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STACKS AND SPLITS OF PARTIALLY ORDERED SETS

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The dimension of a partially ordered set (X, P) is the smallest positive integer t for which there exists a function f which assigns to each $x \in X$ a sequence $\{f(x)(i): 1 \leq i \leq t\}$ of real numbers so that $x \leq y$ in P if and only if $f(x)(i) \leq f(y)(i)$ for each $i = 1, 2, \ldots, t$. The interval dimension of (X, P) is the smallest integer t for which there exists a function F which assigns to each $x \in X$ a sequence $\{F(x)(i): 1 \leq i \leq t\}$ of closed intervals of the real line $\mathbf{R}$ so that $x < y$ in P if and only if $a < b$ in $\mathbf{R}$ for every $a \in F(x)(i)$, $b \in F(y)(i)$, and $i = 1, 2, \ldots, t$. For $t \geq 2$, a partially ordered set (poset) is said to be t-irreducible (resp. t-interval irreducible) if it has dimension t (resp. interval dimension t), and every proper subposet has dimension (resp. interval dimension) less than t. The only 2-irreducible poset is a two element anti-chain, and the only 2-interval irreducible poset is the free sum of two chains each having two points. In sharp contrast, the collection $\mathfrak{R}$ of all 3-irreducible posets consists of 9 infinite families and 18 odd examples, and the collection $\mathfrak{R}_I$ of all 3-interval irreducible posets is sufficiently complex to have avoided complete determination as of this date. Trotter and Moore determined $\mathfrak{R}$ from Gallai's forbidden subgraph characterization of comparability graphs. David Kelly independently determined $\mathfrak{R}$ by a lattice theoretic argument combined with the characterization of planar lattices Kelly and Ivan Rival had previously obtained. In this paper, we introduce a new operation called a stack which we will apply to posets of height one. In some ways the stack operation is an inverse of the split operation on posets previously defined by Kimble. These operations behave predictably with respect to dimension and interval dimension. In particular, the stack of a poset of height one plays a role in interval dimension theory which is analogous to the role played by the completion by cuts in dimension theory. As a consequence, we can exploit the similarities to Kelly's approach to the determination of $\mathfrak{R}$ to produce a relatively compact argument to determine the collection $\mathfrak{R}(I, 1)$ of all 3-interval irreducible posets of height one. This characterization problem has immediate combinatorial connections with a wide range of well-known forbidden subgraph problems including interval graphs, rectangle graphs, and circular arc graphs.

1. Introduction

Throughout this paper, we will use the notation and terminology of [13], [14] and [15] for partially ordered sets (posets), dimension, and interval dimension. For the sake of completeness we give here the central definitions, notations, and conventions. Formally a *poset* $\mathbf{X}$ consists of a pair (X, P) where X is a nonempty set (always finite in this paper) and P is a reflexive, antisymmetric, and transitive relation on X. P is called a *partial order* on X. The notations $(x, y) \in P$, xPy, and $x \leq y$ in $\mathbf{X}$ are used interchangeably. Similarly, we write $x < y$ in P or $x < y$ in $\mathbf{X}$ when $x \leq y$ in P and $x \neq y$. When neither (x, y) nor (y, x) is in P, we say x and y are incomparable and write xIy in P or xIy in $\mathbf{X}$. We denote the set of all

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incomparable pairs by I_P. A poset (X, P) is called an antichain if xIy in P for every $x, y \in X$ with $x \neq y$.

A partial order L is called a *linear order* (also *total order* or *simple order*) when $I_L = \emptyset$ and the poset (X, L) is called a *chain*. We denote an n-element chain by $\mathbf{n}$. If P and Q are partial orders on a set X and $P \subset Q$, then Q is called an *extension* of P. A linear order L which is an extension of P is called a *linear extension of P*. A theorem of Szpilrajn's [8] asserts that for any partial order P, the collection $\mathscr{C}$ of all linear extensions of P is nonempty and that $\bigcap \mathscr{C} = P$. The *dimension* [2] of a poset (X, P), denoted $\mathrm{Dim}(X, P)$ or $\mathrm{Dim}(\mathbf{X})$, is the smallest positive integer t for which there exist linear extensions $L_1, L_2, \ldots, L_t$ of P whose intersection is P. A poset has dimension one if and only if it is a chain.

Alternately, the dimension [7] of (X, P) is the smallest positive integer t for which there exists a function f which assigns to each $x \in X$ a sequence $f(x)(1)$, $f(x)(2), \ldots, f(x)(t)$ of real numbers so that $x \leq y$ in P if and only if $f(x)(i) \leq f(y)(i)$ in $\mathbf{R}$ for $i = 1, 2, \ldots, t$. The function P is called an *embedding* of (X, P) in $\mathbf{R}^t$.

We denote by $\bar{R}$, the transitive closure of a relation R. Now let (X, P) be a poset. Then a set $\{(b_i, a_i)\colon 1 \leq i \leq m\} \subset I_P$ is called a *TM-cycle of length m* when $(a_i, b_j) \in P$ if and only if $j = i + 1$ (cyclically) for $i = 1, 2, \ldots, m$. In [14] Trotter and Moore prove that if $S \subset I_P$, then $\overline{P \cup S}$ is a partial order on X if and only if S contains no TM-cycles. Therefore, we can define the dimension of a poset (X, P) which is not a chain as the smallest positive integer t for which there exists a partition $I_P = S_1 \cup S_2 \cup \cdots \cup S_t$ so that no S_i contains a TM-cycle.

If (X, P) is a poset and Y is a nonempty subset of X, then the *restriction of P to* Y, denoted $P(Y)$, is defined by $P(Y) = P \cap (Y \times Y)$. It is clear that $P(Y)$ is a partial order on Y, and we say $(Y, P(Y))$ is a *subposet* of (X, P). Obviously $\mathrm{Dim}(Y, P(Y)) \leq \mathrm{Dim}(X, P)$ whenever $\emptyset \neq Y \subset X$. The subposet $(Y, P(Y))$ is called a proper subposet of (X, P) when $\emptyset \neq Y \neq X$. For an integer $t \geq 2$, a poset (X, P) is *t-irreducible* if it has dimension t but every proper subposet has dimension less than t. Trivially, the only 2-irreducible poset is a two element antichain.

For a poset $\mathbf{X} = (X, P)$ and a point $x \in X$, we denote by $\mathbf{X} - x$ the subposet $(X - \{x\}, P(X - \{x\}))$. A poset $\mathbf{X} = (X, P)$ is then t-irreducible if $\mathrm{Dim}(X) = t$ and $\mathrm{Dim}(\mathbf{X} - x) < t$ for every $x \in X$. By convention, a one point poset is the only 1-irreducible poset.

If (X, P) and (Y, Q) are posets, then we say (X, P) is isomorphic to (Y, Q) when there exists a function $f\colon X \xrightarrow[\text{onto}]{1\text{-}1} Y$ so that $(x_1, x_2) \in P$ if and only if $(f(x_1), f(x_2)) \in Q$. In this paper, we do not distinguish between isomorphic posets and write $\mathbf{X} = \mathbf{Y}$ when $\mathbf{X}$ and $\mathbf{Y}$ are isomorphic. Similarly, we say $\mathbf{Y}$ is contained in $\mathbf{X}$ (or $\mathbf{X}$ contains $\mathbf{Y}$) and write $\mathbf{Y} \subset \mathbf{X}$ when $\mathbf{Y}$ is isomorphic to a subposet of $\mathbf{X}$.

If $I_1 = [u_1, v_1]$ and $I_2 = [u_2, v_2]$ are closed intervals of $\mathbf{R}$, we say I_1 is *dominated* by I_2, and write $I_1 \lhd I_2$, when $v_1 < u_2$ in $\mathbf{R}$. Trotter and Bogart [13] defined the *interval dimension* of a poset (X, P) denoted $\mathrm{IDim}(X, P)$ as the smallest positive integer t for which there exists a function F which assigns to each $x \in X$ a sequence $F(x)(1), F(x)(2), \ldots, F(x)(t)$ of closed intervals of the real line so that for distinct points $x, y \in X$, we have $x < y$ in P if and only if $F(x)(i) \lhd F(y)(i)$ for $i = 1, 2, \ldots, t$. The function F is called a *t-interval representation of* (X, P). A poset with interval dimension one is called an *interval order*. For $t \geq 2$, a poset is said to be *t-interval irreducible* if it has interval dimension t and every proper subposet has interval dimension less than t. By convention, a one point poset is considered to be 1-interval irreducible. One of the central goals of this paper is to determine the collection $\mathfrak{R}(I, 1)$ of all 3-interval irreducible posets of height one.

If P is a partial order on a set X, we denote by P^d the partial order on X defined by $P^d = \{(y, x): (x, y) \in P\}$. P^d is called the *dual* of P. When $\mathbf{X} = (X, P)$, we let $\mathbf{X}^d$ denote (X, P^d). It is easy to see that $\mathrm{Dim}(\mathbf{X}) = \mathrm{Dim}(\mathbf{X}^d)$ and $\mathrm{IDim}(\mathbf{X}) = \mathrm{IDim}(\mathbf{X}^d)$ in every poset $\mathbf{X}$.

A *fence* in a poset (X, P) is a sequence $x_1, x_2, \ldots, x_n$ of points ($n \geq 2$) from X with $(x_i, x_j) \notin I_P$ if and only if $|i - j| \leq 1$. The length of the fence is $n - 1$ and x_1 and x_n are called the end-points. A poset $\mathbf{X}$ is said to be *connected* if for every pair x, y of distinct points, there exists a fence in $\mathbf{X}$ with endpoints x and y. The distance in $\mathbf{X}$ from x to y is the minimum length of a fence in $\mathbf{X}$ with endpoints x and y. If $\mathbf{X}$ is not connected, then a connected subposet $\mathbf{F}$ of $\mathbf{X}$ is a *component* of $\mathbf{X}$ when the only connected subposet of $\mathbf{X}$ containing $\mathbf{F}$ is $\mathbf{F}$ itself. A disconnected poset $\mathbf{X}$ is the free sum of its components $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_t$ and we write $\mathbf{X} = \mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_t$.

Fishburn proved [3] that the only 2-interval irreducible poset is the disconnected poset $\mathbf{2} + \mathbf{2}$. However for $t \geq 3$, every t-irreducible and every t-interval irreducible poset is connected [13].

The height of a poset is one less than the maximum number of points in a chain contained in the poset. For any poset $\mathbf{X}$, we have $\mathrm{IDim}(\mathbf{X}) \leq \mathrm{Dim}(\mathbf{X})$, but there exist interval orders of arbitrarily large dimension [1]. Such posets must have large height. For posets of height one, there cannot be a great disparity between dimension and interval dimension.

Lemma 1. *If* (X, P) *is a poset of height one, then* $\mathrm{IDim}(X, P)$ *is the smallest positive integer t for which there exists a family* $\{L_1, L_2, \ldots, L_t\}$ *of linear extensions of P so that if x is a maximal element and y is a nonmaximal element with xIy in P, then there is at least one i for which* $(x, y) \in L_i$.

Proof. Suppose first that $\mathrm{IDim}(\mathbf{X}) = t$ and let F be an interval coordinatization of length t for $\mathbf{X}$. For each $i \leq t$, let $S_i = \{(x, y) \in I_P$: x is a maximal, y is nonmaximal, and $F(y)(i) \not\lhd F(x)(i)\}$. Now suppose that for some $i \leq t$, S_i contains a TM-cycle $\{(x_j, y_j)$: $1 \leq j \leq m\}$ of length m. The poset (X, Q) where $Q = \{(x, x): x \in X\} \cup \{(x, y) \in X \times X$: $F(x)(i) \lhd F(y)(i)\}$ is an interval order but the subposet of (X, Q) determined by x_1, x_2, y_1, and y_m is $\mathbf{2} + \mathbf{2}$. The contradiction shows that no S_i contains a TM-cycle. Therefore for each $i \leq t$, there exists a linear extension L_i of $\overline{P \cup S_i}$. These linear extensions clearly meet the requirements of our theorem.

On the other hand, suppose that we have a family $\{L_1, L_2, \ldots, L_s\}$ of linear extensions of P so that for every incomparable pair x, y with x maximal and y nonmaximal, there exists an $i \leq s$ so that $(x, y) \in L_i$. We show that $\mathrm{IDim}(\mathbf{X}) \leq s$. For each $i \leq s$, let $W_i = \{(y, x) \in I_P \cap L_i$: x is maximal and y is nonmaximal$\}$. It is easy to see that no W_i contains a TM-cycle, and that the poset (X, Q_i) where $Q_i = \overline{P \cup W_i}$ is an interval order. Then for each $i \leq s$, we choose intervals $\{F(x)(i)$: $x \in X\}$ which provide a representation of the interval order (X, Q_i). The function F so constructed is an interval representation of length s for (X, P).

Lemma 2. *If* (X, P) *is a poset of height one, then* $\mathrm{Dim}(X, P) \leq 1 + \mathrm{IDim}(X, P)$.

Proof. Suppose $\mathrm{IDim}(X, P) = t$ and choose linear extension $L_1, L_2, \ldots, L_t$ as described in Lemma 1. Now let L_1 order the maximal elements of (X, P) by $x_1 < x_2 < \cdots < x_m$ and the nonmaximal elements $y_1 < y_2 < \cdots < y_n$. Then let L_{t+1} be the order $y_n < y_{n-1} < \cdots < y_2 < y_1 < x_m < x_{m-1} < \cdots < x_2 < x_1$. Clearly $P = L_1 \cap L_2 \cap \cdots \cap L_t \cap L_{t+1}$ so that $\mathrm{Dim}(X, P) \leq t + 1$.

Let $\mathbf{X} = (X, P)$ be a poset of height one. Then let B denote the set of maximal elements and A the set of minimal elements. Note that $X = A \cup B$, but that we may have $A \cap B \neq \emptyset$. We associate with $\mathbf{X}$ an indexed family $\mathcal{F}_{\mathbf{X}}$ of subsets of A defined by $\mathcal{F}_{\mathbf{X}} = \{U(b)$: $b \in B\}$ where $U(b) = \{a \in A$: $a < b$ in $P\}$. Note that it is possible for $U(b_1) = U(b_2)$ when b_1 and b_2 are distinct elements of B. Also note that if $b \in A \cap B$, then $U(b) = \emptyset$ For example, we associate with the poset $\mathbf{X}$ shown in Fig. 1 the family $\mathcal{F}_{\mathbf{X}} = \{U(i)$: $1 \leq i \leq 5\}$ where $U(1) = U(2) = \emptyset$, $U(3) = U(4) = \{6, 7, 8\}$ and $U(5) = \{8\}$.

Conversely, if $\mathcal{F} = \{U(b)$: $b \in B\}$ is any indexed family of sets, we associate with $\mathcal{F}$ a poset $\mathbf{X}_{\mathcal{F}}$ of height one whose maximal elements are the elements of B and whose minimal elements are $\{b \in B$: $U(b) = \emptyset\} \cup (\bigcup \mathcal{F})$. The partial order on $\mathbf{X}_{\mathcal{F}}$ is defined by $a < b$ in $\mathbf{X}_{\mathcal{F}}$ if and only if $a \in U(b)$ for every $a \in \bigcup \mathcal{F}$, $b \in B$. For example, let $\mathcal{F} = \{U(i)$: $1 \leq i \leq 8\}$ where $U(1) = \{1, 2, 3, 4\}$, $U(2) = \{1, 3, 4, 5\}$, $U(3) = \{4, 5\}$, $U(4) = \{2, 5\}$, $U(5) = \{2, 3\}$, $U(6) = U(7) = \{4, 5\}$, and $U(8) = \emptyset$. Then the poset $\mathbf{X}_{\mathcal{F}}$ is shown in Fig. 2.

