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A GENERALIZATION OF TURÁN'S THEOREM TO DIRECTED GRAPHS

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We consider an extremal problem for directed graphs which is closely related to Turán's theorem giving the maximum number of edges in a graph on n vertices which does not contain a complete subgraph on m vertices. For an integer $n \geqslant 2$, let $\mathbf{T}_n$ denote the transitive tournament with vertex set $X_n = \{1, 2, 3, \ldots, n\}$ and edge set $\{(i, j): 1 \leqslant i < j \leqslant n\}$. A subgraph H of $\mathbf{T}_n$ is said to be m-locally unipathic when the restriction of H to each m element subset of X_n consisting of m consecutive integers is unipathic. We show that the maximum number of edges in a m-locally unipathic subgraph of $\mathbf{T}_n$ is $\binom{q}{2}(m-1)^2 + q(m-1)r + \lfloor \frac{1}{4}r^2 \rfloor$ where $n = q(m-1)+r$ and $\lceil \frac{1}{2}(m-1) \rceil \leqslant r < \lceil \frac{3}{2}(m-1) \rceil$. As is the case with Turán's theorem, the extremal graphs for our problem are complete multipartite graphs. Unlike Turán's theorem, the part sizes will not be uniform. The proof of our principal theorem rests on a combinatorial theory originally developed to investigate the rank of partially ordered sets.

1. Introduction

For integers, n, k with $n \geqslant k \geqslant 2$, let $g(n, k)$ be the maximum number of edges in a graph G on n vertices which does not contain a complete subgraph on k vertices. Then let $n = (k-1)q + r$ where $0 \leqslant r < k-1$ and consider the complete multipartite graph $G(n, k)$ having $k-1-r$ parts of size q and r parts of size $q+1$. Clearly, $G(n, k)$ has n vertices but does not have a complete subgraph on k vertices. The following well known theorem of P. Turán [9] tell us that the lower bound on $g(n, k)$ provided by the graph $G(n, k)$ is best possible. It also tells us that $G(n, k)$ is the unique extremal graph.

Theorem 1 (Turán). *For integers n, k with $n \geqslant k \geqslant 2$ the maximum number $g(n, k)$ of edges in a graph on n vertices which does not contain a complete subgraph on k*

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vertices is given by:

$$g(n, k) = \binom{k-1-r}{2}q^2 + \binom{r}{2}(q+1)^2 + (k-1-r)rq(q+1)$$

where $n = (k-1)q + r$ and $0 \leq r < k-1$. Furthermore, if G is a graph on n vertices which does not contain a complete subgraph on k vertices, then G has $g(n, k)$ edges if and only if $G = G(n, k)$.

In this paper, we will consider a similar combinatorial problem involving the maximum number of edges in a directed graph which satisfies a particular property. As in Turán's theorem, the extremal graph(s) will complete multipartite graphs, although the part sizes will not all be uniform.

For an integer $n \geq 2$, let $\mathbf{T}_n$ denote the transitive tournament with vertex set $X_n = \{1, 2, \ldots, n\}$ and edge set $\{(i, j): 1 \leq i < j \leq n\}$. A subgraph H of $\mathbf{T}_n$ is said to be *unipathic* if for each pair x, y of distinct vertices, H contains at most one directed path from x to y. Now consider the following elementary extremal problem: What is the maximum number $u(n)$ of edges in a unipathic subgraph of $\mathbf{T}_n$? It is easy to see that this problem is equivalent to a special case of Turán's theorem.

Theorem 2. *For each $n \geq 2$, the maximum number $u(n)$ of edges in a unipathic subgraph of $\mathbf{T}_n$ is given by the formula: $u(n) = \lfloor \frac{1}{4}n^2 \rfloor$. Furthermore, if H is a unipathic subgraph of $\mathbf{T}_n$ having $u(n)$ edges, then the underlying undirected graph determined by H is the complete bipartite graph $K(\lfloor \frac{1}{2}n \rfloor, \lceil \frac{1}{2}n \rceil)$. Moreover, if $n \geq 4$, the vertices in each of the two parts of H occur consecutively in $\{1, 2, 3, \ldots, n\}$.*

Proof. Let H be a unipathic subgraph of $\mathbf{T}_n$ and let G be the underlying undirected graph determined by H. Since H is unipathic, G is triangle-free, i.e., G does not contain K_3. Thus H and G have at most $g(n, 3) = \lfloor \frac{1}{4}n^2 \rfloor$ edges. On the other hand, let $t = \lfloor \frac{1}{2}n \rfloor$ (or $t = \lceil \frac{1}{2}n \rceil$) and consider the subgraph H of $\mathbf{T}_n$ containing the edges $\{(i, j): 1 \leq i \leq t, t+1 \leq j \leq n\}$. Clearly, H is unipathic and contains $\lfloor \frac{1}{4}n^2 \rfloor$ edges, and thus $u(n) = \lfloor \frac{1}{4}n^2 \rfloor$.

Finally, suppose that $n \geq 4$ and let H be a unipathic subgraph of $\mathbf{T}_n$ containing $\lfloor \frac{1}{4}n^2 \rfloor$ edges. It follows from Turán's theorem that the underlying undirected graph G determined by H is the complete bipartite graph $K(\lfloor \frac{1}{2}n \rfloor, \lceil \frac{1}{2}n \rceil)$. Then let A and B denote the subsets of $\{1, 2, 3, \ldots, n\}$ which form the vertex sets of the two parts of G. If $n \geq 4$ and either A or B does not occur consecutively in $\{1, 2, 3, \ldots, n\}$, then there exists integers $a_1, a_2 \in A$, $b_1, b_2 \in B$ for which one of the following statements holds: $a_1 < b_1 < a_2 < b_2$, $a_1 < b_1 < b_2 < a_2$, $b_1 < a_1 < a_2 < b_2$, or $b_1 < a_1 < b_2 < a_2$. In each of the four cases, H would fail to be unipathic even though it is triangle-free.

We note that although $u(3) = \lfloor 3^2/4 \rfloor = 2$, there are three extremal graphs corresponding to $A=\{1\}$, $A=\{1,2\}$ and $A=\{1,3\}$ respectively. These graphs are $\{(1,2),(1,3)\}$, $\{(1,3),(2,3)\}$, and $\{(1,2),(2,3)\}$. $\square$

Now let n and m be integers with $n \geq m \geq 2$. A subgraph H of $\mathbf{T}_n$ is said to be *m-locally unipathic* when the restriction of H to each subset of V_n containing m consecutive vertices is unipathic. On the other hand, H is said to be *m-locally triangle-free* when the restriction of H to each subset of X_n containing m consecutive vertices is triangle-free. Then let $u(n,m)$ be the maximum number of edges in an m-locally unipathic subgraph of $\mathbf{T}_n$ and $\Delta(n,m)$ the maximum number of edges in an m-locally triangle-free subgraph of $\mathbf{T}_n$.

We have already observed that $u(n,n)=\Delta(n,n)=g(n,3)=\lfloor\frac{1}{4}n^2\rfloor$ for every $n\geq 2$. Furthermore, it is easy to see that $u(n,2)=\Delta(n,2)=\binom{n}{2}$ for every $n\geq 2$ and that $\Delta(n,m)\geq u(n,m)$ for every $n\geq m\geq 2$.

In view of Theorem 2, it is reasonable to conjecture that the extremal graphs for $u(n,m)$ are complete multipartite graphs for all $n\geq m\geq 2$ with the vertices in each part occurring consecutively in $\{1,2,3,\ldots,n\}$ (except for the case $(n,m)=(3,3)$). Analysis of the properties of such graphs suggests the following scheme. For arbitrary integers $m\geq 2, q\geq 0, r\geq 0$ with $n=q(m-1)+r\geq 2$, we construct a complete multipartite subgraph $H(m,q,r)$ of $\mathbf{T}_n$. We begin by setting

$$V_0=\{1,2,3,\ldots,\lfloor\tfrac{1}{2}r\rfloor\},$$
$$V_{q+1}=\{n-\lceil\tfrac{1}{2}r\rceil+1,\, n-\lceil\tfrac{1}{2}r\rceil+2,\, n-\lceil\tfrac{1}{2}r\rceil+3,\ldots,n\},$$

and

$$V_i=\{(i-1)(m-1)+\lfloor\tfrac{1}{2}r\rfloor+j : 1\leq j\leq m-1\} \quad \text{for} \quad i=1,2,\ldots,q.$$

Finally, we define $H(m,q,r)$ as the complete multipartite graph having $q+2$ parts $V_0, V_1, V_2, \ldots, V_{q+1}$ with edge set $\{(j_1,j_2)$: There exist i_1, i_2 with $j_1\in V_{i_1}, j_2\in V_{i_2}$ and $0\leq i_1<i_2\leq q+1\}$. Note that $H(m,q,r)$ is a bipartite graph when $q=0$. Also note that V_0 contains the first $\lfloor\frac{1}{2}r\rfloor$ vertices of $\{1,2,3,\ldots,n\}$ and V_{q+1} contains the last $\lceil\frac{1}{2}r\rceil$ vertices of $\{1,2,3,\ldots,n\}$. We then denote by $\hat{H}(m,q,r)$ the complete multipartite graph obtained by reversing the roles of V_0 and V_{q+1}, i.e., in $\hat{H}(m,q,r)$, V_0 contains the first $\lceil\frac{1}{2}r\rceil$ vertices of $\{1,2,3,\ldots,n\}$ and V_{q+1} contains the last $\lfloor\frac{1}{2}r\rfloor$. Note that $H(m,q,r)$ and $\hat{H}(m,q,r)$ have the same number of edges. In fact, $H(m,q,r)=\hat{H}(m,q,r)$ when r is even. For convenience, we let $h(m,q,r)$ denote the number of edges in $H(m,q,r)$. Note that

$$h(m,q,r)=\binom{q}{2}(m-1)^2+q(m-1)r+\lfloor\tfrac{1}{4}r^2\rfloor.$$

It is easy to see that $H(m,q,r)$ and $\hat{H}(m,q,r)$ are m-locally unipathic. Furthermore, it is straightforward to verify that for fixed values of n and m with $n\geq m\geq 2$, if we choose $q\geq 0$ and $r\geq 0$ so that $n=q(m-1)+r$, then the maximum value of $h(m,q,r)$ is achieved when $\lceil\frac{1}{2}(m-1)\rceil\leq r<\lceil\frac{3}{2}(m-1)\rceil$. When $m\geq 3$, this maximum value is achieved by a unique triple (m,q,r) unless $r=\frac{1}{2}(m-1)$ in which case, we have $h(m, q, r) = h(m, q-1, r+m-1)$. For example $H(5, 3, 2)$ and $H(5, 2, 6)$ each have $h(5, 3, 2) = h(5, 2, 6) = 73$ edges; and $H(11, 1, 5)$ and $H(11, 0, 15)$ each have 56 edges. When $m = 2$, we observe that the maximum value of $h(m, q, r)$ is $\binom{n}{2}$, and this maximum value is achieved if and only if r is 0, 1, or 2. We also observe that $H(2, n, 0) = H(2, n-1, 1) = H(2, n-2, 2) = \mathbf{T}_n$ for every $n \geq 2$. These observations are summarized in the following result.

