CONGRESSUS NUMERANTIUM XIX

Proceedings of the Eighth Southeastern Conference On Combinatorics, Graph Theory, And Computing

Louisiana State University
Baton Rouge
February 28 - March 3, 1977

SOME COMBINATORIAL PROBLEMS FOR PERMUTATIONS

William T. Trotter, Jr.

Department of Mathematics and Computer Science

University of South Carolina

Columbia, South Carolina 29208

1. Introduction

For integers m,k with $m \geq k \geq 2$, Dushnik defined N(m,k) as the least positive integer t for which there exist permutations $\sigma_1, \sigma_2, \ldots \sigma_t$ of $\{1,2,\ldots,m\}$ so that for every k-element subset $A \subset \{1,2,\ldots,m\}$ and each $a_o \in A$, there is at least one i for which $\sigma_i(a_o) < \sigma_i(a)$ for all $a \in A$ with $a \neq a_o$. The permutations $\sigma_1, \sigma_2, \ldots, \sigma_t$ are said to k-filter $\{1,2,\ldots,m\}$. Dushnik's interest in the computation of N(m,k) came from the fact that N(m,k) is the dimension of the partially ordered set consisting of all one element and k - 1 element subsets of an m element set ordered by inclusion [1]. Dushnik derived two inequalities for N(m,k) from which it is possible to determine N(m,k) exactly when k is relatively large compared to m. Spencer used a probabilistic argument to obtain inequalities for N(m,k) for fixed k with m large [2]. In this paper, we will concentrate on determining N(m,k) when both m and k are relatively small. In doing so, we will obtain modest improvements in the results of Dushnik and Spencer for large values of m and k.

2. Spencer's Inequalities

We define f(k,t) for integers $k \geq 3$ and $t \geq 1$ as the

PROC. 8TH S-E CONF. COMBINATORICS, GRAPH
THEORY, AND COMPUTING, pp. 619-632.

largest integer m for which there exist permutations $\sigma_1, \sigma_2, \ldots \sigma_t$ which k-filter $\{1,2,\ldots,m\}$. The Erdös-Szekeres theorem states that any two permutations of $n^2 + 1$ integers have a monotonic subsequence of size $n + 1$. It follows by induction that $f(k,t) \leq 2^{2^{t-1}}$ and $N(m,k) \geq 1 + \log_2\log_2 m$ for each $k \geq 3$. Spencer [2] proved that $f(3,2t) > 2^{\binom{2t-1}{t}}$ and $f(3,2t+1) \geq 2^{\binom{2t}{t-1}}$ and therefore $N(m,3) < \log_2\log_2 m + \frac{1}{2}\log_2\log_2\log_2 m + \log_2(\sqrt{2\pi}) + (1)$. However, an examination of the construction used by Spencer to produce the inequality for $f(3,2t)$ reveals that for each pair of integers $i_1, i_2 \in \{1,2,\ldots,m\}$ where $m = 2^{\binom{2t-1}{t}}$, there are exactly t permutations in which i_1 precedes i_2 and exactly t permutations in which i_2 precedes i_1. In particular Spencer's construction produces 4 permutations $\sigma_1,\sigma_2,\sigma_3,\sigma_4$, which 3-filter (after relabeling) $\{5,6,\ldots,12\}$ so that for each pair $i_1, i_2 \in \{5,6,\ldots,12\}$ there are two permutations in which i_1 precedes i_2. We now extend $\sigma_1,\sigma_2,\sigma_3,\sigma_4$ to permutations of $\{1,2,\ldots,12\}$ by letting i be the first element in σ_i and $\{1,2,3,4\} - \{i\}$ the last 3-elements (the ordering is irrelevant) in σ_i for $i = 1,2,3,4$. We now show that $\sigma_1,\sigma_2,\sigma_3,\sigma_4$ 3-filter $\{1,2,3,\ldots,12\}$. Consider a 3-element set A with distinguished element $a_o \in A$. If $a_o \in T = \{1,2,3,4\}$, there is nothing to show. Now suppose $a_o \in M = \{5,6,\ldots,12\}$. If $A - \{a_o\} \subset T$, say $A - \{a_o\} = \{i_1,i_2\}$, then we may choose $i_3 \in \{1,2,3,4\} - \{i_1,i_2\}$ and thus a_o precedes i_1 and i_2 in σ_{i_3}. If $A - \{a_o\} \subset M$, then Spencer's construction applies. Finally, suppose

$A - \{a_o\} = \{i_1,i_2\}$ where $i_1 \in T$ and $i_2 \in M$. Then there are two permutations in which a_o precedes i_2 and at least one of these is not σ_{i_1}. In this permutation, a_o precedes i_1 and i_2.