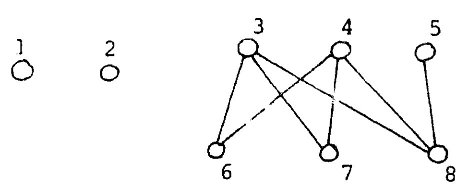


Fig. 1.

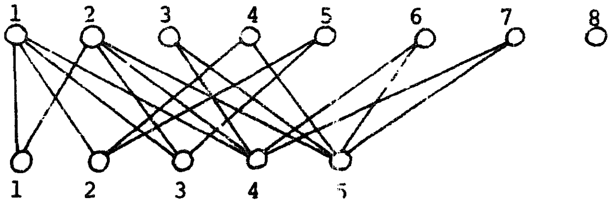


Fig. 2.

2. Splits and stacks

Kimble [6] defined the *split* of a poset $\mathbf{X} = (X, P)$, denoted Split($\mathbf{X}$), as the poset of height one with maximal elements $\{x': x \in X\}$ and minimal elements $\{x'': x \in X\}$ with $x'' < y'$ in Split($\mathbf{X}$) if and only if $x \leq y$ in $\mathbf{X}$.

Theorem 3 [15]. Dim($\mathbf{X}$) = IDim(Split($\mathbf{X}$)) *for every poset* $\mathbf{X}$.

Proof. Let P and Q denote the partial orders on $\mathbf{X}$ and Split($\mathbf{X}$) respectively. Suppose first that IDim(Split($\mathbf{X}$)) $= t$ and let $L_1, L_2, \ldots, L_t$ be the linear orders guaranteed by Lemma 1. For each $i \leq t$, let $S_i = \{(x, y) \in I_P: (x', y'') \in L_i\}$. Then it is easy to see that $I_P = S_1 \cup S_2 \cup \cdots \cup S_t$ and that no S_i contains a TM-cycle, i.e., Dim $\mathbf{X} \leq t$.

Similarly if Dim($\mathbf{X}$) $= s$ and $I_P = S_1 \cup S_2 \cup \cdots \cup S_s$ is a partition so that no S_i contains a TM-cycle, then we define for each $i \leq s$, $W_i = \{(y', x'') \in I_Q: (y, x) \in S_i\}$. It follows easily that no W_i contains a TM-cycle so that we may choose for each $i \leq s$ a linear extension L_i of $\overline{Q \cup W_i}$. These extensions clearly satisfy the requirements of Lemma 1, and thus IDim(Split($\mathbf{X}$)) $\leq s$.

The reader should note that if $\mathbf{X}$ is a poset of arbitrary height and $\mathbf{Y}$ = Split($\mathbf{X}$), then $\mathbf{Y}$ contains twice as many points as $\mathbf{X}$, but IDim ($\mathbf{Y}$) = Dim($\mathbf{X}$), and therefore Dim($\mathbf{Y}$) $\leq 1 +$ Dim($\mathbf{X}$). We now describe a construction for proceeding in the reverse direction. We begin with a poset $\mathbf{X}$ of height one and associate with $\mathbf{X}$ a poset $\mathbf{Y}$ (to be called the stack of $\mathbf{X}$), which may have arbitrarily large height, so that IDim($\mathbf{X}$) = Dim($\mathbf{Y}$). However, in this case, $\mathbf{X}$ and $\mathbf{Y}$ will contain the same number of points.

Let $\mathbf{X} = (X, P)$ be a poset of height one. Then let B denote the set of nonminimal elements and A the set of nonmaximal elements. Also let L be an arbitrary linear order on X. For each $a \in A$, let $G(a) = \{b \in B: a < b \text{ in } P\}$, and for each $b \in B$, let $U(b) = \{a \in A: a < b \text{ in } P\}$ Then define an extension Q of P by:

$$\begin{aligned} Q = P &\cup \{(b_1, b_2) \in B \times B: U(b_1) \subsetneq U(b_2)\} \\ &\cup \{(a_1, a_2) \in A \times A: G(a_2) \subsetneq G(a_1)\} \\ &\cup \{(b, a) \in (B \times A) \cap I_p: a' < b' \text{ in } P \text{ for every } b' \in G(a), a' \in U(b)\} \\ &\cup \{(b_1, b_2) \in (B \times B) \cap L: U(b_1) = U(b_2)\} \\ &\cup \{(a_1, a_2) \in (A \times A) \cap L: G(a_1) = G(a_2)\}. \end{aligned}$$

The poset (X, Q) is called the *stack* of X, P and is denoted $\text{Stack}(X, P)$ or $\text{Stack}(\mathbf{X})$.

The definition of the stack of a poset $\mathbf{X} = (X, P)$ requires us to choose an arbitrary linear order L on X for the purpose of 'breaking ties'. The linear order L does not have to be a linear extension of P. Furthermore, it is obvious that the posets determined by different choices of L are isomorphic, and since we choose not to distinguish between isomorphic posets in this paper, we also do not indicate that the stack of a poset depends on the linear order L.

We will find it convenient to extend this construction to posets of height zero (antichains) by defining $\text{Stack}(\mathbf{X}) = \mathbf{X}$ when $\mathbf{X}$ is an antichain. We now pause to summarize some elementary properties of stacks. Let $\mathbf{X} = (X, P)$ be a poset of height one and let $\text{Stack}(\mathbf{X}) = (X, Q)$. Let B denote the set of nonminimal elements and let A denote the set of nonmaximal elements. Then the following statements hold:

(i) $\mathbf{X} \subset \text{Split}(\text{Stack}(\mathbf{X}))$;

(ii) $P \cap (A \times B) = Q \cap (A \times B)$;

(iii) $\text{Stack}(\mathbf{X})^d = \text{Stack}(\mathbf{X}^d)$;

(iv) If $\mathbf{X} = \mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_t$, then $\text{Stack}(\mathbf{X}) = \text{Stack}(\mathbf{F}_1) + \text{Stack}(\mathbf{F}_2) + \cdots + \text{Stack}(\mathbf{F}_t)$.

The reader should note that we always have $P \cap (B \times A) = \emptyset$ but it is possible for $Q \cap (B \times A)$ to be nonempty. Furthermore, the stack of a connected poset $\mathbf{X} = (X, P)$ can be defined in terms of extensions. Let $\mathscr{C}$ denote the collection of all extensions P' of P which satisfy the property that $P \cap (A \times B) = P' \cap (A \times B)$. If we partial order $\mathscr{C}$ by inclusion, then the partial Q on X in $\text{Stack}(X, P') = (X, Q)$ is a maximal element in $(\mathscr{C}, \subset)$. Moreover, any two maximal elements Q_1, Q_2 in $(\mathscr{C}, \subset)$ generate isomorphic posets (X, Q_1) and (X, Q_2).

To illustrate the preceding definition, we present diagrams of a poset and its stack in Fig. 3.

The reader may enjoy the task of verifying directly that for the poset $\mathbf{X}$ shown in Fig. 3, we have $\text{IDim}(\mathbf{X}) = \text{Dim}(\text{Stack}(\mathbf{X})) = 2$, but that $\text{Dim}(\mathbf{X}) = 3$.

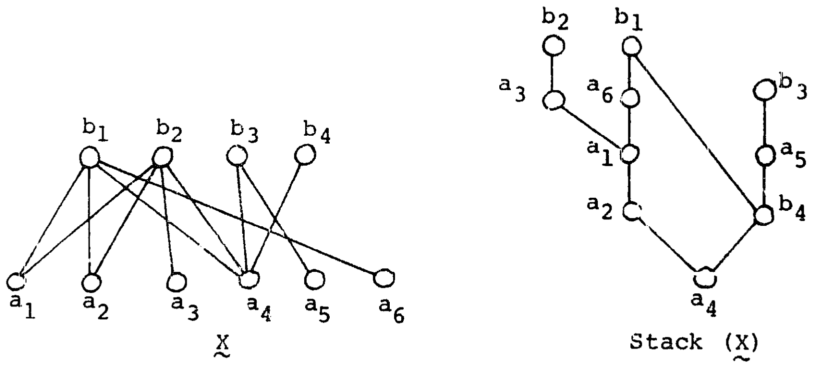


Fig. 3.

Let $\mathbf{X}=(X,P)$ be a poset, and let $A_1, A_2 \subset X$. We say that A_1 is *incomparable* to A_2 in $\mathbf{X}$ and write $A_1 I A_2$ when $a_1 I a_2$ for every $a_1 \in A_1$, $a_1 \in A_2$.

Lemma 4. *Let* $\mathbf{X}=(X,P)$ *be a connected poset of height one and let* $x_1, x_2 \in X$ *with* $x_1 I x_2$ *in* Stack($\mathbf{X}$). *Then there exist two 2-element chains* $C_1, C_2 \subset X$ *so that* $x_1 \in C_1$, $x_2 \in C_2$, *and* $C_1 I C_2$ *in both* $\mathbf{X}$ *and* Stack($\mathbf{X}$).

Proof. Suppose first that x_1 and x_2 are maximal elements. Then since $x_1 I x_2$ in Stack($\mathbf{X}$), we must have $U(x_1) \not\subset U(x_2)$ and $U(x_2) \not\subset U(x_1)$. Then there exist minimal elements $y_1, y_2 \in X$ with $y_1 \leq x_1$ in $\mathbf{X}$, $y_1 \subset x_2$ in $\mathbf{X}$, $y_2 < x_2$ in $\mathbf{X}$, and $y_2 \not< x_1$ in $\mathbf{X}$. Then set $C_1=\{x_1, y_1\}$ and $C_2=\{x_1, y_2\}$. It is easy to see that C_1 and C_2 are chains and that $C_1 I C_2$ in $\mathbf{X}$ and Stack($\mathbf{X}$).

The argument is dual when x_1 and x_2 are both minimal elements of $\mathbf{X}$. Now suppose that x_1 is a maximal element of $\mathbf{X}$ and y_1 is a minimal element of $\mathbf{X}$. Since $x_1 I y_1$ in Stack($\mathbf{X}$), we know that there exists a minimal element $y_2 \in X$ and a maximal element $x_2 \in X$ so that $x_1 > y_2$ in $\mathbf{X}$, $x_2 > y_1$ in $\mathbf{X}$, and $x_2 I y_2$ in $\mathbf{X}$. Then set $C_1=\{x_1, y_2\}$ and $C_2=\{x_2, y_1\}$. It follows that C_1 and C_2 are chains and that $C_1 I C_2$ in $\mathbf{X}$ and Stack($\mathbf{X}$).

With the assistance of the preceding lemma, we can now prove that the phenomena observed for the poset in Fig. 3 is representative of the general situation.

Theorem 5. *If* $\mathbf{X}$ *is a poset of height one, then* IDim($\mathbf{X}$) = Dim(Stack($\mathbf{X}$)) *unless* $\mathbf{X}$ *is a disconnected interval order in which case* IDim($\mathbf{X}$) = 1 *and* Dim(Stack($\mathbf{X}$)) = 2.

Proof. Since $\mathbf{X} \subset$ Split(Stack($\mathbf{X}$)), we know that IDim($\mathbf{X}$) $\leq$ IDim(Split(Stack($\mathbf{X}$))) = Dim(Stack($\mathbf{X}$)), so it remains only to show that Dim(Stack($\mathbf{X}$)) $\leq$ IDim($\mathbf{X}$) except when $\mathbf{X}$ is a disconnected interval order. To dispense of the exceptional case, we note that $\mathbf{X}$ is a disconnected interval order and $\mathbf{X}=\mathbf{F}_1+\mathbf{F}_2+\cdots+\mathbf{F}_t$ is the decomposition of $\mathbf{X}$ into components, then IDim($\mathbf{F}_i$) = 1 and Dim($\mathbf{F}_i$) ≤ 2 for $i=1,2,\ldots,t$. Thus IDim($\mathbf{X}$) = 1 and Dim($\mathbf{X}$) = 2. So in what follows we will assume that $\mathbf{X}$ is not a disconnected interval order. It follows from Property (iv) that we may then assume that $\mathbf{X}$ is connected; otherwise, we apply the following argument to the components of $\mathbf{X}$.

Let $\mathbf{X}=(X,P)$ and Stack($\mathbf{X}$) = (X,Q). Also let B denote the set of maximal elements of $\mathbf{X}$, and let A denote the set of minimal elements of $\mathbf{X}$. Then suppose that IDim($\mathbf{X}$) = t and let $L_1, L_2, \ldots, L_t$ be linear extensions of $\mathbf{X}$ so that $(B \times A) \cap I_p \subset L_1 \cup L_2 \cup \cdots \cup L_t$ as provided by Lemma 1. Next, for each $i=1,2,\ldots,t$, let $S_i = L_i \cap (B \times A) \cap I_Q$. Then suppose that for some $i \leq t$, S_i contains a TM-cycle $\{(b_j, a_j)\colon 1 \leq j \leq m\}$ with respect to Q. Since Q is an extension of P, we know that $\{(b_j, a_j)\colon 1 \leq j \leq m\} \subset I_P$. Furthermore, since $P \cap (A \times B) = Q \cap (A \times B)$, we know that $(a_j, b_k) \in P$ if and only if $k = j+1$ (cyclically). However, these statements imply that $\{(b_j, a_j): 1 \leq j \leq m\}$ is a TM-cycle with respect to P, which is a contradiction since L_i is a linear extension of P. We conclude S_i contains no TM-cycle with respect to Q, and thus for each $i \leq t$, we may choose a linear extension M_i of $\overline{Q \cup S_i}$.

We now show that $Q = M_1 \cap M_2 \cap \cdots \cap M_t$. It suffices to choose an arbitrary pair $(x, y) \in I_Q$ and show that there exists an integer $i \leq t$ so that $(x, y) \in M_i$. To see that this is indeed true, we choose two 2-elements chains $C_1, C_2 \subset X$ so that $x \in C_1$, $y \in C_2$, and $C_1 I C_2$ in $\mathbf{X}$ and Stack($\mathbf{X}$). Then let b denote the maximal element of $\mathbf{X}$ in C_1, and let a denote the minimal element of $\mathbf{X}$ in C_2. Then there exists an integer i so that $(b, a) \in L_i$. Hence $(b, a) \in S_i$ and $(b, a) \in M_i$. Furthermore $x \leq b < a \leq y$ in M_i, and the proof is complete.

In [9], the author defined for $n \geq 3$ and $k \geq 0$, the (generalized) crown $\mathbf{S}_n^k$ as the poset of height one with maximal elements $x_1, x_2, \ldots, x_{n+k}$ and minimal elements $y_1, y_2, \ldots, y_{n+k}$ with $y_i < x_j$ if and only if $j \neq 1, i+1, \ldots, i+k$ (cyclically). The dimension of the crown $\mathbf{S}_n^k$ is given by the formula $\mathrm{Dim}(\mathbf{S}_n^k) = \lceil 2(n+k)/(k+2) \rceil$. It is easy to see that a poset $\mathbf{X}$ contains a crown $\mathbf{S}_n^k$ if and only if $\mathbf{S}_n^k$ is contained in the split of $\mathbf{X}$. Furthermore, if $\mathbf{X}$ is a poset of height one and contains a crown $\mathbf{S}_n^k$, then the stack of $\mathbf{X}$ also contains $\mathbf{S}_n^k$. Although the converse of this statement is false, we can still salvage a weaker result which is sufficient for our purposes.

Following the notation of Kelly [5], we use the symbol $\mathbf{A}_n$ to denote the poset $\mathbf{S}_3^n$. The maximal elements in $\mathbf{A}_n$ are labeled $b_1, b_2, \ldots, b_n$ and the minimal elements are labeled $a_1, a_2, \ldots, a_n$ with $a_i < b_i$ and $a_i < b_{i-1}$ (cyclically) in $\mathbf{A}_n$. Note that $\mathrm{Dim}(\mathbf{A}_n) = \mathrm{IDim}(\mathbf{A}_n) = 3$. Furthermore, since $\mathbf{A}_n - x$ is a fence, which has dimension 2, for every $x \in \mathbf{A}_n$, we observe that $\mathbf{A}_n$ is 3-interval irreducible and 3-irreducible.