Theorem 3. *Let $n \geq m \geq 2$. Then the maximum number $u(n, m)$ of edges in an m-locally unipathic subgraph of $\mathbf{T}_n$ satisfies the inequality:*

$$u(n, m) \geq h(m, q, r) = \binom{q}{2}(m-1)^2 + q(m-1)r + \lfloor \tfrac{1}{4}r^2 \rfloor,$$

where

$$n = q(m-1) + r \quad \text{and} \quad \lceil \tfrac{1}{2}(m-1) \rceil \leq r < \lceil \tfrac{3}{2}(m-1) \rceil.$$

In the remaining sections of this paper, we will show that the inequality in Theorem 3 is best possible. We will also show that if $(n, m) \neq (3, 3)$, then the complete multipartite graphs $H(m, q, r)$ and $\hat{H}(m, q, r)$ are the only extremal graphs except when $r = \frac{1}{2}(m-1)$. When $r = \frac{1}{2}(m-1)$ and r is even, there are two extremal graphs: $H(m, q, r)$ and $H(m, q-1, r+m-1)$. When $r = \frac{1}{2}(m-1)$ and r is odd, there are four extremal graphs: $H(m, q, r)$, $\hat{H}(m, q, r)$, $H(m, q-1, r+m-1)$ and $\hat{H}(m, q-1, r+m-1)$. Sections 2 and 3 will be devoted to the theoretical preliminaries, and the proof of the principal theorem will be presented in Section 4. In Section 5, we will present a brief discussion of the concept of rank for partially ordered sets and the specific problem which motivated our investigation of m-locally unipathic subgraphs of $\mathbf{T}_n$.

2. The digraph of nonforcing pairs for a partially ordered set

In this paper, a *partially ordered set* (poset) is a pair (X, P) where X is a finite set and P is an irreflexive transitive binary relation on X. The notations $(x, y) \in P$, $x > y$ in P, and $y < x$ in P are used interchangeably. The notations $x \leq y$ in P and $y \geq x$ in P mean $x > y$ in P or $x = y$ and we write $x I y$ in P when $x \neq y$, $x \not< y$ in P, and $y \not< x$ in P. We also let $I_p = \{(x, y) : x I y \text{ in } P\}$. A poset (X, P) is called a *totally ordered set* (also a linearly ordered set or chain) when $I_p = \emptyset$.

Throughout this paper, we adopt the following conventions concerning directed graphs. We denote an edge from a vertex x to a vertex x to a vertex y by (x, y) and we specify a digraph by its edge set. It is then understood that the vertex set of a digraph, when not explicitly described, is the set of endpoints of the edges. We may therefore view a binary relation on a set X as a digraph. For example, when $a_1, a_2, \ldots, a_{t+1}$ are distinct, we say that the sequence $\{(a_i, a_{i+1}) : 1 \leq i \leq t\}$ is a *directed path* of length t from a_1 to a_{t+1}. When $a_1, a_2, \ldots, a_t$ are distinct and

$a_1 = a_{t+1}$, we say that the sequence $\{(a_i, a_{i+1}) : 1 \leq i \leq t\}$ is a *directed cycle* of length t. A digraph H is said to be *acyclic* when it does not contain any directed cycles. A digraph H is said to be unipathic when it contains at most one directed path from x to y for every pair of vertices x, y, i.e., if $P_1 = \{(u_i, u_{i+1}) : 1 \leq i \leq t\}$ and $P_2 = \{(v_j, v_{j+1}) : 1 \leq j \leq s\}$ are paths in H, $u_1 = v_1 = x$, and $u_{t+1} = v_{s+1} = y$, then $s = t$ and $(u_i, u_{i+1}) = (v_i, v_{i+1})$ for $i = 1, 2, \ldots, t$.

To assist in distinguishing directed and undirected graphs, we will continue the notational convention adopted in Section 1. Specifically, we will use the letter G to denote an undirected graph and the letters H and N to denote directed graphs. We will then use "primes" or subscripts when we are discussing more than one such graph.

When X is a set, we let $|X|$ denote the number of elements in X, and when H is a digraph, we let $|H|$ denote the number of edges in H.

Now let (X, P) be a poset and let $(x, y) \in I_p$. We say that (x, y) is a *nonforcing pair* when $P \cup \{(x, y)\}$ is a partial order on X, i.e., $z > x$ in P implies $z > y$ in P, and $z < y$ in P implies $z < x$ in P for every $z \in X$. We then let N_p be the digraph (binary relation) of all nonforcing pairs. To illustrate this definition, we provide in Fig. 1 the Hasse diagram of a poset (X, P) and the digraph N_p associated with (X, P).

Note that in general, the digraph N_p may contain directed cycles. In order to extract an acyclic subgraph of N_P, we adopt the following convention for "breaking ties". Let L be an arbitrary linear order on X. Then define the *acyclic digraph of nonforcing pairs* N_P^* by

$$N_P^* = \{(x, y) \in N_P : (y, x) \notin N_P \text{ or } (x, y) \in L\}.$$

It is straightforward to verify that N_P^* is acyclic, so we may adopt the same convention used for Hasse diagrams in providing a diagram for N_P^*, i.e., we require that each edge $(a, b) \in N_P^*$ be represented by a nonhorizontal arc with the point corresponding to a having larger y-coordinate then the point corresponding to b. In order to avoid drawing arrows, it is then understood that the direction of an edge is from top to bottom on the page. For example, if L is defined for the poset drawn in Fig. 1 by $a > b > c > d > e > f > g > h$ in L, then we may represent N_P^* as shown in Fig. 2.

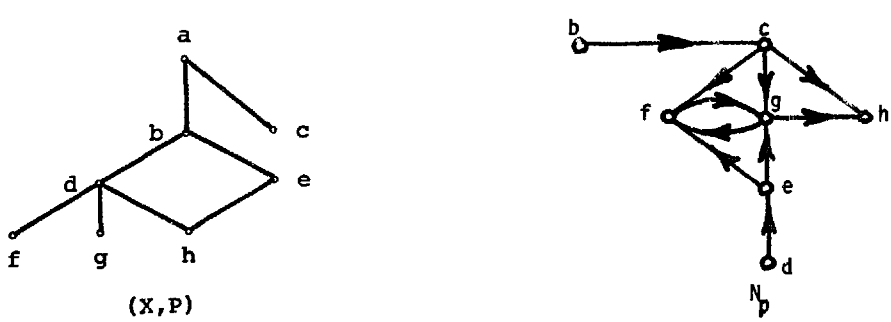

Fig. 1.

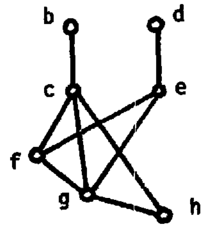

Fig. 2.

The notation N_P^* does not indicate the particular linear order L used in its definition since it is easy to see that the subgraphs of N_P determined by different linear orders are isomorphic.

A subgraph $H \subseteq N_P^*$ is said to be *unipathic relative to P* (we also say H is a U_P^* graph) when the following condition is satisfied: For each pair x, y of distinct vertices, if H contains two nonidentical paths from x to y, then $(x, y) \notin N_P^*$.

For example, the subgraph $H \subseteq N_P^*$ shown in Fig. 3 is unipathic relative to P (but it is not unipathic). Note that H contains nonidentical paths from a to d, but $(a, d) \notin N_P^*$. In fact $a > d$ in P.

We next present some elementary but important lemmas which detail the interplay between the partial order P and the acyclic digraph of nonforcing pairs. The proofs are immediate consequences of the definitions and are therefore omitted.

Lemma 4. $P \cap N_P^* = \emptyset$.

Lemma 5. *$P \cup N_P^*$ is a partial order on X.*

Lemma 6. *If* $\{(u_i, u_{i+1}) : 1 \leq i \leq t\} \subseteq P \cup N_P^*$, *and* $(u_{i_0}, u_{i_1}) \in P$ *for some* i_0, i_1 *with* $1 \leq i_0 < i_1 \leq t+1$, *then* $u_1 > u_{t+1}$ *in P.*

It follows from Lemmas 4 and 5 that a subgraph H of N_P^* is a U_P^* graph if and only if it satisfies the following condition: For each pair x, y of distinct vertices, if H contains two nonidentical paths from x to y, then $x > y$ in P. These lemmas

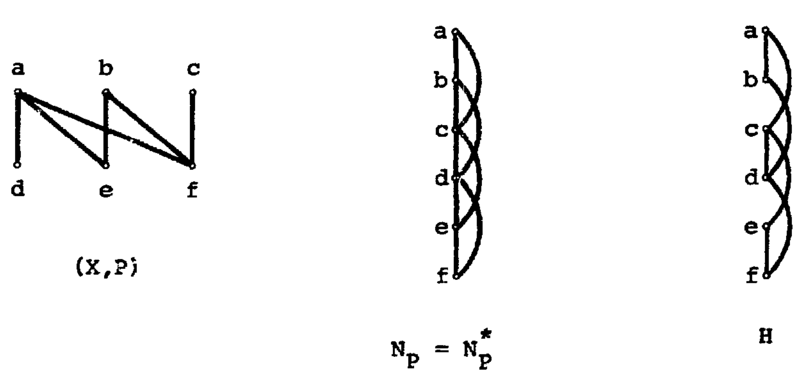

Fig. 3.

also allow us to make the following observation concerning graphs which are not U_P^* graphs. If $H \subseteq N_P^*$ and H is not a U_P^* graph, then there exists an edge $(x, y) \in N_P^*$ for which H contains two nonidentical paths P_1 and P_2 from x to y. Although these two paths are nonidentical, they may have vertices in common other than x and y and may also have common edges. On the other hand, if we examine all edges (x, y) of N_P^* for which H contains two or more nonidentical paths from x to y, and then choose an edge $(x, y) \in N_P^*$ and nonidentical paths P_1 and P_2 from x to y for which the sum $|P_1|+|P_2|$ of the lengths of P_1 and P_2 is as small as possible, then it is easy to see that P_1 and P_2 have no vertices in common other than x and y. A U_P^* graph H is called a *maximal* U_P^* graph when there does not exist a U_P^* graph H' whose edge set contains the edge set of H as a proper subset. A U_P^* graph H is called a *maximum* U_P^* graph when no U_P^* graph contains more edges than H.