Theorem 1: $f(3,4) = 12$.

Proof: The argument given above shows that $f(3,4) \geq 12$. We now show that $f(3,4) \leq 12$. Suppose to the contrary that we have permutations $\sigma_1,\sigma_2,\sigma_3,\sigma_4$ which 3-filter $\{1,2,3,\ldots,13\}$. Without loss of generality [1], we may assume that i is the first integer of σ_i and $\{1,2,3,4\} - \{i\}$ are the last three elements of σ_i. i.e. $\sigma_i(i) = 1$ and $\sigma_i^{-1}\{11,12,13\} = \{1,2,3,4\} - \{i\}$.

Let $T = \{1,2,3,4\}$ and $M = \{5,6,7,\ldots,13\}$. For each distinct pair $m_1, m_2 \in M$, there must be exactly two permutations in which m_1 precedes m_2 for if m_2 precedes m_1 in every permutation except possibly σ_i, then we cannot handle $\{\underline{m}_1, i, m_2\}$, i.e. there is no permutation in which m_1 precedes both i and m_2.

Now let $x \in M$, $A \subset M$, and suppose that x precedes every element of A in two of the permutations σ_{i_1} and σ_{i_2}. Then the other two permutations σ_{i_3} and σ_{i_4} are dual on M. Furthermore σ_{i_1} and σ_{i_2} are also dual on M. For if $m_1, m_2 \in M$ and m_1 precedes m_2 in σ_{i_3} and σ_{i_4}, then we cannot handle $\{\underline{m}_2, m_1, x\}$. Once we know that σ_{i_3} and σ_{i_4} are dual on M, the argument given in the preceding paragraph implies that σ_{i_1} and σ_{i_2} are also dual on M.

Now let $x \in M$, $A \subset M$, and suppose that x precedes every element of A in two of the permutations σ_{i_1} and σ_{i_2}. Suppose further that $|A| \geq 5$. Then the Erdös-Szekeres theorem implies that there exists a 3-element subset $\{m_1,m_2,m_3\} \subset A$ on which σ_{i_2} and σ_{i_3} are monotonic, i.e. σ_{i_2} and σ_{i_3} either order these three elements in the same order or in dual order. Hence we may assume that m_2 is between m_1 and m_3 in both σ_{i_2} and σ_{i_3}. But since σ_{i_1} and σ_{i_2} are dual on A and σ_{i_3} and σ_{i_4} are also dual on A, we may conclude that m_2 is between m_1 and m_3 in all four permutations which implies that we cannot handle $\{\underline{m_2},m_1,m_3\}$. The contradiction allows us to conclude that if $\sigma_i^{-1}(2) = x \in M$, then $\sigma_j(x) \geq 6$ for $j \in \{1,2,3,4\} - \{i\}$. Therefore, we may assume without loss of generality that $\sigma_i^{-1}(2) = 4 + i$ for each $i = 1,2,3,4$.

Similarly, if $\sigma_i^{-1}(3) = x$, then $\sigma_j(x) \geq 5$ for $j \in \{1,2,3,4\} - \{i\}$ and we may assume without loss of generality that $\sigma_i^{-1}(3) = 8 + i$ for each $i = 1,2,3,4$. Then the first 3 positions of the permutations are:

σ_1: $[1,5,9]$

σ_2: $[2,6,10]$

σ_3: $[3,7,11]$

σ_4: $[4,8,12]$

Then without loss of generality we may assume that 13 precedes 11 and 12 in σ_1 and that 13 precedes 9 and 10 in σ_3. We may then assume that 13 precedes 9 and 11 in σ_2. Similarly we then assume without loss of generality that 13 precedes 10 and 12 in σ_1. It follows that 9,10, and 11 precede 13 in σ_4.