Theorem 6. *If $\mathbf{X}$ is a poset of height one and* Stack($\mathbf{X}$) *contains a crown $\mathbf{A}_n$ for some $n \geq 0$,* then there exists an integer *m with $0 \leq m \leq n$ so that $\mathbf{X}$ contains the crown $\mathbf{A}_m$.*

Proof. Choose the smallest integer $k \geq 0$ for which Stack($\mathbf{X}$) contains $\mathbf{A}_k$. We will then show that $\mathbf{X}$ also contains $\mathbf{A}_k$.

Of all of the copies of $\mathbf{A}_k$ contained in Stack($\mathbf{X}$), choose one for which the integer $t = |\{b: 1 \leq i \leq k, b_i \text{ is nonminimal in } \mathbf{X}\}| + |\{a_i: 1 \leq i \leq k, a_i \text{ is nonmaximal in } \mathbf{X}\}|$ is as large as possible. If $t = 2k+6$, then the points in this copy of $\mathbf{A}_k$ in Stack($\mathbf{X}$) also form a copy of $\mathbf{A}_k$ in $\mathbf{X}$. So we may assume that $t < 2k+6$. In view of the duality of the stack, it is clear that we may assume without loss of generality that b_{k+1} is not a maximal element in $\mathbf{X}$. Since $a_1 I b_{k+2}$ in Stack($\mathbf{X}$), it follows from Lemma 4 that there exists a maximal element x in $\mathbf{X}$ so that $x > b_{k+2}$ and $x I a_1$ in both $\mathbf{X}$ and Stack($\mathbf{X}$). If $x I a_i$ in Stack($\mathbf{X}$) for each $i = 2, 3, \ldots, k+1$, then x can replace b_{k+2} in this copy of $\mathbf{A}_k$. So we must have that $x > a_i$ in Stack($\mathbf{X}$) for some i with $2 \leq i \leq k+1$. Choose the smallest integer i_0 so that $2 \leq i_0 \leq k+1$ and $x > a_{i_0}$

in Stack($\mathbf{X}$). Then it follows that the subposet of Stack($\mathbf{X}$) generated by $\{a_1, a_2, \ldots, a_{i_0}, a_{k+3}\} \cup \{b_1, b_2, \ldots, b_{i_0-1}, x, b_{k+3}\}$ is $\mathbf{A}_{i_0-2}$. But since $i_0 \leq k+1$, we also have $i_0 - 2 < k$ which is a contradiction.

The reader may easily verify that if $\mathbf{X}$ is the poset shown in Fig. 2, then $\mathbf{X}$ contains $\mathbf{A}_0$ but not $\mathbf{A}_1$. On the other hand, the stack of $\mathbf{X}$ contains both $\mathbf{A}_0$ and $\mathbf{A}_1$.

For a poset $\mathbf{X}$ and a point $x \in \mathbf{X}$, we denote by $I(x; \mathbf{X})$ the subposet $\{y \in X: xIy$ in $\mathbf{X}\}$. The following result will play a central role in our characterization theorem for 3-irreducible posets of height one.

Theorem 7. *Let $\mathbf{X}$ be a poset of height one and suppose that $\mathbf{X}$ does not contain the crown $\mathbf{A}_n$ for any $n \geq 0$. If y is a minimal (maximal) element of $\mathbf{X}$ and $\mathbf{F}$ is a connected subposet of $I(y$; Stack($\mathbf{X}$)), then there exists an element x which is maximal (minimal) in $\mathbf{X}$ so that $x > y$ in $\mathbf{X}$ and $\mathbf{F} \subset I(x$; Stack($\mathbf{X}$)).*

Proof. We prove the theorem when x is maximal in $\mathbf{X}$. The argument when x is minimal is dual. We now proceed by induction on $|\mathbf{F}|$. The result follows from Lemma 4 when $|\mathbf{F}| = 1$. We assure validity when $|\mathbf{F}| \leq k$ and consider the case $|\mathbf{F}| = k+1$. Clearly, we may also assume y is nonmaximal in $\mathbf{X}$.

Now suppose that m_0 is a minimal element of $\mathbf{F}$ and that $\mathbf{F} - m_0$ is disconnected. Let $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_t$ be the components of $\mathbf{F} - m_0$. Then for each $i = 1, 2, \ldots, t$, the subposets $\mathbf{G}_i = \mathbf{F}_i \cup \{m_0\}$ and $\mathbf{G}_i' = \mathbf{F} - \mathbf{F}_i$ are connected subposets of $I(y$; Stack($\mathbf{X}$)). Choose maximal elements x, x' from $\mathbf{X}$ with $x > y, x' > y, \mathbf{G}_i \subset I(x$; Stack($\mathbf{X}$)), and $\mathbf{G}_i' \subset I(x'$; Stack($\mathbf{X}$)). Then we may assume without loss of generality that $\mathbf{F}_1 \subset I(x'$; Stack($\mathbf{X}$)), and for some $j \leq t$ with $1 \neq j$, $F_j \subset I(x$; Stack($\mathbf{X}$)). Of the points in $\mathbf{F}_1$ which are comparable to x', choose one whose distance to m_0 in $\mathbf{G}_1$ is as small as possible and call it b. Similarly, of all the points in $\mathbf{F}_j$ which are comparable to x, choose one whose distance to m_0 in $\mathbf{G}_j$ is as small as possible and call it b'. Then let $a_1, a_2, \ldots, a_m$ be a fence in $\mathbf{G}_i$ from b to m_0 and let $a_1', a_2', \ldots, a_n'$ be a fence in $\mathbf{G}_j$ from m_0 to b'. It follows immediately that $x, y, x', a_1, a_2, \ldots, a_m, a_2', \ldots, a_n'$ is a crown $\mathbf{A}_s$ in Stack($\mathbf{X}$) for some $s \geq 0$. The contradiction allows us to conclude that $\mathbf{F} - m_0$ is connected for every minimal element $m_0 \in \mathbf{F}$.

Now let m_0 be an arbitrary minimal element of $\mathbf{F}$. If m_0 is the least element of $\mathbf{F}$, then we may choose a maximal element $x \in X$ with $x > y$ and $\mathbf{F} - m_0 \subset I(x$; Stack($\mathbf{X}$)) and conclude that $\mathbf{F} \subset I(x$; Stack($\mathbf{X}$)). Therefore we may assume that F contains distinct minimal elements m_0 and m_1. Choose maximal elements $x, x' \in X$ so that $x > y, x' > y$, $\mathbf{F} - m_0 \subset I(x$, Stack($\mathbf{X}$)) and $\mathbf{F} - m_1 \subset I(x'$, Stack($\mathbf{X}$)). Then we may assume that $x > m_1$ and $x' > m_0$ in Stack($\mathbf{X}$). Now let $a_1, a_2, \ldots, a_n$ be a fence in $\mathbf{F}$ from m_0 to m_1. Then it follows that $x, y, x', a_1, a_2, \ldots, a_n$ is a crown $\mathbf{A}_s$ in Stack($\mathbf{X}$) for some $s \geq 0$. This contradiction completes the proof of our theorem.

3. Three-irreducible posets

Because we intend to follow a similar path through a complex maze of case arguments, we adopt Kelly's notation and labelling for the collection $\mathfrak{R}$ of all 3-irreducible posets.

Theorem 8 [5 and 15]. *The collection $\mathfrak{R}$ of all 3-irreducible posets is given by:*

$$\mathfrak{R} = \{\mathbf{B}, \mathbf{B}^d, \mathbf{C}, \mathbf{C}^d, \mathbf{D}, \mathbf{D}^d, \mathbf{C}X_1, \mathbf{C}X_1^d, \mathbf{C}X_2, \mathbf{C}X_2^d, \mathbf{C}X_3, \mathbf{C}X_3^d, \mathbf{E}X_1, \mathbf{E}X_1^d, \mathbf{E}X_2, \mathbf{F}X_1, \mathbf{F}X_1^d, \mathbf{F}X_2\} \cup \{\mathbf{A}_n, \mathbf{E}_n, \mathbf{F}_n^d, \mathbf{F}_n, \mathbf{G}_n, \mathbf{H}_n, \mathbf{I}_n, \mathbf{I}_n^d, \mathbf{J}_n : n \geq 0\}$$

as shown in Fig. 4.

4. Interval representable families of sets

In view of the association between indexed families of nonempty sets and posets of height one as discussed in Section 1, it is natural to make the following definitions. If $\mathcal{F} = \{A_\alpha : \alpha \in \mathcal{A}\}$ and $\mathcal{G} = \{B_\beta : \beta \in \mathcal{B}\}$ are indexed families, we say that $\mathcal{F}$ is *isomorphic* to G when there exist functions $f: \bigcup \mathcal{F} \to^{1-1}_{\text{onto}} \bigcup \mathcal{G}$ and $g: \mathcal{A} \to^{1-1}_{\text{onto}} \mathcal{B}$ so that $x \in A_\alpha$ if and only if $f(x) \in B_{g(\alpha)}$ for every $x \in \bigcup \mathcal{F}$ and $\alpha \in \mathcal{A}$. As is the case with posets, we do not distinguish between isomorphic families and we write $\mathcal{F} = \mathcal{G}$ when $\mathcal{F}$ is isomorphic to $\mathcal{G}$.

On the other hand we say that $\mathcal{F}$ is a *derived subfamily* of $\mathcal{G}$ when there exist functions $f: \mathbf{F} \to^{1-1} \bigcup \mathcal{G}$ and $g: \mathcal{A} \to^{1-1} \mathcal{B}$ so that $x \in A_\alpha$ if and only if $f(x) \in B_{g(\alpha)}$, for every $x \in \bigcup \mathcal{F}$ and $\alpha \in \mathcal{A}$. When either f or g fails to be surjective, we say that $\mathcal{F}$ is a *proper* derived subfamily. In particular, if $\beta_0 \in \mathcal{B}$, we denote by $G - B_{\beta_0}$ the proper derived subfamily $\{B_\beta : \beta \in \mathcal{B}, \beta \neq \beta_0\}$. Similarly if $x \in \bigcup \mathcal{G}$, we denote by $\mathcal{G} - x$ the proper derived subfamily $\{B_\beta - \{x\} : \beta \in \mathcal{B}\}$.

When $\mathbf{F}$ is a derived subfamily of $\mathcal{G}$, we say that $\mathcal{G}$ contains $\mathcal{F}$ (also, $\mathcal{F}$ is contained in $\mathcal{G}$) and write $\mathcal{F} \subset \mathcal{G}$.

If $\mathcal{F} = \{A_\alpha : \alpha \in \mathcal{A}\}$ is an indexed family, we let $\mathcal{F}^d$ be the indexed family defined by $\mathcal{F}^d = \{U(b): b \in \bigcup \mathcal{F}\} \cup \{U(\alpha): \alpha \in \mathcal{A}$ and $A_\alpha = \emptyset\}$ where $U(b) = \{\alpha : b \in A_\alpha\}$ when $b \in \bigcup \mathcal{F}$ and $U(a) = \emptyset$ when $\alpha \in \mathcal{A}$ and $A_\alpha = \emptyset$. Note that if $\mathbf{X}$ is a poset of height one and $G = \mathcal{F}_{\mathbf{X}}$, then $\mathcal{G}^d = \mathcal{F}_{\mathbf{X}^d}$.

For a bounded subset $S \subset \mathbf{R}$, let $[S]$ denote the smallest closed interval of $\mathbf{R}$ which contains S. Similarly, if F assigns to each $x \in S$ an interval $F(x) \subset \mathbf{R}$, let $[F(S)]$ denote the smallest closed interval of $\mathbf{R}$ which contains $\bigcup \{F(x): x \in S\}$. An indexed family $\mathcal{F} = \{A_\alpha : \alpha \in \mathcal{A}\}$ is said to be *point representable* when there exists a function $f: \bigcup \mathcal{F} \to \mathbf{R}$ so that $b \in [f(A_\alpha)]$ if and only if $b \in A_\alpha$ for every $b \in \bigcup \mathcal{F}$ and $\alpha \in \mathcal{A}$. The function f is called a point representation of $\mathcal{F}$. Note that a derived subfamily of a point representable family is also point representable.

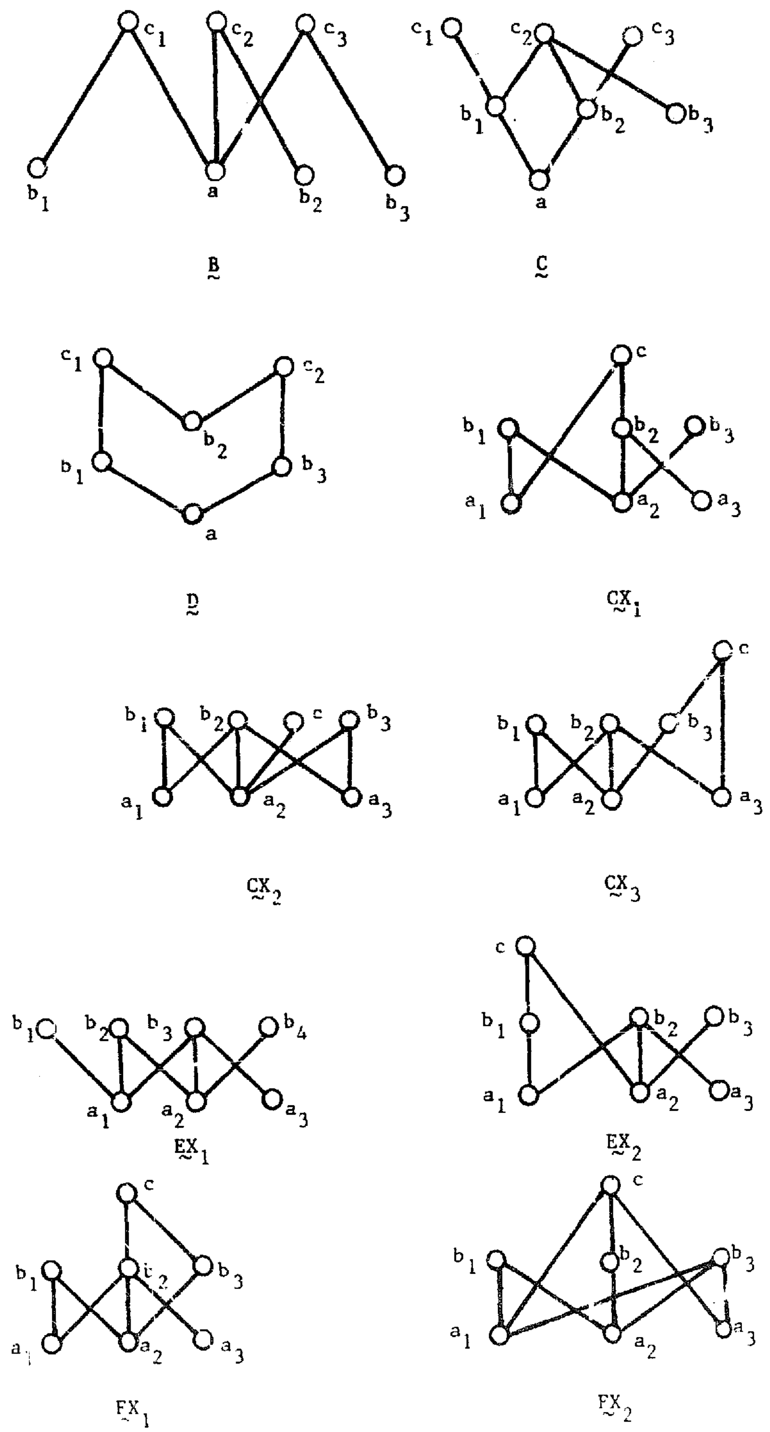
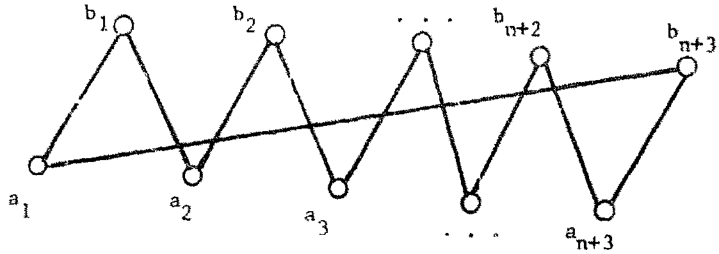
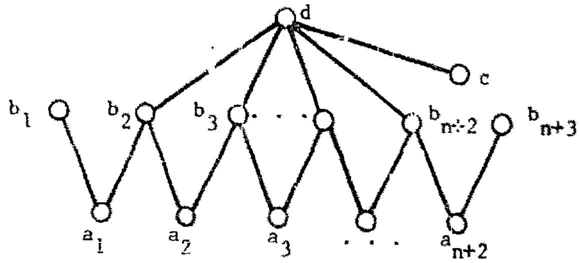


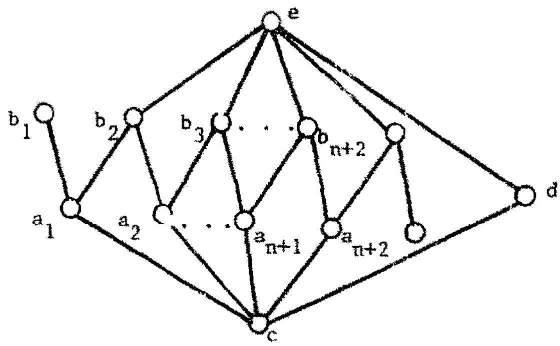
Fig. 4.