Maximal and maximum U_P^* graphs are important concepts in the theory of rank of partially ordered sets, and we refer the reader to [2–6] for details. In particular, we note that (except for certain degenerate cases) the rank of a partially ordered set (X, P) equals the number of edges in a maximum U_P^* graph. In Section 5, we will return to this concept and employ the solution of our extremal problem to compute the rank of a class of partially ordered sets.

3. Exchange theorems for U_P^* graphs

In this section, we develop two exchange theorems for U_P^* graphs. These theorems establish conditions under which it is possible to exchange edges between a U_P^* graph H and $N_P^* - H$ so as to produce a new U_P^* graph.

Theorem 7. *Let (X, P) be a poset and $\{a_1, a_2, a_3\}$ a subset of X for which $\{(a_1, a_2), (a_1, a_3), (a_2, a_3)\} \subseteq N_P^*$. If H is a U_P^* graph and $\{(a_1, a_2), (a_2, a_3)\} \subseteq H$, then $H' = (H - \{(a_2, a_3)\}) \cup \{(a_1, a_3)\}$ and $H'' = (H - \{(a_1, a_2)\}) \cup \{(a_1, a_3)\}$ are U_P^* graphs.*

Proof. We show that H' is a U_P^* graph. The argument for H'' is dual. Suppose to the contrary that H' is not a U_P^* graph. Then there exists an edge $(x, y) \in N_P^*$ for which H' contains nonidentical paths

$$P_1 = \{(u_1, u_{i+1}) : 1 \leq i \leq t\} \quad \text{and} \quad P_2 = \{(v_j, v_{j+1}) : 1 \leq j \leq s\}$$

from x to y. Without loss of generality, we may assume that the edge $(x, y) \in N_P^*$ and the paths P_1 and P_2 have been chosen so that $s + t$ is minimum. We may then assume that x and y are the only two points belonging to both P_1 and P_2.

Since H is a U_P^* graph, we may assume without loss of generality that $(a_1, a_3) \in P_1$, say $(a_1, a_3) = (u_{i_0}, u_{i_0+1})$. Then it follows that H contains the paths $P_1' = (P - \{(a_1, a_3)\}) \cup \{(a_1, a_2), (a_2, a_3)\}$ and P_2 from x to y and we must therefore have $P_1' = P_2$, which is impossible. The contradiction completes the proof. □

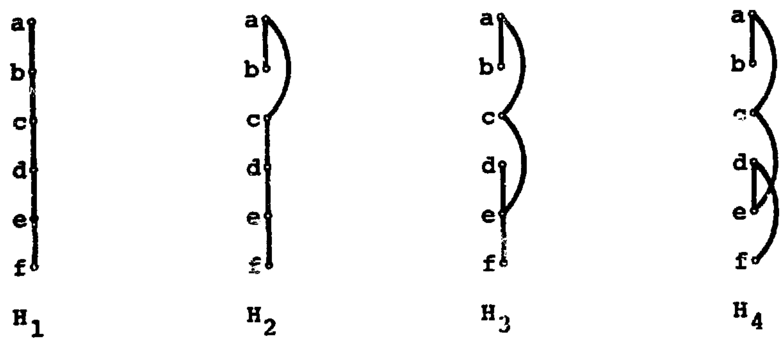

Fig. 4.

To illustrate the preceding theorem, consider the poset (X, P) shown in Fig. 3 and the sequence of U_P^* graphs shown in Fig. 4.

Observe that H_{i+1} is obtained from H_i for $i = 1, 2, 3$ by an exchange permitted by Theorem 7. Also note that H_1 is a maximal (but not maximum) U_P^* graph, but that H_2 is not maximal since $H_2 \cup \{(b, d)\}$ is also a U_P^* graph. Therefore, an exchange of edges permitted by Theorem 7 may destroy the property of being a maximal U_P^* graph. For brevity, we say that a U_P^* graph H does not admit a Type 1 exchange when $\{(a_1, a_2), (a_2, a_3)\} \subseteq H$ implies $(a_1, a_3) \notin N_P^*$ (and therefore $a_1 > a_3$ in P) for all $a_1, a_2, a_3 \in X$.

Our next exchange theorem describes a somewhat more complicated exchange.

Theorem 8. *Let (X, P) be a poset, $\{a_1, a_2, a_3, a_4\} \subseteq X$, and $A = \{(a_i, a_j) : 1 \le i < j \le 4\} \subseteq N_P^*$. Further, suppose that H is a U_P^* graph for which $\{(a_1, a_2), (a_3, a_4)\} \subseteq H$, and then let $G(a_2) = \{z \in X : (z, a_2) \in H$ and $(z, a_3) \in N_P^* - H\}$ and $L(a_3) = \{w \in X : (a_3, w) \in H$ and $(a_2, w) \in N_P^* - H\}$. If H does not admit a Type 1 exchange, then the graph $H' = (H - \{(z, a_2) : z \in G(a_2)\} - \{(a_3, w) : w \in L(a_3)\}) \cup \{(z, a_3) : z \in G(a_2)\} \cup \{(a_2, w) : w \in L(a_3)\}$ is a U_P^* graph.*

Proof. Suppose to the contrary that H' is not a U_P^* graph and choose an edge $(x, y) \in N_P^*$ for which H' contains nonidentical paths $P_1 = \{(u_i, u_{i+1}) : 1 \le i \le t\}$ and $P_2 = \{(v_j, v_{j+1} : 1 \le j \le s\}$ from x to y. As in Theorem 7, we assume that the edge (x, y) and the paths P_1 and P_2 have been chosen so that $s + t$ is minimum. Then let $S_1 = \{(z, a_3) : z \in G(a_2)\}$ and $S_2 = \{(a_2, w) : w \in L(a_3)\}$.

Since H is a U_P^* graph, it is clear that $(P_1 \cup P_2) \cap (S_1 \cup S_2) \ne \emptyset$. On the other hand, it is clear that $|P_i \cap (S_1 \cup S_2)| \le 1$ for $i = 1, 2$. In view of the obvious symmetry and duality, we may therefore reduce the remainder of the argument to the following three cases. Only in the third case will we require the additional hypothesis that H admits no Type 1 exchanges.

Case 1. $|P_1 \cap S_1| = 1$ and $|P_2 \cap S_2| = 1$.

In this case, we may assume that $(z, a_3) \in P_1 \cap S_1$, $(z, a_3) = (u_{i_0}, u_{i_0+1})$, $(a_2, w) \in P_2 \cap S_2$, and $(a_2, w) = (v_{j_0}, v_{j_0+1})$. Recall that $P \cup N_P^*$ is a partial order on X. It follows that $z > a_2 > a_3 > w$ in $P \cup N_P^*$, which implies that $v_1 = x \ne a_2$. Therefore,

$P_3=\{(v_j, v_{j+1}): 1\leq j<j_0\}$ is a path in $H\cap H'$ from x to a_2. However, H also contains the path $P_4=\{(u_i, u_{i+1}): 1\leq i<i_0\}\cup\{(z, a_2)\}$ from x to a_2. Since $(z, a_2)\in P_4-P_3$, we conclude that $P_3\neq P_4$ which contradicts the assumption that H is a U_P^* graph.

Case 2. $|P_1\cap S_1|=1$ and $|P_2\cap S_1|=1$.

Choose i_0, j_0 so that $(z, a_3)=(u_{i_0}, u_{i_0+1})\in P_1\cap S_1$ and $(z', a_3)=(v_{j_0}, v_{j_0+1})\in P_2\cap S_2$. Since $s+t$ is minimum, we must have $i_0=t$ and $j_0=s$, i.e., $u_{t+1}=y=v_{s+1}=a_3$. Then it follows that H contains the nonidentical paths $P_1'=\{(u_i, u_{i+1}): 1\leq i\leq t-1\}\cup\{(z, a_2)\}$ and $P_2'=\{(v_j, v_{j+1}): 1\leq j\leq s-1\}\cup\{(z', a_2)\}$ from s to a_2, and therefore $x>a_2$ in P. Since (a_2, a_3) is an edge in N_P^*, we conclude from Lemma 6 that $x>y$ in P, which is a contradiction.

Case 3. $|P_1\cap S_1|=1$ and $|P_2\cap(S_1\cup S_2)|=0$.

Choose i_0 so that $(z, a_3)=(u_{i_0}, u_{i_0+1})\in P_1\cap S_1$. Now suppose that $i_0<t$. Then $(u_{i_0+1}, u_{i_0+2})=(a_3, u_{i_0+2})\in H'$ implies that $(a_2, u_{i_0+2})\notin N_P^*-H$, i.e., either $(a_2, u_{i_0+2})\in H$ or $a_2>U_{i_0+2}$ in P. If $(a_2, u_{i_0+2})\in H$, then H contains the nonidentical paths $P_3=\{(u_i, u_{i+1}): 1\leq i<i_0\}\cup\{(z, a_2), (a_2, u_{i_0+2})\}\cup\{(u_i, u_{i+1}): i_0+2\leq i\leq t\}$ and P_2 from x to y. On the other hand, if $a_2>u_{i_0+2}$ in P, then it follows from Lemma 6 that $x>y$ in P which is a contradiction. We may therefore assume that $i_0=t$ and $a_3=y=u_{t+1}=v_{s+1}$. At this stage of the argument, we require that H admits no Type 1 exchanges. Since $P_2\cap(S_1\cup S_2)=\emptyset$, i.e., $P_2\subseteq H$, we know that $s=1$, $x=v_1$, and $y=a_3=v_2$. since P_1 and P_2 are edge disjoint and $(z, a_3)\in P_1$, we know that $x\neq z$ and $t\geq 2$. Therefore $t=2$ and $P_1=\{(x, z), (z, a_3)\}$. Furthermore, we know that $\{(x, z), (z, a_2)\}\subseteq H$ and since H admits no Type 1 exchanges, we must have $x>a_2$ in P, which in turn implies that $x>a_3=y$ in P. The contradiction completes the proof of this case and the theorem as well. □

We illustrate the preceding theorem with the U_P^* graphs in Fig. 5.

