)$ and $Y \subseteq U(A)$ so that $|X|+|Y| = n-c-1$. We observe that since A is a maximal antichain in P, in the subposet $P_0 = A \cup X \cup Y$, $X = D_{P_0}(A)$ and $Y = U_{P_0}(A)$.

Without loss of generality, we may assume that $|X| \leq |Y|$. Set $\sigma = |X|$ and $|Y| = \sigma+\tau$ where $\tau \geq 0$. From Lemma 5, we know that $\dim(P_0) \leq 1+\tau+\lceil 4\sigma/3\rceil \leq 2 + \tau + 4\sigma/3$. On the other hand, since $|A| \geq n - c$, we know that there are at most $2c+2$ points of P which do not belong to P_0. Therefore, $\dim(P_0) \geq n-c-(2c+2) = n - (3c + 2)$. Since $n - c - 1 = |X| + |Y| = 2\sigma + \tau$, so that $n = c + 1 + 2\sigma + \tau$, it follows that

$$(c + 1 + 2\sigma + \tau) - (3c + 2) \leq \dim(P_0) \leq 2 + \tau + 4\sigma/3.$$

This implies that $2\sigma/3 \leq 2c + 3$, so that $\sigma \leq 3c + 4$.

We now focus on the bipartite poset $P_1 = A \cup Y$. Since $\dim(P_0) \geq n - (3c + 2)$ and $\sigma \leq 3c + 4$, we know $\dim(P_1) \geq n - (6c + 6)$.

In order to be consistent with the material developed in the preceding section for bipartite posets, we relabel the set Y as B and reuse (in the computer science tradition) the symbol P for the bipartite poset $A \cup B$. With its updated definition, we know $\dim(P) \geq n - (6c + 6)$, so that $\text{Idim}(P) \geq n - (6c + 7) \geq n - 13c$.

Now let d be the size of the largest standard example contained in P. Choose a copy T of S_d in P with minimal elements $A_0 = \{a_1, a_2, \ldots, a_d\} \subseteq A$ and maximal elements $B_0 = \{b_1, b_2, \ldots, b_d\} \subseteq B$. Of course, we also intend that $a_i \parallel b_i$ in P for $i = 1, 2, \ldots, d$. If $d \geq n - f(c)$, the conclusion of the theorem has been established. So we will assume that $d < n - f(c)$ and argue to a contradiction.

Since $f(c) + 1 \leq n - d$ and $d + |A - A_0| = |A| \geq n - c$, we see that $|A - A_0| \geq f(c)-c+1$. Let $A-A_0 = A_1 \cup A_2 \cup \cdots \cup A_{16c}$ be any partition of $A-A_0$ into parts as equal in size as division will allow. For each $i = 1, 2, \ldots, 16c$, let P_i now denote the bipartite subposet $A_i \cup (B - B_0)$. If $\text{Idim}(P_i) < |A_i|$ for each $i = 1, 2, \ldots, 16c$, then $\text{Idim}(P-T) \leq |A-A_0|-16c$. Since $|A| = d+|A-A_0|$ and $|A| \leq n+c+1 \leq n+2c$, using Lemma 2 we get $\text{Idim}(P) \leq n - 14c$, which is false.

After a relabelling, we may assume that $\mathrm{Idim}(P_1) = |A_1|$. Therefore, by Lemma 3, $P - T$ contains the standard example whose dimension is $|A_1|$. We know that $|A_1| \geq \lfloor (f(c) - c + 1)/16c \rfloor$, so we can safely say $|A_1| \geq f(c)/17c = t$. Note also that $t = 5s$. Choose a copy T' of S_t contained in $P - T$ and label the elements of T' as $W_0 = \{w_1, w_2, \ldots, w_t\}$ and $Z_0 = \{z_1, z_2, \ldots, z_t\}$ so that for all $i, j = 1, 2, \ldots, t$, $w_i < z_j$ in P if and only if $i \neq j$.

We associate with the bipartite subposet $(A_0 \cup B_0) \cup (W_0 \cup Z_0)$ an auxiliary graph G which is a bipartite graph. The graph G has vertex set $U \cup V$, where $U = \{u_1, \ldots, u_t\}$ and $V = \{v_1, \ldots, v_d\}$. In G, we have an edge $u_i v_j$ when the subposet $\{w_i, z_i, a_j, b_j\}$ is *not* the standard example S_2.