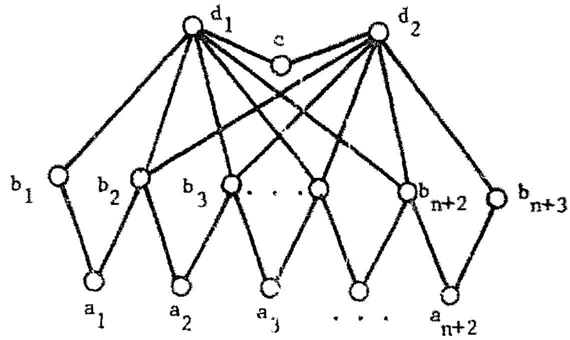
$\underset{\sim}{A}_n$



$\underset{\sim}{E}_n$

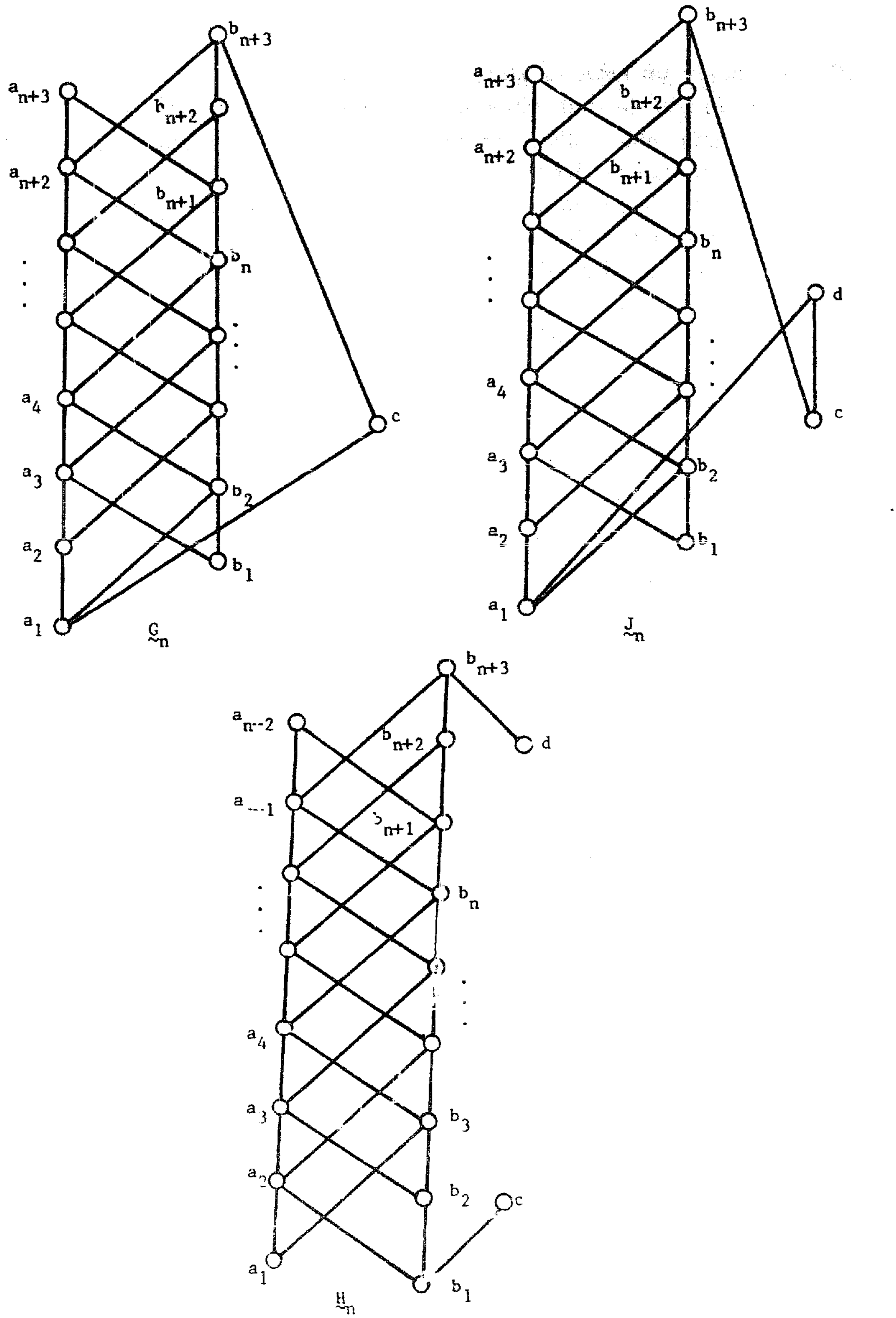


$\underset{\sim}{F}_n$



$\underset{\sim}{I}_n$

Fig. 4 (cont.).



b_{n+3}
a_{n+3}
b_{n+2}
a_{n+2}
b_{n+1}
b_n
a_4
c
a_3
b_2
a_2
b_1
a_1
G_n
b_{n+3}
a_{n+3}
b_{n+2}
a_{n+2}
b_{n+1}
b_n
d
a_4
c
a_3
b_2
a_2
b_1
a_1
J_n
b_{n+3}
a_{n-2}
b_{n+2}
d
b_{n+1}
b_n
a_4
a_3
b_3
a_2
b_2
c
a_1
b_1
H_n

Fig. 4 (cont.).

Then let $\mathcal{PR}$ denote the collection of all families which satisfy the following two properties.

(P_1) $\mathcal{F}$ is not point representable.

(P_2) Every proper derived subfamily of $\mathcal{G}$ is point representable.

In other words, $\mathcal{PR}$ is the collection of all critical non-point-representable families. We refer the reader to [10] where the collection $\mathcal{PR}$ is determined. In this paper, we are more interested in the extension of the concept of point representability to intervals. An indexed family $\mathcal{F} = \{A_\alpha : \alpha \in \mathcal{A}\}$ is said to be *interval representable* when there exists a function F which assigns to each $b \in \bigcup \mathcal{F}$ a closed interval $F(b)$ of $\mathbf{R}$ so that $b \in [F(A_\alpha)]$ if and only if $b \in A_\alpha$ for every $b \in \bigcup \mathcal{F}$ and $a \in \mathcal{A}$. The function F is called an interval representation of $\mathcal{F}$. We consider a point in $\mathbf{R}$ as a degenerate closed interval so that if $\mathcal{F}$ is point representable, then $\mathcal{F}$ is also interval representable. As before, we note that a derived subfamily of an interval representable family is also interval representable. Now let $\mathcal{IR}$ denote the colllection of all indexed families which satisfy the following two properties:

(I_1) $\mathcal{F}$ is not interval representable.

(I_2) Any proper derived subfamily of $\mathcal{F}$ is interval representable.

In other words, $\mathcal{IR}$ is the collection of all critical non-interval representable families. It is easy to see that property I_2 can be replaced by:

(I_2') $\mathcal{F} - A_\alpha$ and $\mathcal{F} - b$ are interval representable for every $\alpha \in \mathcal{A}$ and every $b \in \bigcup \mathcal{F}$.

The following fundamental theorems explain the interconnections between interval representable families and posets of height one and interval dimensions at most two.

Theorem 9 [15]. *Let $\mathcal{F}$ be an indexed family of nonempty sets and let $\mathbf{X}_\mathcal{F}$ be the poset of height one associated with $\mathcal{F}$. Then $\mathcal{F}$ is interval representable if and only if* $\mathrm{IDim}(\mathbf{X}_\mathcal{F}) \leq 2$.

Recall that we use the symbol $\mathcal{R}(I, 1)$ to denote the collection of all 3-interval irreducible posets of height one.

Theorem 10 [15]. *Let $\mathcal{F}$ be an indexed family of sets and let $\mathbf{X}_\mathcal{F}$ be the poset of height one associated with $\mathcal{F}$. Then $\mathcal{F} \in \mathcal{IR}$ if and only if $\mathbf{X}_\mathcal{F} \in \mathcal{R}(I, 1)$.*

Interestingly, it follows from Theorems 9 and 10 that if $\mathcal{F}$ is interval representable, so is $\mathcal{F}^d$. Thus $\mathcal{F} \in \mathcal{IR}$ if and only if $\mathcal{F}^d \in \mathcal{IR}$. On the other hand, it is possible for $\mathcal{F}$ to be point representable while $\mathcal{F}^d$ is not.

In Table 1 as given below, we define indexed families of nonempty sets which we will subsequently show all belong to $\mathcal{IR}$. The formulation of the table is simplified by the following conventions. For an integer $n \geq 0$, let

$$\mathcal{T}_n = \{\{a_1, a_2\}, \{a_2, a_3\}, \{a_3, a_4\}, \ldots, \{a_{n+1}, a_{n+2}\}, \{a_{n+2}, a_{n+3}\}\}.$$

Note that $\mathcal{T}_n$ is interval representable and that $\mathbf{X}_{\mathcal{T}_n}$ is a fence containing $2n+5$ points.

Next for an integer $k\geq 1$, let $U_k=\{b_1, b_2, \ldots, b_k\}$ and $V_k=\{a_1, a_2, \ldots, a_k\}$. Also let $U_0=V_0=\emptyset$. Then define $\mathcal{W}_n$ for each $n\geq 0$ by

$$\mathcal{W}_n=\{U_k\cup V_{k-1}\colon 1\leq k\leq n+3\}\cup\{U_{k-2}\cup V_k\colon 2\leq k\leq n+3\}.$$

Also define $\mathcal{W}'_n$ for each $n\geq 0$ by

$$\mathcal{W}'_n=\{U_k\cup V_{k-2}\colon 2\leq k\leq n+3\}\cup\{U_{k-1}\cup V_k\colon 1\leq k\leq n+2\}.$$

In this case, $\mathcal{W}$ and $\mathcal{W}'_n$ are interval representable although the reader may prefer to delay an attempt to verify that this is so.

Table 1

1. For $n\geq 0$, $\mathcal{A}_n=\mathcal{T}_n\cup\{\{a_1, a_{n+3}\}\}$
2. For $n\geq 0$, $\mathcal{E}_n=\mathcal{T}_{n+1}\cup\{V_{n+5}-\{a_1, a_{n+4}\}, \{a_{n+5}\}\}$
3. For $n\geq 0$, $\mathcal{I}_n=\mathcal{T}_n\cup\{V_{n+4}-\{a_{n+3}\}, V_{n+4}-\{a_1\}\}$
4. For $n\geq 0$, $\mathcal{F}_n=\{A\cup\{n+5\}\colon A\in\mathcal{T}_n\}\cup\{\{n+3\}, \{n+4, n+5\}, V_{n+5}-\{a_1\}\}$
5. For $n\geq 0$, $\mathcal{G}_n=\mathcal{W}_n\cup\{\{a_1, c\}\}$
6. For $n\geq 0$, $\mathcal{H}_n=\mathcal{W}'_n\cup\{\{b_1, c\}, \{b_{n+3}\}\}$
7. $\mathcal{O}_1=\{\{a_1, a_3, a_5\}, \{a_1, a_2\}, \{a_3, a_4\}, \{a_5, a_6\}$
8. $\mathcal{O}_2=\{\{a_1\}, \{a_1, a_2, a_3, a_4\}, \{a_2, a_4, a_5\}, \{a_2, a_3, a_6\}\}$
9. $\mathcal{O}_3=\{\{a_1, a_2\}, \{a_3, a_4\}, \{a_5\}, \{a_1, a_2, a_3\}, \{a_1, a_3, a_5\}\}$

We will now proceed to show that the families defined in Table 1 belong to $\mathcal{IR}$. For the reader's convenience, the argument will be presented as a series of lemmas. We will take full advantage of duality and symmetry. For example, each of the families in the collections $\{\mathcal{A}_n\colon n\geq 0\}$, $\{\mathcal{F}_n\colon n\geq 0\}$, $\{\mathcal{G}_n\colon n\geq 0\}$, and $\{\mathcal{H}_n\colon n\geq 0\}$ is self dual, i.e., if $\mathcal{F}$ is any one of these families, then in order to show that $\mathcal{F}$ satisfies Property I'_2, it suffices to show that $\mathcal{F}-x$ is interval representable for every $x\in\bigcup\mathcal{F}$. Similarly, we note that we can eliminate some cases by observing that $\mathcal{E}_n-1=\mathcal{E}_n-(n+4)$, $\mathcal{I}_n-1=\mathcal{I}_n-(n+3)$, etc.

Lemma 11. *$\mathcal{A}_n\in\mathcal{IR}$ for every $n\geq 0$.*

Proof. Let $n\geq 0$; then $\mathbf{X}_{\mathcal{A}_n}=\mathbf{A}_n=\mathrm{Stack}(\mathbf{X}_{\mathcal{A}_n})$, and thus $3=\mathrm{Dim}(\mathbf{A}_n)=\mathrm{Dim\,Stack}(\mathbf{X}_{\mathcal{A}_n})=\mathrm{IDim}(\mathbf{X}_{\mathcal{A}_n})$. We conclude that $\mathcal{A}_n$ is not interval representable.

On the other hand, we observe that $\mathcal{A}_n$ is self dual and that $\mathcal{A}_n-a_i=\mathcal{A}_n-a_{n+3}$ for $1\leq i\leq n+3$. However, the function f defined by $f(a_i)=i$ is a point representation of $\mathcal{A}_n-a_{n+3}$, and since $\mathcal{A}_n-a_{n+3}$ is point representable, it is also interval representable. Thus $\mathcal{A}_n\in\mathcal{IR}$.

Lemma 12. *$\mathcal{E}_n\in\mathcal{IR}$ for every $n\geq 0$.*

Proof. Let $n \geq 0$. Then $\mathbf{E}_n \subset \text{Stack}(\mathbf{X}_{\mathscr{E}_n})$ and thus $\mathscr{E}_n$ is not interval representable.

On the other hand, the function f defined by $f(a_i) = i$ shows that $\mathscr{E}_n - (n+5)$ and $\mathscr{E}_n - (V_{n+5} - \{a_1, a_{n+4}\})$ are point representable.