We call the exchange of edges in Theorem 8 a Type 2 exchange. For example, we leave it to the reader to verify that for the poset shown in Fig. 5, the graphs H_1

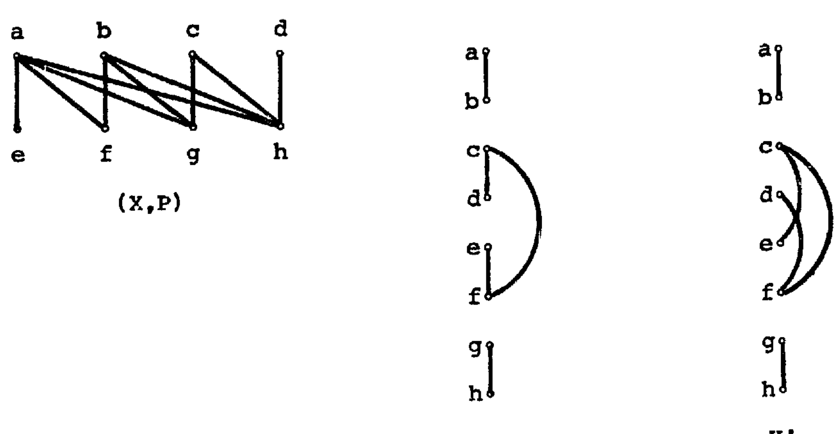

Fig. 5.

Fig. 6.

and H_2 as shown in Fig. 6 are maximal U_P^* graphs which do not admit Type 1 or Type 2 exchanges, and that H_1 is the unique maximum U_P^* graph for this poset.

4. The extremal problem

In this section, we will apply the theory developed in the preceding two sections to determine the maximum number of edges in a U_P^* graph of a carefully constructed poset (X, P). As a consequence, we will solve the original extremal problem: the determination of $u(n, m)$.

For integers n, m with $n \geq m \geq 2$, let $\mathbf{X}(n, m) = (X(n, m), P(n, m))$ be the poset defined by $X(n, m) = \{1, 2, 3, \ldots, n\}$ and $P(n, m) = \{(i, j): 1 \leq i < i + m \leq j \leq n\}$. For example the poset in Figs. 3 and 4 is (after relabeling) $\mathbf{X}(6, 3)$, and the poset in Figs. 5 and 6 is (after relabeling) $\mathbf{X}(8, 4)$. To determine the acyclic digraph of nonforcing pairs for $\mathbf{X}(n, m)$, we use the linear order $L = \mathbf{T}_n = \{(i, j): 1 \leq i < j \leq n\}$ to break the ties. Thus $N_P^* = \{(i, j) \in \mathbf{T}_n : j > i + m\}$. We then define $w(n, m)$ to be the maximum number of edges in a U_P^* subgraph for the poset $\mathbf{X}(n, m)$ and reduce the determination of $u(n, m)$ to the determination of $w(n, m)$. The equivalence of the two problems is easily established by the following lemma which is an immediate consequence of the definitions and the fact that $|P(n, m)| = \binom{n-m+1}{2}$.

Lemma 9. *Let $n \geq m \geq 2$ and $(X, P) = \mathbf{X}(n, m)$. Then a subgraph $H \subseteq N_P^*$ is a U_P^* graph if and only if $H \cup P(n, m)$ is a m-locally unipathic subgraph of $\mathbf{T}_n$. Furthermore, $u(n, m) = w(n, m) + \binom{n-m+1}{2}$.*

Lemma 9 allows us to apply the exchange theorems developed in Section 2 to m-locally unipathic subgraphs of $\mathbf{T}_n$. We will selectively apply these exchange theorems in the proof of the principal theorem.

The next lemma establishes some combinatorial identities which we will require in future arguments.

Lemma 10. *The following identities hold:*

(i) $h(m, q, r) = h(m-1, q, r-2) + \binom{q+2}{2} + (n-q-2)(q+1)$

when $m \geqslant 3$, $q \geqslant 0$, $r \geqslant 2$, *and* $n = m(q-1)+r \geqslant 3$.

(ii) $h(m, q, r) = h(m-1, q, r-1) + \binom{q+1}{2} + (n-q-1)q + \lfloor \frac{1}{2}r \rfloor$

when $m \geqslant 3$, $q \geqslant 0$, $r \geqslant 1$, *and* $n = m(q-1)+r \geqslant 3$.

(iii) $h(4p+1, q, 2p) = h(4p, q-1, 6p-2) + \binom{q+1}{2} + (n-q-1)q$

when $p \geqslant 1$, $q \geqslant 1$, *and* $n = 4pq+2p$.

(iv) $h(4p+3, q, 2p+1) = h(4p+2, q-1, 6p+1) + \binom{q+1}{2} + (n-q-1)q$

when $p \geqslant 1$, $q \geqslant 1$, *and* $n = 4pq+2q+2p+1$.

Proof. To establish the first identity, consider the complete multipartite graph $H = H(m, q, r)$ having $h(m, q, r)$ edges. Using the notation of Section 1, we label the $q+2$ parts of H by $V_0, V_1, V_2, \ldots, V_q, V_{q+1}$ with $|V_0| = \lfloor \frac{1}{2}r \rfloor$, $|V_{q+1}| = \lceil \frac{1}{2}r \rceil$, and $|V_i| = m-1$ for $i = 1, 2, \ldots, q$. Then let S be a $q+2$ element subset of $X_n = \{1, 2, 3, \ldots, n\}$ chosen so that S contains one element from each of the sets $V_0, V_1, V_2, \ldots, V_{q+1}$, and let H' be the restriction of H to $X_n - S$. Then $|H'| = h(m-1, q, r-2)$. Now consider the edges in $H - H'$. There are $\binom{q+2}{2}$ edges in $H - H'$ with both endpoints in S, and there are $(n-q-2)(q+1)$ edges in $H - H'$ with one endpoint in S and the other in $X_n - S$. The identity follows since $H = H' \cup (H - H')$.

To establish the second identity, we modify the argument given above as follows. We choose a $q+1$ element subset $S \subseteq X_n$ consisting of one element from each of the sets $V_1, V_2, \ldots, V_{q+1}$, and let H' be the restriction of H to $X_n - S$. Then $|H'| = h(m-1, q, r-1)$. There are $\binom{q+1}{2}$ edges in $H - H'$ with both endpoints in S, there are $\lfloor \frac{1}{2}r \rfloor (q+1)$ edges in $H - H'$ with one endpoint in V_0, and there are $(n - \lfloor \frac{1}{2}r \rfloor - q - 1)q$ edges in $H - H'$ with one endpoint in $X_n - S - V_0$ and the other in S. The desired identity follows as in the previous paragraph since $H = H' \cup (H - H')$.

To establish the third identity, we consider a $q+1$ element subset of X_n containing exactly one element from $V_1, V_2, \ldots, V_{q+1}$. Let $S \cap V_i = \{x_i\}$ for $i = 1, 2, \ldots, q+1$. Then let $V_q = V'_q \cup V''_q$ where $|V'_q| = 3p$, $|V''_q| = p$, and $x_q \in V'_q$. Let H' be the restriction of H to $X_m - S$, and let H'' be the complete $q+1$ multipartite graph whose parts are $V_0 \cup (V_{q+1} - \{x_{q+1}\}) \cup V''_q V_1 - \{x_1\}$, $V_2 - \{x_2\}, \ldots, V_{q-1} - \{x_{q-1}\}$, $V'_q - \{x_q\}$. Since $|V_0 \cup (V_{q+1} - \{x_{q+1}\}) \cup V''_q| = |V'_q - \{x_q\}| = 3p-1$ and $|V_i - \{x_i\}| = 4p-1$ for $i = 1, 2, \ldots, q-1$, we conclude that $|H''| = h(4p, q-1, 6p-2)$. But H'' is formed from H' by adding $(3p-1)p$ edges between vertices in V'_q and V''_q and deleting $3p^2 - 2p$ edges with both endpoints in

$V_0 \cup (V_{q+1} - \{x_{q+1}\}) \cup V_q''$. From the second identity we have $|H| = |H'| + \binom{q+1}{2} + (n-q-1)q + p$. and since $|H'| = |H''| - p$, the desired identity follows.

The proof of the last identity is similar and is omitted in the interests of brevity. □

For an edge $(i, j) \in \mathbf{T}_n$, we define the *length* of (i, j) to be $j - i$. Note that each edge in $P(n, m)$ has length at least m, and we may therefore view the edges in $P(n, m)$ as "long" edges. Furthermore, if H is a m-locally unipathic subgraph of $\mathbf{T}_n$ having $u(n, m)$ edges, then $P(n, m) \subseteq H$. On the other hand, there are limitations on the number of "short" edges a m-locally unipathic subgraph of $\mathbf{T}_n$ can contain. For example, the restriction of a m-locally unipathic subgraph to a set of m consecutive vertices contains at most $\lfloor \frac{1}{4} m^2 \rfloor$ edges. The next lemma also limits the number of short edges.

Lemma 11. *Let $n \geq m \geq 2$ and let H be a m-locally unipathic subgraph of $\mathbf{T}_n$. Also let i be an integer with $1 \leq i < i + m - 1 \leq n$. Then H contains at most $m - 1$ edges from the $2m - 1$ element set*

$$K\{(i, x) : i < x \leq i + m - 1\} \cup \{(y, i + m - 1) : i \leq y < i + m - 1\}.$$

Proof. Suppose first that H contains the edge $(i, i + m - 1)$. Since the restriction of H to the set $\{i, i+1, i+2, \ldots, i+m-1\}$ is unipathic, it follows that for each j with $i < j < i + m - 1$, H contains at most one edge from the pair $\{(i, j), (j, i + m - 1)\}$. Since there are $m - 2$ integers between i and $i + m - 1$, we conclude that H contains at most $1 + (m - 2) = m - 1$ edges from K.

On the other hand, suppose that $(i, i + m - 1) \notin H$. If it is still true that H contains at most one edge from the pair $\{(i, j), (j, i + m - 1)\}$ for each j with $i < j < i + m - 1$, then it follows that H contains at most $m - 2$ edges from K. So we may assume that there exists an integer j_0 with $i < j_0 < i + m - 1$ for which H contains both (i, j_0) and $(j_0, i + m - 1)$. Since the restriction of H to $\{i, i+1, i+2, \ldots, i+m-1\}$ is unipathic, it follows that for all j with $i < j < i + m - 1$ and $j \neq j_0$, H contains at most one edge from the pair $\{(i, j), (j, i + m - 1)\}$. Therefore $|H \cap K| \leq 2 + (m - 3) = m - 1$. □

We next introduce a technique for considering subsets S of X_n for which the identities in Lemma 10 as well as the restriction on the number of short edges given in Lemma 11 will be applicable. This technique will allow us to construct an inductive argument for the principal theorem utilizing the following convention. If $S \subseteq X_n$ and $|S| = s$ with $0 < s < n$, then the restriction of $\mathbf{T}_n$ to $X_n - S$ is isomorphic to $\mathbf{T}_{n-s}$. Given integers m_1, m_2 with $n \geq m_1 \geq 2$ and $n - s \geq m_2 \geq 2$, we may consider a m_1-locally unipathic subgraph H of $\mathbf{T}_n$ and its restriction H' to $X_n - S$. We may then ask whether H' is a m_2-locally unipathic subgraph of $\mathbf{T}_{n-s}$.