We then conclude that 13 precedes 10 and 5 in σ_3, 13 precedes 9 and 6 in σ_3, 13 precedes 11 and 5 in σ_2, and 13 precedes 11 and 6 in σ_2. Thus 9 precedes 5 and 13 in σ_4, 10 precedes 6 and 13 in σ_4, and both 5 and 6 precede 13 in σ_4.

Similarly 13 precedes 6 and 7 in σ_1 and 13 precedes 5 and 7 in σ_2. Therefore, 11 precedes 7 and 13 in σ_4 and 7 precedes 13 in σ_4.

Then without loss of generality, we may assume 12 precedes 13 in σ_2 and 13 precedes 12 in σ_3. Thus 11 precedes 12 in σ_1 and 9 precedes 12 in σ_3. Also, we must have that 11 precedes 7 and 12 in σ_1 and that 9 precedes 5 and 12 in σ_3. Therefore, 5 precedes 9 in σ_2 and 7 precedes 11 in σ_2.

Finally, we note that 9 must precede 5 and 11 in σ_4 and 11 must precede 7 and 9 in σ_4 which is a contradiction and completes the proof of our theorem.

The role of the Erdös-Szekeres theorem in Spencer's inequality and in the preceding theorem suggests the following modified problem. For integers $k \geq 3$, $t \geq 2$, we define $f^*(k,t)$ as the largest positive integer m for there exist t permutations $\sigma_1,\sigma_2,\ldots,\sigma_t$ which k-filter $\{1,2,\ldots,m\}$ with the additional requirement that $\sigma_1 = \hat{\sigma}_2$, i.e. σ_1 and σ_2 are dual. For example $f(3,3) = 4$ but $f^*(3,3) = 3$.

<u>Theorem 2</u>: $f(3,4) = 6$.

Proof: Consider the permutations $\sigma_1,\sigma_2,\sigma_3$, and σ_4 of $\{1,2,3,4,5,6\}$ defined as follows:

σ_1: [1,2,3,4,5,6]
σ_2: [6,5,4,3,2,1]
σ_3: [3,2,5,4,6,1]
σ_4: [4,5,2,3,6,1]

These permutations 3-filter {1,2,3,...,6} and $\sigma_1 = \hat{\sigma}_2$ so $f^*(3,4) \geq 6$. Now suppose that $f^*(3,4) \geq 7$ and let $\sigma_1, \sigma_2, \sigma_3$, and σ_4 3-filter {1,2,3,...,7} with $\sigma_1 = \hat{\sigma}_2$. We assume that σ_1: [1,2,3,4,5,6,7]. Suppose $\sigma_3(2) \leq 2$; then there is a four element subset $S \subset \{3,4,5,6,7\}$ such that 2 precedes each element of S in σ_1 and σ_3. If $x,y \in S$ and x precedes y in both σ_2 and σ_4, then we do not handle {$\underline{y}$,2,x}. Therefore, we may assume that σ_4 and σ_2 are dual on S. We then restrict each of the permutations to this four element set and discard σ_4 since it is now identical to σ_1. We obtain 3 permutations which 3-filter a four element set which is not possible when σ_1 and σ_2 are dual. Therefore, we may assume that 2 and 6 are not the first or second elements in either σ_3 or σ_4. As before we assume that 1 and 7 occupy the last two positions of σ_3 and σ_4.

Now if 4 does not occupy the first position in either σ_3 or σ_4, then we cannot handle {$\underline{4}$,3,5}. Therefore, we may assume that $\sigma_3(1) = 5$ and that $\sigma_4(1) = 3$. We may assume that $\sigma_3(3) = 5$ and $\sigma_4(4) = 5$. Therefore, $\sigma_3(5) = \sigma_4(5) = 2$. But this implies that we cannot handle {$\underline{2}$,1,5} and the contradiction completes the proof of our theorem.

We can now obtain a slight improvement in the Erdös-Szekeres inequality since it follows trivially that $f(k,t) \leq f^*(k,t-1)$ $f(k,t-1)$. For example, we conclude that $f(3,5) \leq 6 \cdot 12 = 72$

where the original Erdös-Szekeres bound only provides the inequality $f(3,5) \leq 2^{2^4} = 2^{16}$. Similarly we have $f^*(k,t) \leq [f^*(k,t-1)]^2$ and thus $f^*(3,5) \leq 36$.