The function f defined by $f(a_{n+1}) = 1$, $f(a_1) = n+5$, and $f(a_i) = i$ otherwise shows that $\mathscr{E}_n - \{a_1, a_2\}$ and $\mathscr{E}_n - a_1$ are point representable. Therefore $\mathscr{E}_n - \{a_{n+3}, a_{n+4}\}$ and $\mathscr{E}_n - a_{n+4}$ are also point representable. If $i < m < n+4$, then the function f defined by $f(a_{n+5}) = n + \frac{1}{2}$ and $f(a_i) = i$ otherwise shows that $\mathbf{E}_n - \{a_m, a_{m+1}\}$ and $\mathscr{E}_n - a_m$ are point representable.

Finally, we note that $\mathscr{E}_n - \{a_{n+5}\}$ is not point representable, but the function F defined by $F(a_{n+5}) = [2, n+4]$ and $F(a_i) = i$ otherwise is an interval representation of $\mathscr{E}_n - \{a_{n+5}\}$. Thus $\mathscr{E}_n \in \mathscr{IR}$.

Lemma 13. *$\mathscr{I}_n \in \mathscr{IR}$ for every $n \geq 0$.*

Proof. Let $n \geq 0$. If $n = 0$, then $\mathbf{D} \subset \text{Stack}(\mathbf{X}_{\mathscr{I}_0})$, and if $n > 0$, then $\mathbf{I}_{n-1} \subset \text{Stack}(\mathbf{X}_{\mathscr{I}_n})$. Thus $\mathscr{I}_n$ is not interval representable.

On the other hand, the function f defined by $f(a_i) = i$ shows that $\mathscr{I}_n - (V_{n+4} - a_{n+3})$ and $\mathscr{I}_n - a_{n+4}$ are point representable. Therefore $\mathscr{I}_n - (V_{n+4} - a_1)$ is also point representable. When $1 \leq m < n+3$, the function f defined by $f(a_{n+4}) = m + \frac{1}{2}$ and $f(a_i) = i$ otherwise shows that $\mathscr{I}_n - \{a_m, a_{m+1}\}$ and $\mathscr{I}_n - a_m$ are point representable. Since $\mathscr{I}_n - a_{n+3} = \mathscr{I}_n - a_1$, we conclude that $\mathscr{I}_n - a_1$ is also point representable. Thus $\mathscr{I}_n \in \mathscr{IR}$.

Lemma 14. *$\mathscr{F}_n \in \mathscr{IR}$ for every $n \geq 0$.*

Proof. Let $n \geq 0$. Then $\mathbf{F}_n \subset \text{Stack}(\mathbf{X}_{\mathscr{F}_n})$ so $\mathscr{F}_n$ is not interval representable. On the other hand, the function f defined by $f(a_i) = i$ shows that $\mathscr{F}_n - a_{n+5}$ is point representable. The function F defined by $F(a_{n+5}) = n + \frac{5}{2}$ and $F(a_i) = [i, i+n+3]$ otherwise shows that $\mathscr{F}_n - a_{n+4}$ is interval representable. When $1 \leq m \leq n+3$, the function F defined by $F(a_{n+5}) = n + \frac{5}{2}$, $F(a_{n+4}) = [m, m+n+3]$, and $F(a_i) = [i, i+n+3]$ shows that $\mathscr{F}_n - a_m$ is interval representable. Since $\mathscr{F}_n$ is self dual, we conclude that $\mathscr{F}_n \in \mathscr{IR}$.

We pause here to introduce some notation which will simplify the arguments for $\mathscr{G}_n$ and $\mathscr{H}_n$. For each integer $n \geq 0$, let F_n be the interval representation of $\mathscr{W}_n$ defined by $F_n(b_k) = [-1-2k, 2k-1]$ and $F_n(a_k) = [2-2k, 2k]$. Then let G_n be the interval representation of $\mathscr{W}_n$ defined by

$$G_n(b_k) = [-4-2k-2n, 2n+2k+8] \quad \text{and}$$

$$G_n(a_k) = [-5-2k-2n, 2n+2k+5].$$

Note if $1 \leq k_1, k_2 \leq n+3$, then $G_n(b_{k_1}) \cap G_n(a_{k_1}) \subset F_n(b_{k_2}) \cup F_n(a_{k_2})$.

Lemma 15. *$\mathscr{G}_n \in \mathscr{IR}$ for every $n \geq 0$.*

Proof. Let $n \geq 0$. Then $\mathbf{G}_n \subset \text{Stack}(\mathbf{X}_{\mathscr{G}_n})$ so $\mathscr{G}_n$ is not interval representable.

On the other hand the function F_n defined above is an interval representation of $\mathscr{W}_n = \mathscr{G}_n - \{a_1, c\}$. Now choose an integer m with $1 \leq m \leq n+3$. Then the function F defined by

$$F(a_k) = F_n(a_k) \quad \text{and} \quad F(b_k) = F_n(b_k) \quad \text{when} \quad 1 \leq k \leq m;$$
$$F(a_k) = G_n(a_k) \quad \text{and} \quad F(b_k) = G_n(b_k) \quad \text{when} \quad m < k \leq n+3;$$
$$F(a_m) = F_n(a_m), \qquad F(b_m) = G_n(b_m)$$

and $F(c) = [-\frac{1}{4}, 4n+15]$ is an interval representation of $\mathscr{G}_n - (U_m \cup V_{m-1})$.

Next choose an integer m with $2 \leq m \leq n+3$. Then the function F defined by $F(a_k) = F_n(a_k)$ and $F(b_k) = F_n(b_k)$ when $1 \leq k \leq m$, $F(a_k) = G_n(a_k)$ and $F(b_k) = G_n(b_k)$ when $m \leq k \leq n+3$, and $F(c) = [-\frac{1}{4}, 4n+15]$ is an interval representation of $\mathscr{G}_m - (U_{m-2} \cup V_m)$. Since $\mathscr{G}_n$ is self dual, we conclude that $\mathscr{G}_n \in \mathscr{IR}$.

Lemma 16. *$\mathscr{H}_n \in \mathscr{IR}$ for every $n \geq 0$.*

Proof. Let $n \geq 0$. Then $\mathbf{H}_n \subset \text{Stack}(\mathscr{H}_n)$ so $\mathscr{H}_n$ is not interval representable. On the other hand, the function F defined by $F(a_k) = F_n(b_k)$ and $F(b_k) = F_n(a_k)$ when $1 \leq k \leq n+2$, $F(b_{n+3}) = -4n-10$, and $F(c) = [-\frac{1}{4}, 4n+15]$ provides an interval representation of $\mathscr{H}_n - (U_{n+3} \cup V_{n+1})$. If we modify this definition by making the single change of setting $F(b_{n+3}) = F_n(b_{n+3})$, then we see that $\mathscr{H}_n - \{b_{n+3}\}$ is interval representable. Similarly if we change this representation by defining $F(b_{n+3}) = 4n+15$, then we see that $\mathscr{H}_n - \{b_1, i\}$ is interval representable.

Now let m be an integer with $2 \leq m < n+3$. Then the function F defined by $F(a_k) = F_n(b_k)$ and $F(b_k) = F_n(a_k)$ when $1 \leq k < m$, $F(a_k) = G_n(b_k)$ and $F(b_k) = G_n(a_k)$ when $m \leq k < n+3$, $F(b_{n+3}) = -4m-10$, and $F(c) = [-\frac{1}{4}, 9n+15]$ is an interval representation of $\mathscr{H}_n - (U_m \cup V_{m-2})$. Finally we observe that if $1 \leq m \leq n+2$, then the function F defined by $F(a_k) = F_n(b_k)$ and $F(b_k) = F_n(a_k)$ when $1 \leq k < m$, $F(a_k) = G_n(b_k)$ and $F(b_k) = G_n(a_k)$ when $m < k \leq n+2$, $F(b_m) = F_n(a_m)$, $F(a_m) = G_n(b_m)$, $F(b_{n+3}) = -4n-10$, and $F(c) = [-\frac{1}{4}, 4n+15]$ is an interval representation of $\mathscr{H}_n - (U_{m-1} \cup V_m)$. Since $\mathscr{H}_n$ is self dual, we conclude that $\mathscr{H}_n \in \mathscr{IR}$.

We are left only to consider the families $\mathscr{O}_1, \mathscr{O}_2$, and $\mathscr{O}_3$. These 'odd' families should be viewed as pathological examples. In the interest of brevity we do not include all the details necessary to show that $\mathscr{O}_1, \mathscr{O}_2$, and $\mathscr{O}_3 \in \mathscr{IR}$. We note only that $\mathbf{B}^d \subset \text{Stack}(X_{\mathscr{O}_1})$, $\mathbf{C} \subset \text{Stack}(X_{\mathscr{O}_2})$ and $\mathbf{EX}_2 \subset \text{Stack}(X_{\mathscr{O}_3})$. Thus $\mathscr{O}_1, \mathscr{O}_2$ and $\mathscr{O}_3$ are not interval representable. We leave it to the reader to supply the details necessary to show that every proper derived subfamily of $\mathscr{O}_1, \mathscr{O}_2$ and $\mathscr{O}_3$ is interval representable. The argument can be simplified somewhat by taking advantage of similarity and duality.

5. The determination of $\mathscr{IR}$

We are now ready to present the principal result of the paper—a complete determination of $\mathscr{IR}$, the collection of all critical noninterval-representable families. In view of Theorem 10, we will simultaneously determine the collection $\mathscr{R}(I, 1)$ of a 3-interval irreducible posets of height one.

Theorem 17. *The collection $\mathscr{IR}$ of all critical noninterval-representable families is given by:*

$$\mathscr{IR} = \{\mathcal{O}_1, \mathcal{O}_1^d, \mathcal{O}_2, \mathcal{O}_2^d, \mathcal{O}_3\} \cup \{\mathscr{A}_n, \mathscr{E}_n, \mathscr{E}_n^d, \mathscr{I}_n, \mathscr{I}_n^d, \mathscr{F}_n, \mathscr{G}_n, \mathscr{H}_n : n \geq 0\}$$

where these families are defined in Table 1.

Proof. Let $\mathscr{F} \in \mathscr{IR}$ and $\mathbf{X} = (X, P)$ be the poset of height one defined by $\mathbf{X} = \mathbf{X}_{\mathscr{F}}$. Then $\mathbf{X} \in \mathscr{R}(I, 1)$, i.e., $\mathbf{X}$ is a 3-interval irreducible poset of height one. We let A denote the set of all minimal elements of $\mathbf{X}$ and let B denote the set of all maximal elements. Since $\mathbf{X}$ is connected, note that $A \cap B = \emptyset$. Then let $\mathbf{Y} = \text{Stack}(\mathbf{X}) = (X, Q)$. The poset $\mathbf{Y}$ has dimension 3 and thus contains one or more of the posets in the collection $\mathscr{R}$ of all 3-irreducible posets. Suppose first that $\mathbf{Y}$ contains a crown $\mathscr{A}_n$ for some $n \geq 0$. Then we know that $\mathbf{X}$ also contains a crown $\mathbf{A}_m$ for some m with $0 \leq m \leq n$. Since $\mathbf{X} \in \mathscr{R}(I, 1)$, we conclude that $m = n$ and that $\mathbf{X} = \mathbf{A}_n = \mathbf{Y}$. Thus $\mathscr{F} = \mathscr{A}_n$. We may therefore assume in the remainder of the argument that $\mathbf{Y}$ does not contain a crown $\mathbf{A}_n$. The remainder of the argument is divided into a sequence of cases. In each case, we choose a particular 3-irreducible poset contained in $\mathbf{Y}$ and assume that $\mathbf{Y}$ does not contain the 3-irreducible posets treated in previous cases. The argument follows along the same lines as Kelly's argument for determining $\mathscr{R}$ but there are several additional conditions.

Case 1. $\mathbf{Y}$ contains $\mathbf{D}$.

Choose a copy of $\mathbf{D}$ labeled as in Fig. 4. Then it follows immediately from Theorem 7 that we may assume that $c_1, c_2 \in B$ and $a, b \in A$, for suppose for example that $c_1 \notin B$. Then $\{c_2, b_3\}$ is a connected subposet of $I(c_1 : \mathbf{Y}))$ and thus, there exists an element $c_1' \in B$ with $c_1' > c_1$ in $\mathbf{X}$ and $\{c_2, b_3\} \subset I(c_1'; \mathbf{Y})$. Hence c_1' can replace c_1 in this copy of $\mathbf{D}$. Next choose an element $b_2' \in B$ so that $b_2' > b_2$ in $\mathbf{X}$ and $\{b_1, b_3, a\} \subset I(b_2'; \mathbf{Y})$.

Now suppose that $b_1, b_3 \in A$. Then choose elements $b_1', b_3' \in B$ with $b_1' > b_1$ in $\mathbf{X}$, $b_3' > b_3$ in $\mathbf{X}$,

$$\{b_2, c_2, b_3\} \subset I(b_1'; \mathbf{Y}), \quad \text{and} \quad \{b_1, c_1, b_2\} \subset I(b_3'; \mathbf{Y}).$$

Then it follows that $\mathscr{F} = \mathscr{F}_{\mathbf{X}}$ contains

$$\{\{b_1, a\}, \{a, b_3\}, \{b_1, a, b_2\}, \{a, b_3, b_2\}, \{b_2\}\} = \mathscr{I}_0$$

as a derived subfamily. Thus $\mathscr{F} = \mathscr{I}_0$

Now suppose that $b_1, b_3 \in B$. Then choose elements $b_1'', b_3'' \in A$ so that $b_1 > b_1''$ in $\mathbf{X}$, $b_3 > b_3''$ in $\mathbf{X}$,

$$\{b_2', b_2, c_2, b_3\} \subset I(b_1''; \mathbf{Y}) \quad \text{and} \quad \{b_2', b_2, c_1, b_1\} \subset I(b_3''; \mathbf{Y}).$$

Then it follows that $\mathscr{F}$ contains

$$\{\{b_1'', a\}, \{a, b_3''\}, \{b_1'', a, b_2\}, \{a, b_3'', b_2\}, \{b_2\}\} = \mathscr{I}_0,$$

so $\mathscr{F} = \mathscr{I}_0$.

Next suppose that $b_1 \in A$ and $b_3 \in B$. Then choose elements $b_1' \in B$ and $b_3'' \in A$ so that $b_1' > b_1$ in $\mathbf{X}$, $b_3 > b_3''$ in $\mathbf{X}$,

$$\{b_2', b_2, c_2, b_3\} \subset I(b_1'; \mathbf{Y}) \quad \text{and} \quad \{b_2', b_2, c_1, b_1, b_1'\} \subset I(b_3''; \mathbf{Y}).$$

Then it follows that $\mathscr{F} = \mathscr{I}_0$. Since the case where $b_1 \in B$ and $b_2 \in A$ is symmetric to the preceding case, we have therefore proven that if $\mathbf{Y}$ contains $\mathbf{D}$, then $\mathscr{F} = \mathscr{I}_0$. By duality, we may conclude that if $\mathbf{Y}$ contains $\mathbf{D}^d$, then $\mathscr{F} = \mathscr{I}_0^d$. In what follows we therefore assume that $\mathscr{F}$ does not contain $\mathbf{D}$ or $\mathbf{D}^d$.

Case 2. $\mathbf{Y}$ contains $\mathbf{C}$.

Choose a copy of $\mathbf{C}$ labeled as shown in Fig. 4. From Theorem 7, we may assume that $c_1, c_3 \in B$ and $a, b_3 \in A$. Next suppose that there is no element $c_2' \in B$ with $c_2' \geq c_2$ in $\mathbf{X}$ and $\{c_1, c_3\} \subset I(c_2'; \mathbf{Y})$. Then there exist elements $b_4, b_5 \in B$ so that $b_4 > c_2$ in $\mathbf{X}$, $\{c_3\} \subset I(b_4; \mathbf{Y})$, $b_5 > c_2$ in $\mathbf{X}$, and $\{c_1\} \subset I(b_5; \mathbf{Y})$. But this implies that the subposet of $\mathbf{Y}$ determined by $\{a, c_1, b_4, b_5, c_3, b_3\}$ is $\mathbf{D}$. The contradiction allows us to assume that $c_2 \in B$.