For integers n, m, k with $n \geq m \geq 2$ and $0 \leq k < m$, we let $S(n, m, k) = \{i \in X_n : i \equiv k \pmod{m - 1}\}$ and $s(n, m, k) = |S(n, m, k)|$.

Lemma 12. *Let n, m, k be integers with $n \geq m \geq 3$ and $0 \leq k < m$, and let H be a m-locally unipathic subgraph of $\mathbf{T}_n$. If H' is the restriction of H to $X_n - S(n, m, k)$ and $s = s(n, m, k)$, then H' is a $m-1$-locally unipathic subgraph of $\mathbf{T}_{n-s}$.*

Proof. Let A be a set of $m-1$ vertices which occur consecutively in $\mathbf{T}_{n-s}$. If the vertices in A also occur consecutively in $\mathbf{T}_n$, then since H is m-locally unipathic, it is also $m-1$-locally unipathic, and the restriction of H' to A, which is the same as the restriction of H to A, must be unipathic. On the other hand, if the vertices in A do not occur consecutively in $\mathbf{T}_n$, then it follows that there is a unique element $x \in S(n, m, k)$ so that $A \cup \{x\}$ is a set of m consecutive integers in $\mathbf{T}_n$. As before, the restriction of H' to A must be unipathic, and the argument is complete. □

We pause to detail two exceptional cases. The following result follows immediately from the remarks at the end of Theorem 2.

Lemma 13. $u(3,3) = 2$. *Furthermore, there are exactly three 3-locally unipathic subgraphs of T_3 which have two edges:*

$$\begin{aligned} H(3,0,3) &= \{(1,2),(1,3)\},\\ \hat{H}(3,0,3) &= \{(1,3),(2,3)\},\\ H_0 &= \{(1,2),(2,3)\}. \end{aligned}$$

We next discuss the special case $(n, m) = (5, 4)$. The argument presented here will be generalized to obtain the principal theorem.

Lemma 14. $u(5,4) = 7$. *Furthermore, $H(4,1,2)$ is the unique 4-locally unipathic subgraph of $\mathbf{T}_5$ having 7 edges.*

Proof. Let H be a 4-locally unipathic subgraph of $\mathbf{T}_5$ with $|H| = u(5,4)$. Then let $S_1 = S(5,5,1) = \{1,4\}$, $S_2 = S(5,4,2) = \{2,5\}$, $s_1 = |S_1| = 2$, and $s_2 = |S_2| = 2$. Also let H_i denote the restriction of H to $X_5 - S_i$ and let $L_i = (H - H_i) \cap P(5,4) = \{(1,5)\}$ for $i = 1, 2$. Then set $E_1 = \{(4,5)\} \cap H$ and $E_2 = \{(1,2)\} \cap H$. Finally, let $I_i = H_i - L_i - E_i$ for $i = 1, 2$.

Now H_i is a 3-locally unipathic subgraph of $\mathbf{T}_3$ so $|H_i| \leq 2$ for $i = 1, 2$. Also, we note that $|E_i| \leq 1$ for $i = 1, 2$. We next show that $|I_i \cup L_i| \leq 4$ for $i = 1, 2$. However, this follows immediately since $|L_i| = 1$ and $|I_i| \leq 3$ by Lemma 11 for $i = 1, 2$. Therefore, $7 \leq u(5,4) = |H| = |H_i| + |E_i| + |I_i \cup L_i| \leq 2 + 1 + 4 = 7$. Thus $|H| = u(5,4) = 7$.

We now proceed to show that $H = H(4,1,2)$. We begin by observing that we must have $|H_i| = 2$, $|E_i| = 1$, and $|I_i \cup L_i| = 4$ for $i = 1, 2$. In particular, we know that H_1 and H_2 must be one of the three extremal graphs in Lemma 13, and we know that $\{(1,2), (4,5), (1,5)\} \subseteq H$.

Suppose first that $H_1 = H(3,0,3)$, i.e., $\{(2,3), (2,5)\} \subseteq H$. If H contains the edge $(3,4)$, then H contains nonidentical paths from 1 to 4. The contradiction requires that $(3,4)\notin H$. This in turn implies that $H_2\neq \hat{H}(3,0,3)$ and $H_2\neq H_0$, i.e., $H_2=H(3,0,3)$. Thus H contains the edges $(1,3)$ and $(1,4)$ which in turn implies that H contains nonidentical paths from 1 to 3. The contradiction shows that $H_1\neq H(3,0,3)$. At this point, we may take advantage of the obvious duality to conclude that $H_2\neq \hat{H}(3,0,3)$.

Now suppose that $H_1=H_0=\{(2,3),(3,5)\}$. Then we observe that $H_2=H(3,0,3)=\{(1,3),(1,4)\}$ and $H_2=H_0=\{(1,3),(3,4)\}$ imply that H contains the edge $(1,3)$. This in turn implies that H contains nonidentical paths from 1 to 3. The contradiction shows that $H_1\neq H_0$, and by duality, we may conclude $H_2\neq H_0$. Therefore $H_1=\hat{H}(3,0,3)$ and $H_2=H(3,0,3)$. But these statements imply $H=H(4,1,2)$. □

The next lemma allows us to restrict our attention to m-locally unipathic subgraphs which do not admit Type 1 or Type 2 exchanges. This will simplify subsequent arguments consideraly.

Lemma 15. *Let n, m be integers with $n\geq m\geq 3$. If q and r are integers for which $n=q(m-1)+r$, $\lceil\frac{1}{2}(m-1)\rceil\leq r\leq\lceil\frac{3}{2}(m-1)\rceil$, and $(m,q,r)\neq(3,0,3)$, then $H(m,q,r)$ and $\hat{H}(m,q,r)$ cannot be obtained from a m-locally unipathic subgraph H of $\mathbf{T}_n$ by an exchange of Type 1 or Type 2.*

Proof. Suppose first that $H(m,q,r)$ can be obtained by a m-locally unipathic subgraph H of $\mathbf{T}_n$ by a Type 1 exchange. The argument for $\hat{H}(m,q,r)$ is dual. Choose integers x, y, z with $1\leq x<y<z\leq x+m-1$ for which H contains (x,z) and (y,z) but one of these edges is exchanged for (x,z) to form $H(m,q,r)$. Without loss of generality $H(m,q,r)=(H-\{(x,y)\})\cup\{(x,z)\}$. Choose an integer i so that $x\in V_i$. Since $(x,y)\notin H(m,q,r)$, we know that y also belongs to V_i. Since $z-x\leq m-1$, we know that $z\in V_{i+1}$. If $i>0$, let w denote the largest integer in V_{i-1}. Then $h(m,q,r)$ and H contain the edges (w,x) and (w,y). But H also contains (x,y) which is a contradiction since $y-w\leq m-1$. Therefore $i=0$, $r\geq 3$, and $m\geq 6$. If $|V_1|\geq 2$, we may consider the first two integers in V_1 and choose one of them, say z', with $z'\neq z$. It follows that H contains (x,y) (y,z'), and (x,z'). Since $z'\geq 2+\lfloor\frac{1}{2}r\rfloor$ and $x\geq 1$, we see that $z'-x\leq 1+\lfloor\frac{1}{2}r\rfloor\leq m-1$ which is a contradiction.

Now suppose that $H(m,q,r)$ is obtained from a m-locally unipathic subgraph H of $\mathbf{T}_n$ by a Type 2 exchange. Choose intergers x, y, z, w with $1\leq x<y<z<w\leq x+m-1$ for which $(x,y)\in H$, $(z,w)\in H$, $(x,z)\notin H$, $(y,z)\notin H$, and $H(m,q,r)$ is then obtained from H by a Type 2 exchange which results in (x,y) being exchanged for (x,z) and (z,w) being exchanged for (y,w). (Other exchanges may also be involved but this will not matter.) We then choose an integer i for which $x,y\in V_i$ and $z,w\in V_{i+1}$. Hence H contains both (x,w) and (y,z) but this implies that H contains nonidentical paths from x to w. The contradiction completes the proof. □

We are now ready to present the principal theorem of this paper.

Theorem 16. *Let $n \geqslant m \geqslant 2$. Then the maximum number $u(n, m)$ of edges in a m-locally unipathic subgraph of $\mathbf{T}_n$ is $h(m, q, r)$ where q and r are the unique integers satisfying $n = q(m-1)+r$ and $\lceil \frac{1}{2}(m-1)\rceil \leqslant r < \lceil \frac{3}{2}(m-1)\rceil$. Furthermore:*

(i) *If $m = 2$, then $\mathbf{T}_n$ itself is the unique m-locally unipathic subgraph of $\mathbf{T}_n$ having $u(n, m)$ edges.*

(ii) *If $n = m = 3$, then there are three m-locally unipathic subgraphs of $\mathbf{T}_n$ having $u(n, m)$ edges: $H(3, 0, 3)$, $\hat{H}(3, 0, 3)$, and $H_0 = \{(1, 2), (2, 3)\}$.*

(iii) *If $n \geqslant m \geqslant 3$, $r > \frac{1}{2}(m-1)$, and H is a m-locally unipathic subgraph of $\mathbf{T}_n$ having $u(n, m)$ edges, then either $H = H(m, q, r)$ or $H = \hat{H}(m, q, r)$.*

(iv) *If $n \geqslant m \geqslant 3$, $r = \frac{1}{2}(m-1)$, $(n, m) \neq (3, 3)$, and H is a m-locally unipathic subgraph of $\mathbf{T}_n$ having $u(n, m)$ edges, then either $H = H(m, q, r)$, $H = \hat{H}(m, q, r)$, $H = (m, q-1, r+m-1)$, or $H = \hat{H}(m, q-1, r+m-1)$.*

Proof. We first dispense of the case $m = 2$. In this case, we observe that $\mathbf{T}_n$ itself is the only 2-locally unipathic subgraph of $\mathbf{T}_n$ having $u(n, 2) = \binom{n}{2}$ edges, and the desired result follows since $\mathbf{T}_n = H(2, n-1, 1)$. We may also assume $(n, m) \neq (3, 3)$.