Theorem 3: $f^*(3,5) \leq 27$.

Proof: Suppose $f^*(3,5) \geq 28$ and let σ_1, σ_2, σ_3, σ_4, σ_5 3-filter {1,2,3,...,28} with $\sigma_i(i) = 1$ and $\sigma_2 = \hat{\sigma}_1$. We may assume without loss of generality that 1 and 28 occupy the last two positions in σ_3, σ_4, and σ_5. Now choose a six element subset of {2,3,4,...,27} which is monotonic for σ_2 and σ_3. We may then append 1 or 28 as required to obtain a 7-element subset which is monotonic for σ_2 and σ_3. The restriction of the five permutations to these seven element produces two identical permutations and thus would require that $f^*(3,4) \geq 7$. The contradiction completes the proof.

It is the author's opinion that the precise determination of $f^*(3,5)$ is a manageable problem while the problem for $f(3,5)$ is probably not. Although we will discuss such problems in more detail in the next section, it should be relatively easy to determine $f(4,7)$ and $f(5,9)$.

Furthermore, we note that the balanced nature of Spencer's construction permits the following modest improvement in the lower bound on $f(3,t)$. If we let $g(2t) = 2^{\binom{2t-1}{t}}$ and $g(2t+1) = 2^{\binom{2t}{t-1}}$ then $f(3,2^t) \geq g(2^t) + 2g(2^{t-1}-1) + 2^2 g(2^{t-2}-1)$ $+ \ldots$. This inequality is produced simply by stacking Spencer's construction in the obvious manner with blocks near the top in this stack placed in reverse order in the other permutations. Note that this is simply a generalization of the construction used in Theorem 1.

3. Dushnik's Inequalities

In [1] Dushnik proved that $N(t^2-2,2t-2) = N(t^2-1,2t-2) = t^2-t$ and $N(t^2+t-1,2t-1) = N(t^2+t,2t-1) = t^2$ for every $t \geq 2$. In this section we present some extensions of these results.

Theorem 4: $N(t^2,2t-2) = N(t^2+1,2t-2) = t^2-t$ for every $t \geq 2$.

Proof: It suffices to show that $N(t^2+1,2t-2) \leq t^2-t$ for every $t \geq 2$. The result is trivial when $t = 2$ so we assume $t \geq 3$. We now construct t^2-t permutations $\sigma_1,\sigma_2,\ldots,\sigma_{t^2-t}$ of $\{1,2,\ldots,t^2+1\}$. We will actually specify only a small number of positions of each σ_i. We begin with the standard device; we set $\sigma_i(i) = 1$ and let the last t^2-t-1 positions in σ_i be $\{1,2,\ldots,t^2-t\} - \{i\}$. Now label the elements of $M = \{t^2-t+1,t^2-t+2,\ldots,t^2+1\}$ by $\{m_1,m_2,\ldots,m_{t+1}\}$. We call these elements "middle elements". Now let m_1 and m_2 each occupy 2^{nd} position in t-1 permutations and let each of the other middle elements occupy 2^{nd} position in a block of t-2 permutations. We call m_1 and m_2 big middles and $m_3,\ldots,m_{t+1}$ little middles. Now place each little middle in 3^{rd} position in one of the permutations having m_1 in 2^{nd} position. Do the same for m_2. For each block of permutations containing the little middle element m_i in 2^{nd} position, place each of the other little middle elements in 3^{rd} position. Then put m_2 in 4^{th} position in the m_1 block. In the other permutations having m_i and m_j in 2^{nd} and 3^{rd} positions respectively, put m_1 in fourth position if $i < j$ and m_2 in fourth position if $i > j$. The remaining positions are arbitrary.

Now let A be a 2t-2 element subset of $\{1,2,\ldots,t^2+1\}$ and let $a_o \in A$. We show that there is a permutation in which a_o precedes all the remaining elements of A. Clearly we may assume $a_o \in M$. If $a_o \in \{m_3,m_4,\ldots,m_{t+1}\}$, there is no problem unless A contains the t-2 top elements of the block having a_o in 2^{nd} position and at least one element of the t disjoint pairs of elements in the first two positions of the permutations having a_o in 3^{rd} position. However, this requires A to contain $1+(t-2)+t = 2t-1$ elements.