Next choose an element $b_3' \in B$ so that $b_3' > b_3$ in $\mathbf{X}$ and $\{c_3, b_2, a, b_1, c_1\} \subset I(b_3'; \mathbf{Y})$. Then choose elements $c_1'', c_3'' \in A$ so that $c_1 > c_1''$ in $\mathbf{X}$, $c_3 > c_3''$ in $\mathbf{X}$,

$$\{b_3, b_3', c_2, b_2, c_3\} \subset I(c_1''; \mathbf{Y}), \quad \text{and} \quad \{b_3, b_3', c_2, b_1, c_1, c_1''\} \subset I(c_3''; \mathbf{Y}).$$

If $b_1, b_2 \in A$, then it follows that $\mathscr{F}$ contains

$$\{\{b_3\}, \{b_3, a, b_2, b_1\}, \{a, b_1, c_1''\}, \{a, b_2, c_3''\}\} = \mathscr{O}_2$$

and thus $\mathscr{F} = \mathscr{O}_2$.

Next suppose that $b_1, b_2 \in B$. Then choose elements $b_1'', b_2'' \in A$ so that $b_1 > b_1''$ in $\mathbf{X}$, $b_2 > b_2''$ in $\mathbf{X}$,

$$\{b_2, c_3, c_3''\} \subset I(b_1''; \mathbf{Y}), \quad \text{and} \quad \{b_1'', b_1, c_1, c_1''\} \subset I(b_2''; \mathbf{Y}).$$

If $b_3' > b_1''$ in $\mathbf{X}$ and $b_3' > b_2''$ in $\mathbf{X}$, then $\{b_1, b_3', b_2, b_1'', a, b_2''\}$ generates $\mathbf{A}_0$. If $b_3' > b_1''$ in $\mathbf{X}$ and $b_3' I b_2''$ in $\mathbf{X}$, then $\{c_2, b_3', b_1'', c_1, a, b_2\}$ generates $\mathbf{D}^d$ in $\mathbf{Y}$. By symmetry, we conclude that $b_3' I b_1''$ in $\mathbf{X}$ and $b_3' I b_2''$ in $\mathbf{X}$. And it follows that

$$\mathscr{F} = \{\{b_3\}, \{b_3, a, b_2'', b_1''\}\{a, b_1'', c_1''\}\{a, b_2'', c_3''\}\} = \mathscr{O}_2.$$

We may now add to our list of assumptions the statements that $\mathbf{Y}$ does not contain $\mathbf{C}$ or $\mathbf{C}^d$.

Case 3. $\mathbf{Y}$ contains $\mathbf{CX}_3$.

We choose a copy of $\mathbf{CX}_3$ in $\mathbf{Y}$ with the points labeled as in Fig. 4. Then it follows from Theorem 7 that we may assume without loss of generality that $b_1, c \in B$ and $a_1, a_3 \in A$. Next choose an element $b_2' \in B$ with $b_2' \geq b_2$ in $\mathbf{X}$ and $\{b_3, c\} \subset I(b_2'; \mathbf{Y})$. If $b_2' > b_1$ in $\mathbf{X}$, then $\{b_2', b_1, a_2, b_3, c, a_3\}$ generates a copy of $\mathbf{D}$ in $\mathbf{Y}$. The contradiction shows that $b_2' I b_1$ in $\mathbf{X}$, i.e., we may assume that $b_2 \in B$.

Next suppose that there is no point $a_2'' \in A$ so that $a_2'' \leq a_2$ in $\mathbf{X}$ and $\{a_1, a_3\} \subset I(a_2''; \mathbf{Y})$. Then we may choose elements $a_4, a_5 \in A$ so that $a_2 > a_4$ in $\mathbf{X}$, $a_2 > a_5$ in $\mathbf{X}$, $\{a_3\} \subset I(a_4; \mathbf{Y})$, and $\{a_1\} \subset I(a_5; \mathbf{Y})$. However, it follows that $\{b_2, a_1, a_4, b_3, a_5, a_3\}$ generates a copy of $\mathbf{D}^d$ in $\mathbf{Y}$. The contradiction shows that we may assume that $a_2 \in A$.

Now let us assume that $b_3 \in B$. Then choose elements $a_4 \in A$, $b_4 \in B$ so that $b_4 > a_3$ in $\mathbf{X}$, $b_3 > a_4$ in $\mathbf{X}$,

$$\{a_3, b_2, a_1, b_1\} \subset I(a_4; \mathbf{Y}), \quad \text{and} \quad \{a_4, b_3, a_2, b_1, a_1\} \subset I(b_4; \mathbf{Y}).$$

Then it follows that $\mathscr{F}$ contains

$$\{\{a_1, a_2\}, \{a_2, a_4\}, \{a_1, a_2, a_3\}, \{a_2, a_4, a_3\}, \{a_3\}\} = \mathscr{I}_0,$$

and thus $\mathscr{F} = \mathscr{I}_0$.

So we may therefore assume that $b_3 \in A$. Then choose elements $b_3', a_3' \in B$ so that $b_3' > b_3$ in $\mathbf{X}$, $a_3' > a_3$,

$$\{a_3, b_2, a_1, b_1\} \subset I(b_3'; \mathbf{Y}), \quad \text{and} \quad \{b_3', b_3, a_2, b_1, a_1\} \subset I(a_3'; \mathbf{Y}).$$

Then it follows that

$$\mathscr{F} = \{\{a_1, a_2\}, \{a_2, b_3\}, \{a_1, a_2, a_3\}, \{a_2, b_3, a_3\}\{a_3\}\} = \mathscr{I}_0.$$

We may now assume in what follows that $\mathbf{Y}$ does not contain $\mathbf{CX}_3$, or $\mathbf{CX}_3^d$.

Case 4. $\mathbf{Y}$ contains $\mathbf{CX}_2$.

Choose a copy of $\mathbf{CX}_2$ labeled as shown in Fig. 4. By Theorem 7, we may assume that $a_1, a_3 \in A$ and $c \in B$. The same argument used in Case 3 also allows us to assume $a_2 \in A$. Next, choose an element $b_3' \in B$ with $b_3' \geq b_3$ in $\mathbf{X}$ and $\{b_1, b_2, a_1\} \subset I(b_3'; \mathbf{Y})$. If $b_3' > c$ in $\mathbf{Y}$, then $\mathbf{Y}$ contains $\mathbf{CX}_3$, so we may assume that $b_3' I c$ in $\mathbf{Y}$, i.e., we may assume that $b_3 \in B$. By symmetry, we may also assume that $b_1 \in B$.

Next choose elements $b_2' \in B, c'' \in A$ so that $b_2' \geq b_2$ in $\mathbf{X}$, $c > c''$ in $\mathbf{X}$,

$$\{b_3, a_3, b_2, a_1, b_1\} \subset I(c''; \mathbf{Y}), \quad \text{and} \quad \{c, c''\} \subset I(b_2'; \mathbf{Y}).$$

Then choose an element $b_4 \in B$ so that $b_4 > a_3$ in $\mathbf{X}$ and $\{c, c'', a_2, b_1, a_1\} \subset I(b_4; \mathbf{Y})$. Then it follows that

$$\mathscr{F} = \{\{a_1, a_2\}, \{a_2, c\}, \{a_1, a_2, a_3\}, \{a_2, c, a_3\}, \{a_3\}\} = \mathscr{I}_0.$$

We subsequently assume that $\mathbf{Y}$ does not contain $\mathbf{CX}_2$ or $\mathbf{CX}_2^d$.

Case 5: $\mathbf{Y}$ contains $\mathbf{CX}_1$.

Choose a copy of $\mathbf{CX}_1$ in $\mathbf{Y}$ labelled as shown in Fig. 4. From Theorem 7, we may assume that $b_3 \in B$ and $a_1, a_3 \in A$. Then choose an element $b_1' \in B$ so that $b_1' \geq b_1$ in $\mathbf{X}$ and $\{c, b_2, a_3\} \subset I(b_3; \mathbf{Y})$. If $b_1' > b_3$ in $\mathbf{Y}$, then $\{b_1', b_3, a_2, b_2, c, a'\}$ generates $\mathbf{D}$ in $\mathbf{Y}$ so we may conclude that $b_1' I b_3$ in $\mathbf{Y}$, i.e., we may assume that $b_1 \in B$.

If there is no element $a_2'' \in A$ with $a_2 \geq a''$ in $\mathbf{X}$ and $\{a_1, a_3\} \subset I\ (a_2''; \mathbf{Y})$, then there exist elements $a_4, a_5 \in A$, with $a_2 > a_4$ in $\mathbf{X}$, $a_2 > a_5$ in $\mathbf{X}$, $\{a_1\} \subset I(a_5; \mathbf{Y})$, and $\{a_3\} \subset I(a_4; \mathbf{Y})$. But this implies that the subposet of $\mathbf{Y}$ generated by $\{c, a_3, a_5, b_3, a_4, a_1\}$ in $\mathbf{D}^{\mathrm{d}}$. The contradiction shows that we may assume that $a_2 \in A$.

Now choose elements $a_1' \in B$, $b_3'' \in A$, $a_3' \in B$ so that $a_1' > a_1$ in $\mathbf{X}$, $b_3 > b_3''$ in $\mathbf{X}$ and $a_3' > a_3$ in $\mathbf{X}$.

$$\{b_3, a_2, b_2, a_3\} \subset I(a_1'; \mathbf{Y}), \qquad \{a_3, b_2, c, a_1, a_1', b_1\} \subset I(b_3''; \mathbf{Y}),$$

and

$$\{b_3'', b_3, a_2, b_1, a_1, a_1'\} \subset I(a_3'; \mathbf{Y}).$$

If $b_2 \in B$, then it follows that

$$\mathscr{F}^{\mathrm{d}} = \{\{b_3\}, \{b_3, c, b_1, b_2\}, \{c, b_2, a_3'\}, \{c, b_1, a_1'\}\} = \mathcal{O}_2.$$

If $b_2 \in A$, choose $b_2' \in B$ so that $b_2' > b_2$ in $\mathbf{X}$ and $\{a_1', a_1, b_1\} \subset I(b_2'; \mathbf{Y})$. If $b_2' > b_3''$ in $\mathbf{X}$, then the subposet of Y generated by $\{c, b_2', b_3, a_3', a_2, a_3, b_3''\}$ is $\mathbf{CX}_2$. The contradiction implies that $b_2' I b_3''$ and thus

$$\mathscr{F}^{\mathrm{d}} = \{\{b_3\}, \{b_3, c, b_1, b_2'\}, \{c, b_2', a_3'\}, \{c, b_1, a_1'\}\} = \mathcal{O}_2.$$

We may subsequently assume that $\mathbf{Y}$ does not contain $\mathbf{CX}_1$ or $\mathbf{CX}_1^{\mathrm{d}}$.

Case 6. $\mathbf{Y}$ contains $\mathbf{EX}_2$.

Choose a copy of $\mathbf{EX}_2$ in $\mathbf{Y}$ labeled as shown in Fig. 4. By Theorem 7 we may assume that $b_3 \in B$ and $a_2 \in A$. Since $\mathbf{EX}_2^{\mathrm{d}} = \mathbf{EX}_2$ we may assume without loss of generality that $b_1 \in B$. Next suppose that there is no point $b_2' \in B$ so that $b_2' \geq b_2$ in $\mathbf{X}$ and $\{c, b_3\} \subset I(b_2'; \mathbf{Y})$. Then it follows that there exists points $b_4, b_5 \in B$ with $b_4 > b_2$ in $\mathbf{X}$, $b_5 > b_2$ in $\mathbf{X}$,

$$\{b_3\} \subset I(b_4; \mathbf{Y}), \quad \text{and} \quad \{c, b_1\} \subseteq I(b_5; \mathbf{Y}).$$

This implies that $b_5 > b_3$ in $\mathbf{Y}$. If $b_4 > c$ in $\mathbf{Y}$, then $\{b_4, c, a_2, b_3, b_5, a_3\}$ generates a copy of $\mathbf{D}$ in $\mathbf{Y}$. So we must have $b_4 I c$ in $\mathbf{Y}$, and thus $b_4 > b_1$ in $\mathbf{Y}$. However, this implies that $\{c, b_1, b_4, b_2, a_2, b_3, a_3\}$ generates a copy of $\mathbf{CX}_1$ in $\mathbf{Y}$. The contradiction allows us to assume that $b_2 \in B$ and dually that $a_2 \in A$.

Next choose a point $c' \in B$ with $c' \geq c$ in $\mathbf{X}$ and $\{b_2, a_2\} \subset I(c'; \mathbf{Y})$. If $c' > b_3$ in $\mathbf{Y}$, then $\mathbf{Y}$ contains $\mathbf{D}^{\mathrm{d}}$, so we may assume that $c' I b_3$ in $\mathbf{Y}$, i.e., we may assume that $c' = c \in B$. Dually, we may also assume that $a_1 \in A$.

Then choose points $b_3'' \in A$, $a_3' \in B$ so that $b_3 \geq b_3''$ in $\mathbf{X}$, $a_3' \geq a_3$ in $\mathbf{X}$,

$$\{a_3, b_2, a_1, b_1, c\} \subset I(b_3''; \mathbf{Y}), \quad \text{and} \quad \{b_3'', b_3, a_2, c, b_1, a_1\} \subset I(a_3'; \mathbf{Y}).$$

Finally choose a point $b_1'' \in A$ with $b_1 \geq b_1''$ in $\mathbf{X}$ and $\{a_3', a_3, b_2, a_2, b_3, b_3''\} \subset I(b_1''; \mathbf{Y})$. Then it follows that

$$\mathscr{F} = \{\{a_1, b_1''\}, \{a_2, b_3''\}, \{a_3\}, \{a_1, b_1'', a_2\}, \{a_1, a_2, a_3\}\} = \mathscr{O}_3.$$

In what follows, we therefore assume that $\mathbf{Y}$ does not contain $\mathbf{EX}_2$.

Case 7. $\mathbf{Y}$ contains $\mathbf{FX}_1$.

Choose a copy of $\mathbf{FX}_1$ labelled as shown in Fig. 4. Then it follows from Theorem 7 that we may assume that $b_1, b_3, c \in B$ and $a_3 \in A$. Next choose an element $b_2' \in B$ with $b_2' \geq b_2$ in $\mathbf{X}$ and $\{b_3\} \subset I(b_2'; \mathbf{Y})$. If $b_2' > b_1$ in $\mathbf{Y}$, then $\mathbf{Y}$ contains $\mathbf{D}$, so we may assume that $b_2' I b_1$ in $\mathbf{Y}$, i.e., we may assume that $b_2' = b_2 \in B$. If there is no element $a_2'' \in A$ so that $a_2 \geq a_2''$ in $\mathbf{X}$ and $\{a_1, a_3\} \subset I(a_2''; \mathbf{Y})$, then $\mathbf{Y}$ contains $\mathbf{D}^d$, so we may assume that $a_2 \in A$.

Next choose an element $b_1'' \in A$ so that $b_1 \geq b_1''$ in $\mathbf{X}$ and $\{a_3, b_2, c, b_3\} \subset I(b_1''; \mathbf{Y})$. If $b_1'' > a_1$ in $\mathbf{Y}$, then $\mathbf{Y}$ contains $\mathbf{EX}_2$. So we must have $b_1'' I a_1$ in $\mathbf{Y}$. If $b_1 > a_2$ in $\mathbf{Y}$, then $\mathbf{Y}$ contains $\mathbf{D}$ so we must also have $b_1 I a_2$ in $\mathbf{Y}$.