We then assume validity for all values of m with $m \leqslant p$ where p is some integer with $p \geqslant 2$ and consider the case $m = p+1$. In view of Theorem 2, we may assume $n > m$. Throughout the remainder of the argument, q and r will denote the unique integers for which $n = q(m-1)+r$ and $\lceil \frac{1}{2}(m-1)\rceil \leqslant r < \lceil \frac{3}{2}(m-1)\rceil$.

From this point on, we proceed with an indirect proof. We assume that the theorem is false and let $\mathscr{C}$ denote the set of all counterexamples, i.e., $\mathscr{C}$ is the set of all m-locally unipathic subgraphs of $\mathbf{T}_n$ having $u(n, m)$ edges other than the canonical graphs given in the statement of the theorem. We may then choose a counterexample $H \in \mathscr{C}$ which does not admit either a Type 1 or Type 2 exchange. To see that this is possible, we observe that each time an exchange of either Type 1 or Type 2 is performed, the sum of the lengths of the edges in the graph increases, but of course the number of edges remains the same. On the other hand, it follows that if we choose a graph $H \in \mathscr{C}$ for which the sum of the lengths of the edges in H is maximum, then H does not admit either a Type 1 or Type 2 exchange. Otherwise, the exchange would necessarily transform H into one of the canonical extremal graphs which is impossible by Lemma 15.

It is important to note that the counterexample H satisfies the following two properties.

P_1: If $1 \leqslant x < y < z \leqslant m$, $(x, y) \in H$, and $(w, z) \in H$, then $w < y$.

P_2: If $n-m+1 \leqslant x < y < z$, $(y, z) \in H$, and $(x, w) \in H$, then $y < w$.

We first establish P_1. Suppose to the contrary that $1 \leqslant x < y < z \leqslant m$, $(x, y) \in H$, (w, z) and $y \leqslant w$. Suppose first that $y = w$. Then H contains (x, y) and (y, z) and admits a Type 1 exchange. Now suppose $y < w$. If H contains either (x, w) or (y, z), it admits a Type 1 exchange and if H contains neither (x, w) or (y, z), then it admits a Type 2 exchange. This completes the proof of P_1. The proof of P_2 is dual and is therefore omitted.

At this point, we divide the remainder of the argument into four cases depending on the magnitude of r.

Case 1. $m \leq r < \lceil \frac{3}{2}(m-1) \rceil$.

Let $S_1 = S(n, m, 1)$, $S_2 = S(n, m, 1 + \lfloor \frac{1}{2}(r-m) \rfloor)$, $S_3 = S(n, m, r-m+1)$, and $S_4 = S(n, m, 1 + \lfloor \frac{1}{2} r \rfloor)$. Note that $|S_1| = |S_2| = |S_3| = q+2$ and $|S_4| = q+1$. For convenience, we also let $s_i = |S_i|$ for $i = 1, 2, 3, 4$. Then for $i = 1, 2, 3, 4$, let H_i be the restriction of H to $X_n - S_i$. It follows from Lemma that H_i is a $m-1$-locally unipathic subgraph of $\mathbf{T}_{n-q-2}$ for $i = 1, 2, 3$, and that H_4 is a $m-1$-locally unipathic subgraph of $\mathbf{T}_n - q - 1$.

We next observe that the equation $n = q(m-1) + r$, and the inequality $m \leq r < \lceil \frac{3}{2}(m-1) \rceil$ together imply that the following statements hold:

(a) $n - q - 2 = q(m-2) + r - 2$ and $\lceil \frac{1}{2}(m-2) \rceil \leq r - 2 < \lceil \frac{3}{2}(m-2) \rceil$.

(b) If $r \neq \frac{1}{2}(3m-4)$, then $n - q - 1 = q(m-2) + r - 1$ and $\lceil \frac{1}{2}(m-2) \rceil \leq r - 1 < \lceil \frac{3}{2}(m-2) \rceil$.

(c) If $r = \frac{1}{2}(3m-4)$, then $n - q - 1 = (q+1)(m-2) + r - m + 1)$ and $r - m + 1 = \frac{1}{2}(m-2)$.

It follows from the inductive hypothesis that $|H_i| \leq u(n-q-2, m-1) = h(m-1, q, r-2)$ for $i = 1, 2, 3$. If $r \neq \frac{1}{2}(3m-4)$, then $|H_4| \leq u(n-q-1, m-1) = h(m-1, q, r-1)$. On the other hand, if $r = \frac{1}{2}(3m-4)$, then $|H_4| \leq u(n-q-1, m-1) = h(m-1, q+1, r-m+1)$. But since $r - m + 1 = \frac{1}{2}(m-2)$, we have $|H_4| \leq h(m-1, q+1, r-m+1) = h(m-1, q, r-1)$. We conclude that $|H_4| \leq h(m-1, q, r-1)$ for all values of r treated in this case.

We now describe a method for partitioning each of the sets $H - H_i$ into three subsets. First, we let $L_i = (H - H_i) \cap P(n, m)$ for $i = 1, 2, 3, 4$. Then let a_i be the least integer in S_i and b_i the greatest integer in S_i. We define

$$E_i = H \cap (\{(x, a_i) : 1 \leq x < a_i\} \cup \{(b_i, y) : b_i < y \leq n\})$$

for $i = 1, 2, 3, 4$. Finally, we set $I_i = H - H_i - L_i - E_i$ for $i = 1, 2, 3, 4$. (We use the letters L, E, and I to suggest "long", "exterior", "interior" respectively.)

We now proceed to examine the number of edges in these sets. First, it is easy to see that L_i contains $\binom{s_i}{2} - (s_i - 1)$ edges with both endpoints in S_i. If $x \in X_n - S_i$ and $a_i < x < b_i$, then there are $s_i - 2$ edges in L_i having x as one of its endpoints. If $x \in X_n - S_i$ and either $x < a_i$ or $b_i < x$, then there are $s_i - 1$ edges in L_i having x as one of its endpoints. Therefore,

$$|L_i| = \binom{s_i}{2} - (s_1 - 1) + (s_i - 1)(m-2)(s_i - 2)$$
$$+ [n - (s_i - 1)(m-1) - 1](s_i - 1).$$

Second, we observe that it follows immediately from Lemma 11 that $|I_i| \leq (s_i - 1)(m-1)$.

We conclude that

$$\begin{aligned}|I_i \cup L_i| &\leq \binom{s_i}{2} - (s_i - 1) + (s_i - 1)(m-2)(s_i - 2)\\ &\quad + [n - (s_i - 1)(m-1) - 1](s_i - 1) + (s_i - 1)(m-1)\\ &= \binom{s_i}{2} + (n - s_i)(s_i - 1).\end{aligned}$$

The form of the preceding inequality is not surprising since it is immediate that

$$|I_i \cup L_i| = \binom{s_i}{2} + (n - s_i)(s_i - 1) \quad \text{if } H = h(m, q, r).$$

In this case, note that H contains each of the $\binom{s_i}{2}$ edges with both endpoints in S_i, and if $x \in X_n - S_i$, then there are $s_i - 1$ edges of $H - E_i - H_i$ joining x with a point of S_i.

We may combine these inequalities with the identities in Lemma 10 to obtain the following inequalities.

$$\begin{aligned}h(m, q, r) &\leq u(n, m) = |H| = |H_i| + |I_i \cup L_i| + |E_i|\\ &\leq h(m-1, q, r-2) + \binom{s_i}{2} + (n - s_i)(s_i - 1) + |E_i|\\ &\leq h(m-1, q, r-q) + \binom{q+2}{2} + (n-q-2)(q+1) + |E_i|\\ &\leq h(m, q, r) + |E_i| \quad \text{for} \quad i = 1, 2, 3.\\ h(m, q, r) &\leq u(n, m) = |H| = |H_4| + |I_4 \cup L_4| + |E_4|\\ &\leq h(m-1, q, r-1) + \binom{s_4}{2} + (n - s_4)(s_4 - 1) + |E_4|\\ &\leq h(m, q, r) + |E_4| - \lfloor \tfrac{1}{2} r \rfloor.\end{aligned}$$

We conclude from this that we must have $|E_4| \geq \lfloor \frac{1}{2} r \rfloor$. Now suppose that $|E_i| > 0$ for $i = 1, 2, 3$.

We note that $a_1 = 1$, $a_2 = 1 + \lfloor \frac{1}{2}(r-m) \rfloor$, $a_3 = 1 + r - m$, $a_4 = 1 + \lfloor \frac{1}{2} r \rfloor$, $b_1 = n - r + m$, $b_2 = n - \lceil \frac{1}{2}(r-m) \rceil$, $b_3 = n$, and $b_4 = n - \lceil \frac{1}{2} r \rceil + 1$. Since $|E_1| > 0$ and $|E_3| > 0$, we know that H contains an edge $e_1 = (n - r + m, j)$ where $n - r + m < j \leq n$ and edge $e_3 = (i, 1 + r - m)$ where $1 \leq i < 1 + r - m$. Since $|E_2| > 0$, we know that either H contains an edge $e_2 = (i', 1 + \lfloor \frac{1}{2}(r-m) \rfloor)$ where $1 \leq i' < 1 + \lfloor \frac{1}{2}(r-m) \rfloor$ or an edge $e_2' = (n - \lceil \frac{1}{2}(r-m) \rceil, j')$ where $n - \lceil \frac{1}{2}(r-m) \rceil < j' \leq n$. Now suppose that H contains an edge $e_2 = (i', 1 + \lfloor \frac{1}{2}(r-m) \rfloor)$. Since H satisfies property P_1, it follows that if $(x, a_4) \in E_4$, then $1 \leq x < 1 + \frac{1}{2}(r-m)$. Similarly, since H satisfies P_2, it follows that if $(b_4, y) \in E_4$, then $n - r + m < y \leq n$. It follows that $|E_4| \leq \lfloor \frac{1}{2}(r-m) \rfloor + r - m \lfloor \frac{3}{2}(r-m) \rfloor$ which is impossible since $|E_4| \geq \lfloor \frac{1}{2} r \rfloor$ and $\lfloor \frac{1}{2} r \rfloor > \lceil \frac{3}{2}(r-m) \rceil$. On the other hand, if H contains an edge $e_2' = (n - \lceil \frac{1}{2}(r-m) \rceil, j')$ then we would conclude that if $(x, a_4) \in E_4$, then $1 \leq x < 1 + r - m$, and if $(b_4, y) \in E_4$, then $n - \lceil \frac{1}{2}(r-m) \rceil < y \leq n$. It follows that $|E_4| \leq (r-m) + \lceil \frac{1}{2}(r-m) \rceil = \lceil \frac{3}{2}(r-m) \rceil$. As before, this is impossible since $|E_4| \geq \lfloor \frac{1}{2}r \rfloor$ and $\lfloor \frac{1}{2}r \rfloor > \lceil \frac{3}{2}(r-m) \rceil$.