Claim 1 *In the bipartite graph G, there is a graph matching from U to V, i.e., there is a* 1–1 *function $g : U \to V$ so that $g(u)$ is a neighbor of u for every $u \in U$.*

Proof We use Hall's theorem. For each subset $S \subseteq U$, let $N_G(S)$ be the subset of V consisting of all vertices in V adjacent in G to one or more vertices in S. If the claim is false, then there is a set $S \subseteq U$ with $|S| > |N_G(S)|$. However, if we remove from T all pairs of the form $\{a_i, b_i\}$ with $v_i \in N(S)$ and replace them with the pairs $\{w_j, z_j\}$ with $u_j \in S$, we obtain a standard example whose dimension is $d - |N(S)| + |S|$ which is larger than d. The contradiction completes the proof of the claim.

Without loss of generality, we may assume that the pairs in $A_0 \cup B_0$ have been labelled so that $g(u_i) = v_i$ for all $i = 1, 2, \ldots, t$, i.e., the subposet $\{a_i, b_i, w_i, z_i\}$ is not S_2. Then let Q denote the bipartite poset consisting of all elements of $(A_0 \cup W_0) \cup (B_0 \cup Z_0)$ with subscripts at most t. Then let q be the largest integer so that there is a balanced subposet Q' of $P - Q$ with $|Q'| = 2q$ so that Q' admits a matching. Then let P' be the bipartite poset formed by $Q \cup Q'$, and note that $Q \cup Q'$ is a maximal matching. Using Lemma 2, we have $n - 13c \leq \mathrm{Idim}(P) \leq 2t + q$, so then $(2n + 1) - (4t + 2q) \leq 26c + 1 \leq 27c$, and so we conclude that there are at most $27c$ points of P which do not belong to P'. It follows that $\mathrm{Idim}(P') \geq (n - 13c) - 27c = n - 40c$.

On the other hand, $\mathrm{Idim}(P') \leq q + \mathrm{Idim}(Q)$. Furthermore, since $t = 5s$, we know from Lemma 4 that $\mathrm{Idim}(Q) \leq 9s = 2t - s$. It follows that

$$(2t + q) - 40c \leq n - 40c \leq \mathrm{Idim}(P') \leq q + (2t - s).$$

This implies that $s \leq 40c$ which is false, since $s = 41c$. The contradiction completes the proof.

4 Fractional Dimension

The concept of *fractional dimension* was introduced by Brightwell and Scheinerman in [7], but we elect to use the alternative formulation of this parameter given by Biró et al. in [2]. Let $\{L_1, L_2, \ldots, L_t\}$ be the family of *all* linear extensions of a poset P. Then the fractional dimension of P, denoted $\dim^*(P)$ (some authors use the notation $\mathrm{fdim}(P)$), is the least positive real number d for which there are non-negative real numbers $\{\alpha_i : 1 \leq i \leq t\}$ so that (1) $\sum_{i=1}^{t} \alpha_i = d$; and (2) for every pair (x, y) with $x \parallel y$ in P, $\sum\{\alpha_i : 1 \leq i \leq t, x > y \text{ in } L_i\} \geq 1$.

For fractional dimension, we have the following inequalities, all due to Brightwell and Scheinerman [7].

Theorem 6 *Let P be poset. Then the following inequalities hold.*

1. $\dim^*(P) \leq \dim(P)$.
2. *If* $x \in P$, *then* $\dim^*(P) \leq 1 + \dim^*(P - \{x\})$.
3. *If* $a \in \mathrm{Min}(P), b \in \mathrm{Max}(P)$ *and* $a \parallel b$ *in* P, *then* $\dim^*(P) \leq 1+\dim^*(P-\{a, b\})$.

For a bipartite poset $P = A \cup B$, there is a natural fractional dimension analogue of the inequality $\dim(P) \leq 1 + \mathrm{Idim}(P)$. We let $\mathrm{Idim}^*(P)$ be the least d so that there are non-negative real numbers $\{\alpha_i : 1 \leq i \leq t\}$ so that (1) $\sum_{i=1}^{t} \alpha_i = d$; and (2) for every pair $(a, b) \in A \times B$ with $a \parallel b$ in P, $\sum\{\alpha_i : 1 \leq i \leq t, a > b \text{ in } L_i\} \geq 1$. Clearly $\mathrm{Idim}^*(P) \leq \mathrm{Idim}(P)$; in fact $\mathrm{Idim}^*(P)$ may be zero.

Proposition 6 *For a bipartite poset $P = A \cup B$*, $\dim^*(P) \leq 2 + \mathrm{Idim}^*(P)$.