Then choose elements $b_3'' \in A$, $a_3' \in B$ so that $b_3 \geq b_3''$ in $\mathbf{X}$, $a_3' \geq a_3$ in $\mathbf{X}$,

$$\{a_3, b_2, a_1, b_1, b_1''\} \subset I(b_3''; \mathbf{Y}), \quad \text{and} \quad \{b_3'', b_3, a_2, b_1, b_1'', a_1\} \subset I(a_3'; \mathbf{Y}).$$

If $a_1 \in A$, then

$$\mathscr{F} = \{\{b_1'', a_1, a_2\}, \{a_1, a_3, a_2\}, \{a_3\}, \{b_3'', a_2\}, \{a_1, a_3, b_3'', a_2\}\} = \mathscr{F}_0$$

so we may assume that $a_1 \in B$. Then choose $a_1'' \in A$ so $a_1 > a_1''$ in $\mathbf{X}$ and $\{b_3'', b_3, a_2\} \subset I(a_1''; \mathbf{Y})$. If $a_1'' I a_3'$, then $\mathscr{F} = \mathscr{F}_0$ so we assume that $a_1'' < a_3'$ in $\mathbf{X}$. Then the subposet of $\mathbf{Y}$ generated by $\{b_1'', b_1, a_2, b_3, c, a_1'', a_3'\}$ is $\mathbf{EX}_2$. The contradiction completes the proof of the case, and we assume hereafter that $\mathbf{Y}$ does not contain $\mathbf{FX}_1$ or $\mathbf{FX}_1^d$.

Case 8. $\mathbf{Y}$ contains $\mathbf{EX}_1$.

Choose a copy of $\mathbf{EX}_1$ contained in $\mathbf{Y}$ labelled as shown in Fig. 4. From Theorem 7, we may assume that $b_1, b_4 \in B$ and $a_3 \in A$. Choose an element $b_2' \in B$ with $b_2' \geq b_2$ in $\mathbf{Y}$ and $\{b_3, a_3\} \subset I(b_2'; \mathbf{Y})$. If $b_2' > b_1$ in $\mathbf{Y}$ and $b_2' > b_4$ in $\mathbf{Y}$, then $\mathbf{Y}$ contains $\mathbf{D}^d$. If $b_2' > b_1$ in $\mathbf{Y}$ and $b_2' I b_4$ in $\mathbf{Y}$, then $\mathbf{Y}$ contains $\mathbf{EX}_2$. If $b_2' I b_1$ and $b_2' > b_4$ in $\mathbf{Y}$, then $\mathbf{Y}$ contains $\mathbf{EX}_2$, so we may assume that $b_2' I b_1$ and $b_2' I b_4$ in $\mathbf{Y}$, i.e., we may assume that $b_2 = b_2' \in B$.

Next choose an element $b_3' \in B$ with $b_3' \geq b_3$ in $\mathbf{X}$ and $\{b_2\} \subset I(b_3'; \mathbf{Y})$. If $b_3' > b_1$ in $\mathbf{Y}$ and $b_3' > b_4$ in $\mathbf{Y}$, then $\mathbf{Y}$ contains $\mathbf{D}^d$. If $b_3' > b_4$ in $\mathbf{Y}$ and $b_3' I b_1$ in $\mathbf{Y}$, then $\mathbf{Y}$ contains $\mathbf{FX}_1$. Similarly, if $b_3' > b_1$ in $\mathbf{Y}$ and $b_3' I b_4$ in $\mathbf{Y}$, then $\mathbf{Y}$ contains $\mathbf{FX}_1$. Thus we may assume that $b_3 \in B$. Next choose an element $a_1'' \in A$ with $a_1 \geq a_1''$ in $\mathbf{X}$ and $\{a_2, b_4\} \subset I(a_1''; \mathbf{Y})$. If $a_3 > a_1''$ in $\mathbf{Y}$, then $\mathbf{Y}$ contains $\mathbf{FX}_1^d$. Thus we may assume that $a_1'' I a_3$ in $\mathbf{Y}$, i.e., we may assume that $a_1'' = a_1 \in A$. By symmetry, we may also assume that $a_2 \in A$.

Next choose elements $b_1'', b_4'' \in A$, $a_3' \in B$ so that $b_1 \geq b_1''$ in $\mathbf{X}$, $b_4 \geq b_4''$ in $\mathbf{X}$, $a_3' \geq a_3$ in $\mathbf{X}$,

$$\{b_2, a_2, b_3, b_4, a_3\} \subset I(b_1''; \mathbf{Y}), \qquad \{b_1'', b_1, b_2, a_1, b_3, a_2, a_3\} \subset I(b_4''; \mathbf{Y}),$$

and

$$\{b_4'', b_4, a_2, b_3, a_1, b_2, b_1, b_1''\} \subset I(a_3'; \mathbf{Y}).$$

Then it follows that

$$\mathcal{F} = \{\{b_1'', a_1\}, \{a_1, a_2\}, \{a_2, b_4''\}, \{a_1, a_2, a_3\}, \{a_3\}\} = \mathscr{E}_0.$$

We may assume hereafter that $\mathbf{Y}$ does not contain either $\mathbf{EX}_1$ or $\mathbf{EX}_1^{\mathrm{d}}$.

Case 9. $\mathbf{Y}$ contains $\mathbf{FX}_2$.

Choose a copy of $\mathbf{FX}_2$ in $\mathbf{Y}$ labeled as shown in Fig. 4. From Theorem 7, we may assume that $b_1, b_2 \in B$ and $a_3 \in A$. If there is no element $c' \in B$ with $c' \geq c$ in $\mathbf{X}$ and $\{b_1, b_3\} = I(c'; \mathbf{Y})$, then $\mathbf{Y}$ contains $\mathbf{D}$. So we may assume that $c \in B$. Dually, we may also assume that $a_2 \in A$.

Next choose an element $b_3' \in B$ with $b_3' \geq b_3$ in $\mathbf{X}$ and $\{c, b_2\} \subset I(b_3'; \mathbf{Y})$. If $b_3' > b_1$ in $\mathbf{Y}$, then $\mathbf{Y}$ contains D^{d}. So we may assume that $b_3' = b_3 \in B$. Dually we may assume that $a_1 \in A$. Then choose points $b_1'', b_2'' \in A$ and $a_3' \in B$ so that $b_1 \geq b_1''$ in $\mathbf{X}$, $b_2 \geq b_2''$ in $\mathbf{X}$, $a_3' \geq a_3$ in $\mathbf{X}$,

$$\{b_2, c, b_3\} \subset I(b_1''; \mathbf{Y}), \qquad \{b_1, b_3\} \subset I(b_2''; \mathbf{Y}), \quad \text{and} \quad \{b_1'', a_1, a_2, b_2''\} \subset I(a_3'; \mathbf{Y}).$$

Then it follows that

$$\mathcal{F} = \{\{b_1'', a_1, a_2\}, \{a_1, a_3, a_2\}, \{a_3\}, \{b_2'', a_2\}, \{a_1, a_3, b_2'', a_2\}\} = \mathcal{F}_0.$$

We now assume in what follows that $\mathbf{Y}$ does not contain $\mathbf{FX}_2$.

Case 10. $\mathbf{Y}$ contains $\mathbf{B}$.

Choose a copy of $\mathbf{B}$ in $\mathbf{Y}$ labelled as shown in Fig. 4. From Theorem 7 we may assume that $b_1, b_2, b_3 \in A$. Then choose elements $b_1', b_2', b_3' \in B$ so that $b_1' \geq b_1$ in $\mathbf{X}$, $b_2' \geq b_2$ in $\mathbf{X}$, $b_3' \geq b_3$ in $\mathbf{X}$,

$$\{c_2, a, b_2, c_3, b_3\} \subset I(b_1'; \mathbf{Y}), \qquad \{c_1, b_1, b_1', a, c_3, b_3\} \subset I(b_2'; \mathbf{Y}),$$

and

$$\{c, b_1, b_1', a, c_2, b_2, b_3\} \subset I(b_3'; \mathbf{Y}).$$

Next choose an element $c_1' \in B$ with $c_1' \geq c_1$ in $\mathbf{X}$ and $\{c_2, b_2, b_2'\} \subset I(c_1'; \mathbf{Y})$. If $c_1' > c_3$ in $\mathbf{Y}$, then $\mathbf{Y}$ contains $\mathbf{C}^{\mathrm{d}}$. So we may assume $c_1' I c_3$. If $c_1' > b_3$ in $\mathbf{Y}$, then $\mathbf{Y}$ contains $\mathbf{CX}_1$, so we may assume that $c_1' I b_3$ in $\mathbf{Y}$ and that $c_1' = c_1 \in B$. By symmetry, we may also assume that $c_2, c_3 \in B$.

Next choose an element $a'' \in A$ with $a \geq a''$ in $\mathbf{X}$ and $\{b_1\} \subset I(a''; \mathbf{Y})$. Then it follows that $\{b_1', b_2', b_3'\} \subset I(a''; \mathbf{Y})$. We conclude that

$$\mathcal{F}^{\mathrm{d}} = \{\{c_1, c_2, c_3\}, \{c_1, b_1'\}, \{c_2, b_2'\}, \{c_3, b_3'\}\} = \mathcal{O}_1.$$

In what follows, we therefore assume that $\mathbf{Y}$ does not contain $\mathbf{B}$ or $\mathbf{B}^{\mathrm{d}}$.

Case 11. $\mathbf{Y}$ contains $\mathbf{F}_n$ for some $n \geq 0$.

We also assume that $\mathbf{Y}$ does not contain $\mathbf{F}_m$, $\mathbf{E}_m$, or $\mathbf{E}_m^{\mathrm{d}}$ when $0 \leq m \leq n$. Now choose a copy of $\mathbf{F}_n$ in $\mathbf{Y}$ labelled as shown in Fig. 4. From Theorem 7, we may assume that $e, b_1 \in B$ and $c, a_{n+2} \in A$. Then without loss of generality we may assume that $d \in B$. We then choose an element $d'' \in A$ so that $d \geq d''$ in $\mathbf{X}$ and $\{b_1, b_2, \ldots, b_{n+2}\} \cup \{a_1, a_2, \ldots, a_{n+2}\} \subset I(d''; \mathbf{Y})$.

Now let i be an integer with $2 \leq i \leq n+2$. We try to choose an element $b_i' \in B$ with $b_i' \geq b_i$ in $\mathbf{X}$ and $\{d'', a_j\} \subset I(b_i'; \mathbf{Y})$ when $|i-j|>1$ and $1 \leq j \leq n+2$. Suppose that this is impossible for some integer $i = i_0$. Then choose an element $x \in B$ with $x \geq b_{i_0}$ in $\mathbf{X}$ and $\{d, d''\} \subset I(x; \mathbf{Y})$. Suppose that $x \geq b_1$ in $\mathbf{Y}$. If $x \geq a_{n+2}$ in $\mathbf{Y}$, then $\mathbf{Y}$ contains $\mathbf{D}$. Now choose the largest integer j for which $x \geq a_j$. Then $j < n+2$. It follows that

$$\{x, b_{j+1}, b_{j+2}, \ldots, b_{n+2}\} \cup \{a_j, a_{j+1}, \ldots, a_{n+2}\} \cup \{e, d, c\}$$

generates $\mathbf{F}_m$ in $\mathbf{Y}$ for some m with $0 \leq m < n$. The contradiction allows us to conclude that xIb_1 in $\mathbf{Y}$.

If $x > b_{j_1}$ in $\mathbf{Y}$ and $x > b_{j_2}$ in $\mathbf{Y}$, $j_1 < j_2$, then $x > b_j$ for every j with $j_1 < j < j_2$, otherwise $\mathbf{Y}$ contains a crown or $\mathbf{D}^d$. Similarly, if $x > a_{j_1}$ in $\mathbf{Y}$, $x > a_{j_2}$ in $\mathbf{Y}$ and $j_1 < j_2$, then $x > a_j$ in $\mathbf{Y}$ for every j with $j_1 < j < j_2$.

Now suppose that xIe in $\mathbf{Y}$. Then suppose that xIa_{n+2} in $\mathbf{Y}$ and let j be the largest integer for which $x > a_j$ in $\mathbf{Y}$. Then

$$\{x, b_{j+1}, \ldots, b_{n+2}\} \cup \{a_j, a_{j+1}, \ldots, a_{n+2}\} \cup \{c, d, e\}$$

generates $\mathbf{F}_m$ in $\mathbf{Y}$ where $m < n$. The contradiction shows $x > a_{n+2}$ in $\mathbf{Y}$. Now suppose that xIa_1 in $\mathbf{Y}$ and let j be the smallest integer for which $x > a_j$ in $\mathbf{Y}$. Then

$$\{b_1, b_2, \ldots, b_j\} \cup \{a_1, a_2, \ldots, a_j\} \cup \{d, e, x\}$$

generates $\mathbf{E}_m$ where $m < n$. Then we may assume that $x > a_1$ in $\mathbf{Y}$. However this implies that $\{b_1, a_1, e, x, a_{n+2}, c, d\}$ generates $\mathbf{FX}_1^d$. The contradiction implies that $e > x$ in $\mathbf{Y}$.

Now choose j_1 and j_2 as the smallest and largest integers respectively for which $x > a_{j_1}$ in $\mathbf{Y}$ and $x > a_{j_2}$ in $\mathbf{Y}$. If $j_2 > j_1 + 1$, then the subposet of $\mathbf{Y}$ generated by

$$\{b_1, b_2, \ldots, b_{j_1}, x\} \cup \{a_1, a_2, \ldots, a_{j_1}\} \cup \{a_{j_2}, c, d, e\}$$

is $\mathbf{F}_m$ for some $m < n$. The contradiction shows that $j_2 = j_1 + 1$. Thus we may replace b_i by x in this copy of $\mathbf{F}_n$. By symmetry and duality, we may therefore assume that

$$\{b_1, b_2, \ldots, b_{n+2}\} \subset B \quad \text{and} \quad \{a_1, a_2, \ldots, a_{n+2}\} \subset A.$$

Next choose elements $b_1'' \in A$, $a_{n+2}' \in B$ so that $b_1 \geq b_1''$ in $\mathbf{X}$, $a_{n+2}' \geq a_{n+2}$ in $\mathbf{X}$,

$$\{b_2, b_3, \ldots, b_{n+2}\} \cup \{e, d\} \subset I(b_1''; \mathbf{Y}), \quad \text{and} \quad \{a_1, a_2, \ldots, a_{n+1}\} \cup \{c, d'', b''\} \subset I(a_{n+2}'; \mathbf{Y}).$$

Then it follows that $\mathcal{F} = \mathcal{F}_n$.

Case 12. $\mathbf{Y}$ contains $\mathbf{E}_n$ for some $n \geq 0$.

In this case, we assume that $\mathbf{Y}$ does not contain $\mathbf{E}_m, \mathbf{E}_m^d$ when $0 \leq m < n$. We also assume $\mathbf{Y}$ does not contain $\mathbf{F}_m$ when $0 \leq m \leq n$. Then choose a copy of $\mathbf{E}_n$ in

$\mathbf{Y}$ with the points labeled as shown in Fig. 4. Then it follows directly from Theorem 7 that we may assume that $b_1, b_{n+3} \in B$ and $c \in A$.

Now suppose that $1 \leq i \leq n+1$. Then choose an element a_i'' with $a_i \geq a_i''$ in $\mathbf{X}$ and $\{b_{i+2}\} \subset I(a_i''; \mathbf{Y})$. Suppose that $c > a_i''$ in $\mathbf{Y}$. If $a_1 I a''$, then $\mathbf{Y}$ contains $\mathbf{E}_m^d$ where $m < n$. It follows that we must also have $a_1 > a_i''$. Then it follows that $\mathbf{Y}$ contains $\mathbf{F}_m$ where $0 \leq m < n$. The contradiction allows us to conclude that $c I a_i''$ in $\mathbf{Y}$. If $b_j > a_i''$ for some j with $j \neq i, i+1$, then it follows that $\mathbf{Y}$ contains $\mathbf{E}_m$ where $m < n$.