Now suppose $a_o = m_1$. Then we may assume that A contains the t-1 elements in first position in the m_1 block. Now suppose A contains s other middle elements. Note that $t-s \geq 2$. Then there are $\frac{(t-s)(t-s-1)}{2}$ permutations having m_1 in 4^{th} position where A does not contain either of the elements in 2^{nd} and 3^{rd} position so we may assume A contains the elements in 1^{st} position in each of these permutations. Then $|A| \geq 1+(t-1) + s + \frac{(t-s)(t-s-1)}{2} \geq t + s + (t-s-1) = 2t-1$. The contradiction completes the proof of our theorem.

Example: Here are the first four positions of 12 permutations which 6-filter $\{1,2,\ldots,17\}$

σ_1: $[1,m_1,m_3,m_2]$	σ_7: $[7,m_3,m_4,m_1]$
σ_2: $[2,m_1,m_4,m_2]$	σ_8: $[8,m_3,m_5,m_1]$
σ_3: $[3,m_1,m_5,m_2]$	σ_9: $[9,m_4,m_3,m_2]$
σ_4: $[4,m_2,m_3,m_1]$	σ_{10}: $[10,m_4,m_5,m_1]$
σ_5: $[5,m_2,m_4,m_1]$	σ_{11}: $[11,m_5,m_3,m_2]$
σ_6: $[6,m_2,m_5,m_1]$	σ_{12}: $[12,m_5,m_4,m_2]$

Theorem 5: $N(t^2+t+1, 2t-1) = t^2$ for every $t \geq 2$.

Proof: the result for $t = 2$ follows from Theorem 1. We now assume that $t \geq 3$ and construct t^2 permutations $\sigma_1, \sigma_2, \ldots, \sigma_{t^2}$ which 2t-1 filter $\{1,2,3,\ldots,t^2+t+1\}$. We begin by setting $\sigma_i(i) = 1$ and letting $\sigma_i^{-1}\{t+3, t+4, t+5, \ldots, t^2+t+1\} = \{1,2,3,\ldots,t^2\} - \{i\}$. Then relabel the elements of $M = \{t^{2+1}, t^2+2, \ldots, t^2+t+1\}$ by $m_1, m_2, \ldots, m_{t+1}$. We call m_1 a big middle element and $m_2, m_3, \ldots, m_{t+1}$ little middle elements. Place m_1 in second position in a block of t permutations. Then place each little middle element in second position in a block of t-1 permutations. Then place each little middle element in third position in one of permutations in the m_1 block. For each $i \geq 2$, place each the little middle elements (except m_i) in 3^{rd} position in a permutation in the m_i block. Finally place m_1 in fourth position in all the permutations in the little blocks.

Now consider a 2t-1 element subset $A \subset \{1,2,3,\ldots,t^2+t+1\}$ and an element $a_o \in A$. We show that there is a permutation in which a_o precedes all other elements of A. If $a_o \in \{1,2,\ldots,t^2\}$, there is nothing to show. If $a_o = m_i$ for some $i \geq 2$, then there is no problem unless A contains the t-1 elements which precede m_i in the permutations in the m_i block. Furthermore A must contain an element from each of the t two element sets consisting of the elements which precede m_i when m_i is in third position in one of the other blocks. But this would require A to contain $1 + (t-1) + t = 2t$ elements.

Now consider the case $a_o = m_1$. We may also assume that A contains the t elements preceding m_1 in the permutations in m_1 block. Now suppose that A contains s little middle elements. As before $t-s \geq 2$. So there are $(t-s)(t-s-1)$ permutations containing m_1 in fourth place with m_1 preceded by two little middle elements neither of which belongs to A. So we may assume that A contains each of the first elements of these permutations. However, this requires A to contain at least $1 + t + s + (t-s)(t-s-1) \geq 1 + t + s + t-s > 2t-1$ elements. The proof of our theorem is now complete.

We note that Theorem 4 is best possible when t=3 since we have the following result.