The contradiction allows us to conclude that there must be some $i \in \{1, 2, 3\}$ for which $|E_i| = 0$. Since $r - 2 \neq \frac{1}{2}(m-2)$, we note that this implies in turn that $|H| = u(n, m) = h(m, q, r)$, $|H_i| = h(m-1, q, r-2)$, $|I_i \cup L_i| = \binom{q+2}{2} + (n-q-2) \times (q+1)$ and $|I_i| = (q+1)(m-1)$. Therefore, either $H_i = H(m-1, q, r-2)$ or $H_i = \hat{H}(m-1, q, r-2)$. In either case, it is easy to see that H_i contains $q+1$ edges of length one. Furthermore, if we choose an arbitrary consecutive pair $v_1, v_2 \in S_i$, then there exists a unique edge $(w, w+1) \in H_i$ so that $1 \leq v_1 < w < w+1 < v_2 = v_1 + m \leq n$. Since $|I_i| = (q+1)(m-1)$, it follows that H contains exactly $m-1$ edges from $\{(v_1, x) : v_1 < x \leq v_2\} \cup \{(y, v_2) : v_1 \leq y < v_2\}$. Thus if $v_1 < x < v_2$, then H must contain at least one of (v_1, x) and (x, v_2).

First, suppose that $v_1 < x \leq w$. We show that $(x, v_2) \in H$. To the contrary, assume $(x, v_2) \notin H$; then $(v_1, x) \in H$. Now $(x, w+1)$, $(w, w+1) \in H_i \cap H$ so $(v_1, w) \notin H$, $(w, v_2) \in H$, i.e., H contains $(w, w+1)$, $(w+1, v_2)$, and (w, v_2) which is a contradiction. We conclude that if $v_1 < x \leq w$, then $(x, v_2) \in H$.

A dual argument shows that if $w + 1 \leq y < v_2$, then $(v_1, y) \in H$. We now show that H contains (v_1, v_2). To the contrary, suppose that $(v_1, v_2) \notin H$. Then there exists an integer x with $v_1 < x < v_2$ for which H contains both (v_1, x) and (x, v_2). If $x \leq w$, then H contains (v_1, x), $(x, w+1)$, and $(v_1, w+1)$ which is a contradiction. Similarly, if $w + 1 \leq x$, then H contains (w, x), (x, v_2), and (w, v_2) which is also a contradiction. We conclude that $(v_1, v_2) \in H$.

In the above argument, v_1 and v_2 were an arbitrary consecutive pair from S_i so that we have determined the location of each of the $(q+1)(m-1)$ edges in I_i. Since $E_i = \emptyset$ and $L_i \subseteq P(n, m) \subseteq H$, it follows that if $H_i = H(m-1, q, r-2)$, then $H = H(m, q, r)$, and if $H_i = \hat{H}(m-1, q, r-2)$, then $H = \hat{H}(m, q, r)$. Of course, we have obtained a contradiction since the assumption that H was a counterexample has led to the conclusion that H was not a counterexample. With this observation, the proof of Case 1 is complete.

Case 2. $\frac{1}{2}(m-1) < r < m$.

In view of Lemma 14, we assume $(n, m) \neq (5, 4)$. Consider the three sets $S_1 = S(n, m, 1)$, $S_2 = S(n, m, \lceil \frac{1}{2}r \rceil)$, and $S_3 = S(n, m, r)$. Let $s_i = |S_i|$ and let a_i and b_i denote the least integer and the greatest integer in S_i respectively for $i = 1, 2, 3$. Note that $s_i = q + 1$ for $i = 1, 2, 3$. We then define for each $i = 1, 2, 3$ the subgraphs H_i, I_i, L_i, and E_i exactly as in Case 1. Since $\lceil \frac{1}{2}(m-2) \rceil \leq r - 1 < \lceil \frac{3}{2}(m-2) \rceil$, we know that the following inequality holds.

$$
\begin{aligned}
h(m, q, r) \leq u(n, m) &= |H| = |H_i| + |I_i \cup L_i| + |E_i| \\
&\leq h(m-1, q, r-1) + \binom{q+1}{2} + (n-q-1)q + |E_i| \\
&= h(m, q, r) + |E_i| - \lfloor \tfrac{1}{2}r \rfloor \quad \text{for } i = 1, 2, 3.
\end{aligned}
$$

In particular, we note that $|E_i| \geq \lfloor \frac{1}{2}r \rfloor$ for $i = 1, 2, 3$. If $|E_i| = \lfloor \frac{1}{2}r \rfloor$ for some $i \in \{1, 2, 3\}$, then we know that $H_i = H(m-1, q, r-1)$ or $H_i = \hat{H}(m-1, q, r-1)$ unless $r - 1 = \frac{1}{2}(m-2)$, in which case, we may also have $H_i = H(m-1, q-1, r-m-3)$ or $H_i = \hat{H}(m-1, q-1, r+m-3)$. However, it is easy to show that the requirement that $|E_i| \geq \lfloor \frac{1}{2}r \rfloor$ for $i = 1, 2, 3$ rules out this possibility. To see that this is true, we observe that if $H_i = H(m-1, q-1, r+m-3)$ or $H_i = \hat{H}(m-1, q-1, r+m-3)$ for some $i \in \{1, 2, 3\}$, then H_i contains an edge $(y, y+1)$ where either $y = \lfloor \frac{1}{2}(r+m-1) \rfloor$ or $y = \lceil \frac{1}{2}(r+m-1) \rceil$. Since $|E_3| > 0$, H contains an edge of the form (x, r) where $1 \leq x < r$. But this implies that H violates Property P_2, since $1 \leq x < r < y < y+1 \leq m$. We may therefore assume that either $H_i = H(m-1, q, r-1)$ or $H_i = \hat{H}(m-1, q, r-1)$ whenever $|E_i| = \lfloor \frac{1}{2}r \rfloor$ and $i \in \{1, 2, 3\}$.

Suppose first that $|E_1| = \lfloor \frac{1}{2}r \rfloor$. Then we must have $H_1 = H(m-1, q, r-1)$, for if r is even and $H_1 = \hat{H}(m-1, q, r-1)$, then H_1 contains each of the $\frac{1}{2}r - 1$ edges in the set $\{(n - \frac{1}{2}r + 1, x) : n - \frac{1}{2}r + 1 < x \leq n\}$. However, this implies that $|E_1| \leq \frac{1}{2}r - 1 < \lfloor \frac{1}{2}r \rfloor$ which is a contradiction. Since $H_1 = H(m-1, q, r-1)$, we know that H contains each of the $\lfloor \frac{1}{2}r \rfloor$ edges in the set $\{(n - \lfloor \frac{1}{2}r \rfloor, x) : n - \lfloor \frac{1}{2}r \rfloor < x \leq n\}$, and thus $E_1 = \{(n - r + 1, x) : n - \lfloor \frac{1}{2}r \rfloor < x \leq n\}$. The argument in Case 1 may now be applied to determine the edges in I_i and show that $H = H(m, q, r)$. We may therefore assume that $|E_1| > \lfloor \frac{1}{2}r \rfloor$. Dually, we may assume that $|E_3| > \lfloor \frac{1}{2}r \rfloor$.

It follows that E_2 contains no edges from the set $\{(x, \lceil \frac{1}{2}r \rceil) : 1 \leq x \leq \lceil \frac{1}{2}r \rceil\}$, for otherwise we would conclude that $|E_3| \leq \lfloor \frac{1}{2}r \rfloor$. Therefore $E_2 = \{(n - \lfloor \frac{1}{2}r \rfloor, x) : n - \lfloor \frac{1}{2}r \rfloor < x \leq n\}$ and $|E_2| = \lfloor \frac{1}{2}r \rfloor$. However, this in turn requires that $|E_1| \leq \lfloor \frac{1}{2}r \rfloor$ which is a contradiction. This completes the proof for Case 2.

Case 3. $\frac{1}{2}(m-1) = r$ and r is even.

First set $r = 2p$ and $m = 4p + 1$. We then consider the sets $S_1, S_2, \ldots, S_r$ where $S_i = S(n, m, i)$ and $s_i = q + 1$ for $i = 1, 2, \ldots, r$. Note that

$$u(n-q-1, m-1) = u(4pq + 2q - q - 1, 4p) = h(4p, q-1, 6p-2).$$

It follows that if $|E_i| = 0$ for some $i \in \{1, 2, \ldots, r\}$, then $H_i = H(4p, q-1, 6p-2)$ and the same argument used in Case 1 would allow us to conclude that

$$H = H(m, q-1, r+m-1) = H(4p+1, q-1, 6p).$$

We may therefore assume that $|E_i| > 0$ for $i = 1, 2, \ldots, r$.

Now consider the set $S_{r+1} = S(n, m, 3p)$. Since $s_{r+1} = q$ and

$$u(n-q, m-1) = u(4pq + 2p - q, 4p) = h(4p, q, 2p),$$

we conclude that $|E_{r+1}| \geq 2p$. If $|E_{r+1}| = 2p$, it follows easily that $H = H(m, q, r) = H(4p+1, q, 2p)$. We therefore assume that $|E_{r+1}| > r$.

Next suppose that for some $i \in \{1, 2, \ldots, r\}$, E_i contains an edge (x, i) where $1 \leq x < i$ and an edge $(n - r + i, y)$ where $n - r + i < y \leq n$. Then we would conclude that $|E_{r+1}| \leq (i-1) + r - i = r - 1$. The contradiction shows that for each

$i=1,2,3,\ldots,r$, we either have $E_i\subseteq\{(x,i):1\leqslant x<i\}$ or $E_i\subseteq\{(n-r+i,y):n-r+i<y\leqslant n\}$.

Similar reasoning shows that if H contains an edge of the form $(x,i+1)$ where $1\leqslant x\leqslant i$ and an edge of the form $(n-r+i,y)$ where $n-r+i<y\leqslant n$, then $|E_{r+1}|\leqslant r$. The contradiction shows that we must either have $E_i\subseteq\{(x,i):1\leqslant i<i\}$ for $i=1,2,\ldots,r$ or $E_i\subseteq\{(y,n-r+i):n-r+i<y\leqslant n\}$ for $i=1,2,\ldots,r$, neither of which is possible. The contradiction completes the proof of this case.