Proof Let $\{\alpha_i : 1 \leq i \leq t\}$ be a set of non-negative weights witnessing the value of $\mathrm{Idim}^*(P)$. Then let L and L' be linear extensions of P with $A < B$ in L, $A < B$ in L', $L(A)$ is the dual of $L'(A)$ and $L(B)$ is the dual of $L'(B)$. Then increase the weights of L and L' by 1. The resulting values show $\dim^*(P) \leq \mathrm{Idim}^*(P) + 2$.

We will need the following trivial consequence of Lemma 2.

Lemma 6 *Let $P = A \cup B$ be a bipartite poset, and let Q be a maximal matching with m minimal and m maximal elements in P. Then* $\mathrm{Idim}^*(P) \leq m$.

A trivial consequence of Lemma 5 is the following version for fractional dimension.

Lemma 7 *Let A be a maximal antichain in a poset P which is not an antichain. If $X = D(A)$ and $Y = U(A)$ are antichains, $|X| = s$ and $|Y| = s + t$ where $t \geq 0$, then* $\dim^*(P) \leq 1 + t + \lceil 4s/3 \rceil$.

We next turn our attention to developing analogous versions of Theorem 2 for fractional dimension. We start with the bipartite version.

Theorem 7 *For every positive integer c, if $n > 10(5c + 12)$, and $P = A \cup B$ is a bipartite poset with $|P| \leq 2n + 1$ and* $\dim^*(P) \geq n - c$, *then P contains a standard example S_d with $d \geq n - (5c + 12)$.*

Proof We will assume $|P| = 2n + 1$. Otherwise add loose points which cannot decrease the fractional dimension. In presenting the proof, we will find it convenient to use graph theoretic terminology for the bipartite graph G whose vertex set is $A \cup B$ with G containing an edge (a, b) when $(a, b) \in A \times B$ and $a \parallel b$ in P. In particular, paths and cycles in G will play an important role in our proof.

Next, we will identify a set of linear extensions of P and assign positive weights to these extensions. All other linear extensions will be assigned weight 0.

First, if G is acyclic, set $s = 0$ and $Q_1 = \emptyset$. If G is not acyclic, let s be the largest integer for which there is a balanced subposet Q_1 of P so that $|Q_1| = 2s$ and Q_1 is the union of disjoint cycles. Note that we do not require that the cycles be induced. For each edge (a, b) which is one of the edges on one of the cycles, we choose a linear extension $L(a, b, P)$ reversing a with B and A with b and assign it weight $1/3$.

We note that $P - Q_1$ is acyclic. If $P - Q_1$ does not contain a path on 4 vertices, we set $r = 0$ and $Q_2 = \emptyset$; otherwise, let r be the largest integer for which there is a subposet Q_2 of $P - Q_1$ so that $|Q_2| = 4r$ and Q_2 has a matching $\{u_1, u_2, \dots, u_{2r}\} \cup \{v_1, v_2, \dots, v_{2r}\}$ so that for each $i = 1, 2, \dots, r, u_{2i-1} \parallel v_{2i}$ in P. Note that the maximality of s implies that $u_{2i} < v_{2i-1}$ in P for each $i = 1, 2, \dots, r$. As before, for each $i = 1, 2, \dots, r$, we choose three linear extensions $L(u_{2i-1}, v_{2i-1}, P)$, $L(u_{2i}, v_{2i}, P)$ and $L(u_{2i-1}, v_{2i}, P)$, but now we assign weight $1/2$ to each of them.

If there are no edges in $P - (Q_1 \cup Q_2)$, set $d = 0$ and $Q_3 = \emptyset$; otherwise let d be the largest positive integer for which there is a balanced $2d$-element subposet Q_3 in $P - (Q_1 \cup Q_2)$ with a matching $\{a_1, a_2, \dots, a_d\} \cup \{b_1, b_2, \dots, b_d\}$. In this case, when $d \geq 2$, we note that Q_3 is the standard example S_d. For each $i = 1, 2, \dots, d$, we choose a linear extension $L(a_i, b_i, P)$ and assign it weight 1.

Set $Q_4 = P - (Q_1 \cup Q_2 \cup Q_3)$ and let $q = |Q_4|$. We note that if $(a, b) \in A \times B$, with $a, b \in Q_4$, $a < b$ in P. If $a \in Q_4 \cap A$, choose a linear extension $L(a, B, P)$ and assign it weight $1/2$. Similarly, for each $b \in Q_4 \cap B$, choose a linear extension $L(A, b, P)$ and assign it weight $1/2$.