Theorem 6: $N(11,4) = 7$

Proof: It suffices to show that $N(11,4) > 6$. Suppose that $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5$, and σ_6 4-filter $\{1,2,3,\ldots,11\}$. We assume that $\sigma_i(i) = 1$ and $\sigma_i^{-1}\{7,8,9,10,11\} = \{1,2,3,4,5,6\} - \{i\}$. We then relabel $\{7,8,9,10,11\}$ as m_1, m_2, m_3, m_4, and m_5. We may then assume that m_5 precedes all other m_i's in at least two permutations. If we delete the appearance of m_5 from each of the permutations, we obtain 6 permutations which 4 filter a set of size ten. Using arguments as in [1], it is easy to see that we may assume without loss of generality that the first five positions of these restrictions are:

σ_1: $[1, m_1, m_3, m_2, m_4]$
σ_2: $[2, m_1, m_4, m_2, m_3]$
σ_3: $[3, m_2, m_3, m_1, m_4]$
σ_4: $[4, m_2, m_4, m_1, m_3]$

σ_5: $[5, m_3, m_4, m_1, m_2]$

σ_6: $[6, m_4, m_3, m_2, m_1]$

Furthermore, the deletion of any other m_i must leave this same pattern (although the ordering on the permutations may change). Note that the fourth position in each permutation is occupied by a big middle.

Suppose that m_5 is the leading middle element in σ_1 and σ_2. Then if we delete m_4, we have m_j followed immediately by m_1 in σ_1 and σ_2 and the required pattern is broken. The same argument applies if m_5 is the leading middle element in σ_3 and σ_4. Now suppose m_j is the leading middle element in σ_1 and σ_3. If we delete m_3, then m_1 and m_2 are little middle elements but they then occur in position 4 in σ_1 and σ_3. The contradiction shows that m_5 cannot lead in σ_1 and σ_3. Similarly m_5 cannot lead in σ_1 and σ_4, σ_2 and σ_4, or σ_2 and σ_3.

If m_5 leads in σ_1 and σ_5, delete m_3 to obtain a contradiction. If m_5 leads in σ_5 and σ_6, delete m_1. The other can follow by symmetry. The proof is now complete.

<u>Theorem 7</u>: N(14,4) = 7

Proof: It suffices to show that N(14,4) $\leq$ 7. We describe the first positions of seven permutations which 4-filter {1,2,3,...,14}.

σ_1: [1,8,14,11,9]

σ_2: [2,9,8,12,10]

σ_3: [3,10,9,13,11]

σ_4: [4,11,10,14,12]

σ_5: [5,12,11,8,13]

σ_6: [6,13,12,9,14]

σ_7: [7,14,13,10,8]

On the other hand, Theorem 4 is not best possible when t=3.

Theorem 8: N(14,5) = 9

Proof: We give the first few positions of 9 permutations which 5-filter {1,2,...,14}.

σ_1: [1,10,13,12,11]

σ_2: [2,10,14,12,11]

σ_3: [3,10,11,12]

σ_4: [4,11,13]

σ_5: [5,11,14]

σ_6: [6,12,13]

σ_7: [7,12,14]

σ_8: [8,13,14,12,10,11]

σ_9: [9,14,13,11,10,12]

We conclude with a small table of values for N(m,k).

m \ k	2	3	4	5	6	7	8	9	10	11	12	13	14
2	2												
3	2	3											
4	2	3	4										
5	2	4	4	5									
6	2	4	5	5	6								
7	2	4	6	6	6	7							
8	2	4	6	7	7	7	8						
9	2	4	6	8	8	8	8	9					
10	2	4	6	8	9	9	9	9	10				
11	2	4	7	9	10	10	10	10	10	11			
12	2	4	7	9	10	11	11	11	11	11	12		
13	2	5	7	9	11	12	12	12	12	12	12	13	
14	2	5	7	9	12	12	13	13	13	13	13	13	14

References

1. B. Dushnik, "Concerning a Certain Set of Arrangements", *Proceedings* A.M.S. *1* (1950) 788-796.

2. J. Spencer, "Minimal Scrambling Sets of Simple Orders", *Acta Math. Acad. Sci. Hung.* 22 (1971) 349-353.

Note: The author has succeeded in proving that Theorem 4 is best possible for $t \geq 3$, i.e. $N(t^2+2,2t-2) = t^2-t+1$. Furthermore, Theorem 5 is best possible for $t \geq 4$, i.e. $N(t^2+t+2,2t-1) = t^2+1$.