Case 4. $\frac{1}{2}(m-1)=4$, r is odd, $(n,m)\neq(3,3)$.

First set $r=2p+1$ and $m=4p+3$. As in Case 3, we consider the sets $S_1,S_2,\ldots,S_r$ where $S_i=S(n,m,i)$ and $s_i=q+1$ for $i=1,2,\ldots,r$. Note that

$$u(n-q-1,m-1)=u(4pq+2p+q,4p+2)=h(4p+2,q-1,6p+1).$$

It follows that if $|E_i|=0$ for some $i\in\{1,2,\ldots,r\}$, then either

$$H_i=H(4p+2,q-1,6p+1)\quad\text{or}\quad H_i=H(4p+2,q-1,6p+1).$$

Applying the argument used in the previous cases, we would conclude that either

$$H=H(4p+3,q-1,6p+3)\quad\text{or}\quad H=\hat{H}(4p+3,q-1,6p+3).$$

We therefore assume $|E_i|>0$ for $i=1,2,\ldots,r$.

Now consider the set $S_{r+1}=S(n,m,3p)$. Since $s_{r+1}=q$ and

$$u(n-q,m-1)=u(4pq+2p+q+1,4p+2)=h(4p+2,q,2p+1),$$

we conclude that $|E_{r+1}|\geqslant r$. If $|E_{r+1}|=r$, then it follows easily that

$$H=H(4p+3,q,2p+1)\quad\text{or}\quad H=\hat{H}(4p+3,q,2p+1).$$

We may therefore assume that $|E_{r+1}|>r$. The remainder of the case follows along the same lines as Case 3 and is therefore omitted. With this observation, the proof of our theorem is complete. □

5. The computation of rank

If P and Q are partial orders on a set X and $P\subseteq Q$, we say that Q is an *extension* of P. If Q is also a linear order, then we say Q is a linear extension. A well known theorem of Szpilrajn [7] asserts that if P is a partial order on a set X, then the collection $\mathcal{D}$ of all linear extensions of P is nonempty and $\bigcap\mathcal{D}=P$. A family $\mathcal{F}$ of linear extensions of a partial order P is called a *realizer* of P when $\bigcap\mathcal{F}=P$. A realizer $\mathcal{F}$ of P is said to be *irredundant* when $\bigcap\mathcal{S}\neq P$ for every proper subfamily $\mathcal{S}\subsetneq\mathcal{F}$. Dushnik and Miller [1] defined the *dimension* of a poset (X,P) as the smallest integer t for which there exists a realizer $\mathcal{F}=\{L_1,L_2,\ldots,L_t\}$ of P. Note that if (X,P) has dimension t and $\mathcal{F}=\{L_1,L_2,\ldots,L_t\}$ is a realizer of P, then $\mathcal{F}$ is irredundant. Maurer and Rabinovitch [2] defined the *rank* of (X,P) as the largest integer t for which there exists an irredundant realizer

$\mathcal{F}=\{L_1, L_2, \ldots, L_t\}$ of P and showed that while a n-element antichain has dimension two when $n \geq 2$, it has rank $\lfloor \frac{1}{4}n^2 \rfloor$ when $n \geq 4$. In [6], Rabinovitch and Rival gave a formula for the rank of a distributive lattice. In [3] and [4], Maurer, Rabinovitch, and Trotter developed a general theory of rank based on the graph theoretic concepts discussed in Section 2 of this paper. For the sake of completeness, we state here the principal results of this theory.

For $n \geq 0$, let $\underline{n}$ and $\bar{n}$ denote respectively an n-element chain and antichain. If $\mathbf{X}=(X, P)$ and $\mathbf{Y}=(Y, Q)$ are posets, we define $\mathbf{X}$ *join* $\mathbf{Y}$, denoted $\mathbf{X} \oplus \mathbf{Y}$, as the poset $(X \cup Y, P \cup Q \cup (X \times Y))$, i.e., in $\mathbf{X} \oplus \mathbf{Y}$, every element of X is greater than every element of Y. A poset (X, P) is said to be *rank degenerate* if there exist integers $n, m \geq 0$ such that (X, P) is isomorphic to a subposet of $\underline{n} \oplus \bar{3} \oplus \underline{m}$. The *width* of a poset (X, P) is the maximum number of points in an antichain contained in (X, P).

Theorem 17 [5]. *If (X, P) is rank degenerate, then* $\operatorname{rank}(X, P)=\operatorname{width}(X, P)$.

Theorem 18 [3]. *If (X, P) is not rank degenerate, then the rank of (X, P) equals the maximum number of edges in a $U_{\mathbf{P}}^*$ subgraph of $N_{\mathbf{P}}^*$.*

By combining Theorem 18 and Lemma 9, we can now compute the rank of the family of posets $\{\mathbf{X}(n, m): n \geq m \geq 2\}$. Note that $\mathbf{X}(n, n)=\bar{n}$ for $n \geq 2$ so rank $\mathbf{X}(2,2)=2$, rank $\mathbf{X}(3,3)=3$, and rank $\mathbf{X}(n, n)=\lfloor \frac{1}{4}n^2 \rfloor$ when $n \geq 4$.

Corollary 19. *Let $n>m \geq 2$. Then*

$$\operatorname{rank} \mathbf{X}(n, m)=h(m, q, r)-\binom{n-m+1}{2}$$

where $n=(m-1)q+r$ and $\lceil \frac{1}{2}(m-1) \rceil \leq r < \lceil \frac{3}{2}(m-1) \rceil$.

Proof. Note first that $\mathbf{X}(n, m)$ is not rank degenerate when $n>m$ so that by Theorem 18, the rank of $\mathbf{X}(n, m)$ equals $w(n, m)$, the maximum number of edges in a $U_{\mathbf{P}}^*$ subgraph of $N_{\mathbf{P}}^*$. In view of Lemma 9, we know that

$$w(n, m)=u(n, m)-\binom{n-m+1}{2},$$

and our conclusion follows from Theorem 16 since $u(n, m)=h(m, q, r)$. □

It is of particular interest to consider the special case of the preceding result which occurs when $n=2m$. The family $\{\mathbf{X}(2m, m): m \geq 1\}$ is a collection of posets of height one of particular combinatorial interest. First, the posets are interval orders of height one and secondly, $\mathbf{X}(2m, m)$ is the horizontal split of $\underline{m}$ (see [8] for definitions). $\mathbf{X}(2m, m)$ has dimension two for all $m \geq 2$, and we may examine Corollary 19 in detail to obtain a formula for the rank of $X(2m, m)$.

Corollary 20.

(i) rank $\mathbf{X}(4, 2) = 3$.
(ii) rank $\mathbf{X}(6, 3) = 7$.
(iii) rank $\mathbf{X}(8, 4) = 12$.
(iv) rank $\mathbf{X}(2m, m) = \lfloor \frac{1}{4}(3m^2 - 3) \rfloor$ *for* $m \geq 5$.

Proof. $\mathbf{X}(2m, m)$ is not rank degenerate when $m \geq 2$ so that rank $\mathbf{X}(2m, m) = w(2m, m)$, the maximum number of edges in a U_P^* subgraph of N_P^*. By Lemma 9, we know that

$$w(2m, m) = u(2m, m) - \binom{m+1}{2}.$$

It follows from Theorem 16 that $u(4, 2) = h(2, 2, 2) = 6$ so that

$$w(4, 2) = u(4, 2) - \binom{3}{2} = 6 - 3 = 3.$$

Similarly

$$w(6, 3) = u(6, 3) - \binom{4}{2} = h(3, 2, 2) - \binom{4}{2} = 13 - 6 = 7,$$

and

$$w(8, 4) - \binom{5}{2} = h(4, 2, 2) - \binom{5}{2} = 22 - 10 = 12.$$

On the other hand, when $m \geq 5$, $u(2m, m) = h(m, 1, m+1)$ so that

$$\begin{aligned} w(2m, m) &= h(m, 1, m+1) - \binom{m+1}{2} \\ &= (m-1)(m+1) + \lfloor \tfrac{1}{4}(m+1)^2 \rfloor - \binom{m+1}{2} \\ &= \lfloor \tfrac{1}{4}(3m^2 - 3) \rfloor. \quad \square \end{aligned}$$

Although we do not discuss the details here, it is relatively easy to establish the inequality rank $\mathbf{X}(2m, m) \geq \lfloor \frac{1}{4}(3m^2 - 3) \rfloor$ directly from the definition of rank. This is accomplished by explicitly constructing an irredundant realizer $\mathscr{F}$ for $\mathbf{X}(2m, m)$ with $|\mathscr{F}| = \lfloor \frac{1}{4}(3m^2 - 3) \rfloor$. The problem of establishing the reverse inequality, rank $\mathbf{X}(2m, m) \leq \lfloor \frac{1}{4}(3m^2 - 3) \rfloor$, served as the initial motivating force behind this paper.

6. Open problems

One of the obvious problems remaining to be solved is to investigate further the relationship between $u(n, m)$, the maximum number of edges in an m-locally unipathic subgraph of $\mathbf{T}_n$, and $\Delta(n, m)$, the maximum number of edges in an m-locally triangle free subgraph of $\mathbf{T}_n$. We recall that $\Delta(n, m) \geqslant u(n, m)$ for all $n \geqslant m \geqslant 2$ and that $\Delta(n, n) = u(n, n) = \lfloor \frac{1}{4}n^2 \rfloor$ while $\Delta(n, 2) = u(n, 2) = \binom{n}{2}$. On the other hand, it may happen that $\Delta(n, m) > u(n, m)$. For example, when $n = 9$ and $m = 8$, $u(9, 8) = 20$ and the only extremal graphs are the complete bipartite graphs $H(8, 0, 9)$ and $\hat{H}(8, 0, 9)$. However, it is straightforward to show that $\Delta(9, 8) = 21$ and that $\{(i, j) : 1 \leqslant i \leqslant 4, 5 \leqslant j \leqslant 8\} \cup \{(j, 9) : 5 \leqslant j \leqslant 8\} \cup \{(1, 9)\}$ is an extremal graph.

Several problems involving the digraphs of nonforcing pairs also arise naturally.

(1) What (acyclic) digraphs are the (acyclic) digraphs of nonforcing pairs of a poset?

(2) Characterize maximal and maximum U_P^* graphs.

(3) If $|X| = n$, characterize the set S of integers for which there exists a poset (X, P) so that for every $s \in S$, there exists a maximal U_P^* graph having s edges.

(4) Which posets have the property that every maximum U_P^* graph admits no Type 1 or Type 2 exchanges.

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