Let $(a, b) \in A \times B$ be an incomparable pair, and let w be the sum of weights of linear extensions in which (a, b) is reversed. If $a \in Q_3$ or $b \in Q_3$, then $w \geq 1$. If $a, b \in Q_1$, then $w \geq 1/3 + 1/3 + 1/3$. If $a \in Q_1$ or $b \in Q_1$, but not both, then $w \geq 1/3 + 1/3 + 1/2$. In all other cases, we have $a, b \in Q_2 \cup Q_4$, and then $w \geq 1/2 + 1/2$. In all cases $w \geq 1$.

Let t denote the sum of all the weights we have assigned. The argument above shows that $\mathrm{Idim}^*(P) \leq t$.

It follows that:

$$n - (c + 2) \leq \mathrm{Idim}^*(P) \leq t = 2s/3 + 3r/2 + d + q/2. \tag{1}$$

Recall that $2n + 1 = |P| = |Q_1| + |Q_2| + |Q_3| + |Q_4| = 2s + 4r + 2d + q$, so by the previous inequality,

$$2s + 4r + 2d + q - 1 - 2(c + 2) = 2n - 2(c + 2) \leq 4s/3 + 3r + 2d + q,$$

hence $2s/3 + r \leq 2c + 5$.

Notice that $Q_1 \cup Q_2 \cup Q_3$ admits a maximal matching, so by Lemma 6, we get

$$n - (c + 2) \leq \mathrm{Idim}^*(P) \leq s + 2r + d.$$

Similarly as above, from this we conclude $q \leq 2c + 5$.

We return to (1), and rewrite it to get

$$d \geq n - (c + 2) - (2s/3 + 3r/2 + q/2). \tag{2}$$

Considering the previously proven inequalities $2s/3 + r \leq 2c + 5$ and $q \leq 2c + 5$, inequality (2) is weakest when $s = 0$, $r = 2c + 5$ and $q = 2c + 5$. With these values, it becomes:

$$d \geq n - (c + 2) - (4c + 10) = n - (5c + 12).$$

Next, we present the analogous version for general posets.

Theorem 8 *For every positive integer* c, *if* $n > 10(30c + 52)$ *and* P *is any poset with* $|P| \leq 2n + 1$ *and* $\dim^*(P) \geq n - c$, *then* P *contains a standard example* S_d *with* $d \geq n - (30c + 52)$.

Proof Using the inequalities in Theorem 6 and following along lines from the proof of Theorem 2, we first obtain a subposet P_0 consisting of an antichain A, which is maximum in P, and two other antichains $X \subseteq D(A)$ and $Y \subseteq U(A)$ with $\dim^*(P_0) \geq n - (3c + 2)$. We now use the inequality $\dim^*(P_0) \leq 2 + t + 4s/3$ to conclude that $s \leq 3c + 4$. This implies the bipartite subposet $A \cup Y$ has fractional dimension at least $n - (6c + 6)$. After the relabelling, we have a bipartite poset $P = A \cup B$ with $\mathrm{Idim}^*(P) \geq n - (6c + 8)$.

From the preceding proof, we then conclude that P contains a standard example S_d with

$$d \geq n - (5(6c + 8) + 12) = n - (30c + 52).$$

5 Closing Remarks

As commented previously, our upper bound on $f(c)$ in Theorem 2 shows that $f(c) = O(c^2)$. For a lower bound, consider the poset P associated with a finite projective plane of order q, as discussed in Example 1 in Sect. 1. Then let m be an integer which is large relative to q. Form a poset Q by adding $2m$ new points to P. The new points form a standard example S_m. In Q, all minimal elements of S_m are less than all maximal elements of P, and all maximal elements of S_m are greater than all minimal elements of P. Set $n = m + (q^2 + q + 1)$ and $c = q^{3/2}$. Then $\dim(Q) \geq n - c$. However, Q does not contain a standard example S_d with $d \geq m + q^{3/2} + 2$. Since $q^2 = c^{4/3}$, it follows that $f(c) = \Omega(c^{4/3})$.

With additional work, it is quite possible that the bounds on $f(c)$ may be tightened. Of course, in the fractional dimension setting, it is quite possible that with further effort, the exact answer can be found, especially in the bipartite case.

There are some more modest problems associated with the details of our proofs. One of them is the inequality for bipartite posets: $\dim^*(P) \leq 2 + \mathrm{Idim}^*(P)$. Is there a constant $q < 2$ so that one always has $\dim^*(P) \leq q + \mathrm{Idim}^*(P)$? We tend to believe that this holds when $q = 4/3$. A second problem concerns the inequality of Lemma 4. It is quite possible that this inequality may be strengthened.

A third problem is find the best possible bound in Lemma 5. It is not too difficult to show that the dimension of P is at most $t + s$, when t is sufficiently large compared to s, so the real problem is to find the maximum dimension when t is bounded as a function of s.

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