In summary, we may therefore assume that $\{a_1, a_2, \ldots, a_{n+1}\} \subset A$. By symmetry, we may also assume that $a_{n+2} \in A$. Next choose an integer i with $2 \leq i \leq n+2$. Then choose an element $x \in B$ with $x \geq b_i$ in $\mathbf{X}$ and $\{c\} \subset I(x; \mathbf{Y})$. Suppose that $x I d$ in $\mathbf{Y}$. If $x > a_1$ in $\mathbf{Y}$ and $x > a_{n+2}$ in $\mathbf{Y}$, then $\{b_1, x, d, b_{n+3}, a_1, a_{n+2}, c\}$ is $\mathbf{EX}_1$ if $x I b_1$ in $\mathbf{Y}$, while the same point set generates $\mathbf{EX}_2$ when $x > b_1$ in $\mathbf{Y}$. We may therefore assume without loss of generality that $x I a_{n+2}$. It follows that $\mathbf{Y}$ contains $\mathbf{E}_m$ where $m < n$. The contradiction allows us to conclude that $d > x$ in $\mathbf{Y}$. Thus $b_1 I x$ in $\mathbf{Y}$ and $b_{n+3} I x$ in $\mathbf{Y}$. Finally, we note that if $x > a_j$ for some j with $j \neq i-1, i$, then $\mathbf{Y}$ contains $\mathbf{E}_m$ where $m < n$. So we may assume without loss of generality that $x = b_i \in B$, and therefore $\{b_2, b_3, \ldots, b_{n+2}\} \subset B$.

Next choose elements $b_1'', b_{n+3}'' \in A$ so that $b_1 \geq b_1''$ in $\mathbf{X}$, $b_{n+3} \geq b_{n+3}''$ in $\mathbf{X}$,

$$\{b_2, b_3, \ldots, b_{n+3}, d\} \subset I(b_1''; \mathbf{Y}), \quad \text{and} \quad \{b_1, b_2, \ldots, b_{n+2}, d\} \subset I(b_{n+3}''; \mathbf{Y}).$$

Now assume that $d \in B$. Then it follows easily that $\mathcal{F} = \mathcal{F}_n$. On the other hand, if there is no element $d' \in B$ so that $d' \geq d$ in $\mathbf{X}$ and $\{b_1, b_{n+3}\} \subset I(d'; \mathbf{Y})$, then it follows that $\mathbf{Y}$ contains $\mathcal{I}_{n+1}$. We may therefore assume that $\mathbf{Y}$ does not contain $\mathbf{E}_n$, $\mathbf{E}_n^d$, or $\mathbf{F}_n$ when $n \geq 0$.

Case 13. $\mathbf{Y}$ contains $\mathbf{I}_n$ for some $n \geq 0$.

We also assume that $\mathbf{Y}$ does not contain $\mathbf{I}_m$ when $0 \leq m < n$. Then choose a copy of $\mathbf{I}_n$ labeled as shown in Fig. 4. Then by Theorem 7, we may assume that $d_1, d_2, b_1, b_{n+3} \in B$ and $c \in A$.

Now choose an integer i with $2 \leq i \leq n+2$ and let $b_i' \in B$ with $b_i' \geq b_i$ in $\mathbf{X}$ and $\{c\} \subset I(b_i'; \mathbf{Y})$. Then suppose that $b_i' \geq a_j$ in $\mathbf{Y}$ for some $j \neq i-1, i$. If $b_i' I d_1$ in $\mathbf{Y}$ or $b_i' I d_2$ in $\mathbf{Y}$, then $\mathbf{Y}$ contains $\mathbf{D}^d$ so we may assume that $b_i' < d_1$ in $\mathbf{Y}$ and $b_i' < d_2$ in $\mathbf{Y}$. But this implies that $\mathbf{Y}$ contains $\mathbf{I}_m$ where $0 \leq m < n$. The contradiction allows us to conclude that if $b_i' > a_j$ in Y, then either $j = i$ or $j = i-1$.

Now let i be an integer with $1 < i \leq n+3$. Then choose an element $a_i'' \in A$ with $a_i \geq a_i''$ in $\mathbf{X}$ and $\{c\} \subset I(a_i''; \mathbf{Y})$. If $b_1 > a_i''$ in $\mathbf{Y}$ and $b_{n+3} > a_i''$ in $\mathbf{Y}$, then $\mathbf{Y}$ contains $\mathbf{D}^d$. So we may assume without loss of generality that $b_{n+3} I a_i''$ in $\mathbf{Y}$. Then it follows easily that $\mathbf{Y}$ contains $\mathbf{I}_m$ where $0 \leq m < n$ whenever there exists an integer j with $b_j > a_i''$ in $\mathbf{Y}$ and $j \neq i, i+1$.

Next, we choose elements b_1'', b_{n+3}'' as in the preceding case, and observe that $\mathcal{F} = \mathcal{I}_{n+1}$. In what follows, we therefore add the assumption that $\mathbf{Y}$ does not contain $\mathbf{I}_n$ or $\mathbf{I}_n^d$ when $n \geq 0$.

Case 14. $\mathbf{Y}$ contains $\mathbf{G}_n$ for some $n \geq 0$.

In this case, we also assume that $\mathbf{Y}$ does not contain $\mathbf{J}_m$ or $\mathbf{H}_m$ when $0 \leq m < n$. Then choose a copy of $\mathbf{G}_n$ in $\mathbf{Y}$ labeled as shown in Fig. 4. We will now establish the following statements

(P_1) If $2 \leq i \leq n+2$ and $a_i \in A$, ($a_i \in B$) then there exists an element $a_i' \in B$ ($a_i'' \in A$) so that $a_i' \geq a_i$ in $\mathbf{X}$ ($a_i \geq a_i''$ in $\mathbf{X}$) and the subposet of $\mathbf{Y}$ obtained by replacing a_i by a_i' (a_i by a_i'') is $\mathbf{G}_n$.

(P_2) If $2 \leq i \leq n+2$ and $b_i \in A$ ($b_i \in B$), then there exists an element $b_i' \in B$ ($b_i'' \in A$) so that $b_i' \geq b_i$ in $\mathbf{X}$ ($b_i \geq b_i''$ in $\mathbf{X}$) and the subposet of $\mathbf{Y}$ obtained by replacing b_i by b_i' (b_i by b_i'') is $\mathbf{G}_n$.

We establish Statement P_1. Statement P_2 is dual. Choose an integer i with $2 \leq i \leq n+2$. First, we suppose that $a_i \in A$. Then choose an element $x \in B$ with $x > a_i$ in $\mathbf{X}$ and $\{b_i, b_{i-1}\} \subset I(x; \mathbf{Y})$. Then suppose that $i = n+2$.

Next suppose that xIa_{n+3} in $\mathbf{Y}$ and xIb_{n+3} in $\mathbf{Y}$. If $x > c$ in $\mathbf{Y}$, then $\mathbf{Y}$ contains $\mathbf{FX}_1^d$ so we may assume xIc in $\mathbf{Y}$. However, when $n > 0$, we see that $\mathbf{Y}$ contains $\mathbf{G}_m$ for some $m < n$, and when $n = 0$, we see that $\mathbf{Y}$ contains $\mathbf{FX}_1^d$. Then suppose that $x < a_{n+3}$ in $\mathbf{Y}$ and xIb_{n+3} in $\mathbf{Y}$. Then it follows that xIc in $\mathbf{Y}$. If $n = 0$, we see that $\mathbf{Y}$ contains $\mathbf{D}$, and if $n > 0$, we see that $\mathbf{Y}$ contains $\mathbf{G}_m$ for some $m < n$.

Next suppose that xIa_{n+3} in $\mathbf{Y}$ and that $x < b_{n+3}$ in $\mathbf{Y}$. Then $\mathbf{Y}$ contains $\mathbf{D}^d$. So we may now suppose that $2 \leq i < n+2$. With this assumption on the range of i, we then suppose that xIa_{n+3} in $\mathbf{Y}$ and xIb_{n+3} in $\mathbf{Y}$. If xIc, then $\mathbf{Y}$ contains $\mathbf{G}_m$ where $m < n$, and if $x > c$, then $\mathbf{Y}$ contains $\mathbf{J}_m$ where $m < n$.

Now suppose that $x < a_{n+3}$ in $\mathbf{Y}$ and xIb_{n+3} in $\mathbf{Y}$. If $2 < i$, then $\mathbf{Y}$ contains $\mathbf{G}_m$ where $m < n$, and if $i = 2$, then $\mathbf{Y}$ contains $\mathbf{D}$. Next suppose that xIa_{n+3} in $\mathbf{Y}$ and $x < b_{n+3}$ in $\mathbf{Y}$. If xIb_{i+1} in $\mathbf{Y}$ contains $\mathbf{G}_m$ where $m < n$, and if $x < b_{i+1}$ in $\mathbf{Y}$, then $\mathbf{Y}$ contains $\mathbf{D}$. We are left then with the case where $x < a_{n+3}$ in $\mathbf{Y}$ and $x < b_{n+3}$ in $\mathbf{Y}$. Then let j be the least integer for which $x < b_j$ in $\mathbf{Y}$. Suppose now that $j > i+1$. If xIa_j, then $\mathbf{Y}$ contains $\mathbf{G}_m$ where $m < n$; similarly, if $x < a_j$, then $\mathbf{Y}$ contains $\mathbf{G}_m$ where $m < n$. Now suppose $j = i+1$. If xIa_j in $\mathbf{Y}$, then $\mathbf{Y}$ contains $\mathbf{D}^d$. so we are left with $x < a_j$ in $\mathbf{Y}$. But this implies that x can replace a_i in this copy of $\mathbf{G}_n$.

To complete the proof of Statement P_1, we suppose that $a_i \in B$ and then choose an element $y \in A$ with $y < a_i$ in $\mathbf{X}$ and $\{b_i, b_{i-1}\} \subset I(y; \mathbf{Y})$. We proceed to show that y can replace a_i in this copy of $\mathbf{G}_n$. Similar to the preceding argument, we first suppose that $i = 2$.

We then assume that yIa_1 in $\mathbf{Y}$. If yIc in $\mathbf{Y}$, then $\mathbf{Y}$ contains $\mathbf{CX}_1^d$, and if $y < c$ in $\mathbf{Y}$, then $\mathbf{Y}$ contains $\mathbf{D}^d$. So we must have $y > a_1$ in $\mathbf{Y}$. Then if $y < c$ in $\mathbf{Y}$ and $n > 0$, we see that $\mathbf{Y}$ contains $\mathbf{G}_m$ where $m < n$. If $y < c$ in $\mathbf{Y}$ and $n = 0$, then $\mathbf{Y}$ contains $\mathbf{D}^d$. Therefore, we must have yIc in $\mathbf{Y}$ and thus y can replace a_2 in this copy of $\mathbf{G}_n$. So it remains to consider the case where $2 < i \leq n+2$.

Now suppose that yIa_1 in $\mathbf{Y}$ and yIb_1 in $\mathbf{Y}$. If yIc in $\mathbf{Y}$, then $\mathbf{Y}$ contains $\mathbf{H}_m$ where $m < n$. If $y < c$ in $\mathbf{Y}$ and $i = n+2$, then $\mathbf{Y}$ contains $\mathbf{D}^d$. If $y < c$ in $\mathbf{Y}$ and $i < n+2$, then $\mathbf{Y}$ contains $\mathbf{G}_m$ where $m < n$.

Next suppose that yIa_1 in $\mathbf{Y}$ and $y > b_1$ in $\mathbf{Y}$. Then $\mathbf{Y}$ contains $\mathbf{G}_m$ where $m < n$. Next suppose that $y > a_1$ in $\mathbf{Y}$ and yIb_1 in $\mathbf{Y}$. When $i \neq n+2$, we see that $\mathbf{Y}$

contains $\mathbf{G}_m$ where $m<n$. When $i=n+2$, we see that $\mathbf{Y}$ contains $\mathbf{G}_0$ when yIc in $\mathbf{Y}$ and that $\mathbf{Y}$ contains $\mathbf{D}^d$ when $y<c$ in $\mathbf{Y}$.

Thus, we are left with the case where $y>a_1$ in $\mathbf{Y}$ and $y>b_1$ in $\mathbf{Y}$. Now let j be the largest integer for which $y>a_j$ in $\mathbf{Y}$ and suppose first that $j\neq i-1$. If yIb_{j+1}, then it follows that $\mathbf{Y}$ contains $\mathbf{G}_m$ where $m<n$ although the value of m and the set of points which generates the copy of $\mathbf{G}_m$ depends on the comparabilities between y and b_j and b_{j-1}. If $y>b_{j+1}$ in $\mathbf{Y}$ and $j+1=i-2$, then $\mathbf{Y}$ contains $\mathbf{D}$. If $y>b_{j+1}$ in $\mathbf{Y}$ and $j+1<i-2$, then $\mathbf{Y}$ contains $\mathbf{G}_m$ where $m<n$. So we may now assume that $j=i-1$. If yIb_{j-1} in $\mathbf{Y}$, then $\mathbf{Y}$ contains $\mathbf{G}_m$ where $m<n$. It follows that $y>b_{j-1}$ in $\mathbf{Y}$ and we conclude that y can replace a_i in this copy of $\mathbf{G}_n$. This completes the proof of Statement P_1, and by duality, Statement P_2 is also established.

Finally, we choose elements $a''_{n+3}\in A$, $b'_1\in B$ so that $a_{n+3}\geqslant a''_{n+3}$ in $\mathbf{X}$, $b'_1\geqslant b_1$ in $\mathbf{X}$,

$$\{b_{n+2}, b_{n+3}, c\}\subset I(a''_{n+3}; \mathbf{Y}), \quad \text{and} \quad \{a_1, a_2, c\}\subset I(b'_1; \mathbf{Y}).$$

Then it follows immediately that $\mathcal{F}=\mathcal{G}_n$.

Case 15. $\mathbf{Y}$ contains $\mathbf{J}_n$ for some $n\geqslant 0$.

In this case, we also assume that if $0\leqslant m<n$, then $\mathbf{Y}$ does not contain $\mathbf{G}_m$, $\mathbf{H}_m$, or $\mathbf{J}_m$. We further assume that $\mathbf{Y}$ does not contain $\mathbf{G}_n$. At this point, we proceed to establish the obvious analogues of Statements P_1 and P_2 as given in Case 14. For the sake of brevity, we do not include the details of the argument since they follow essentially the same pattern. The end conclusion is the same, i.e., $\mathcal{F}=\mathcal{G}_m$.

Case 16. $\mathbf{Y}$ contains $\mathbf{H}_n$ for some $n\geqslant 0$.

In this case, we assume that if $0\leqslant m<n$, then $\mathbf{Y}$ does not contain $\mathbf{G}_m$, $\mathbf{H}_m$, or $\mathbf{J}_m$, and that $\mathbf{Y}$ does not contain either $\mathbf{G}_n$ or $\mathbf{J}_n$. As in Case 15, analogues of Statements P_1 and P_2 are established. Subsequently, points $d''\in A$ and $c'\in B$ are chosen in the obvious manner after which it follows that $\mathcal{F}=\mathcal{H}_n$.

Once the reader has supplied the missing details to Cases 15 and 16, the proof of the theorem has been completed.

6. Concluding remarks

At this point, the reader who has successfully navigated the intricate and sometimes tedious list of cases present in the determination of $\mathcal{R}(I;1)$ would most likely agree that in retrospect the proof is lengthy but straightforward. However, it should be recognized that the details of the argument are motivated naturally by the physical appearance of the poset diagrams. The diagrams also assist in verifying the argument. It should be recognized that the determination of $\mathcal{R}(I;1)$ also represents a major contribution to the solutions of the forbidden subgraph characterization problems for circular-arc graphs and rectangle graphs. (See [11 and 15] for a discussion of these problems.) Furthermore, it is the author's opinion that this contribution would most likely have remained inaccessible without the poset structure to suggest reasonable avenues of attack. Finally, the results of this paper shed additional light on the merits of Kelly's solution to the problem of the determination of $\mathfrak{R}$. Consequently, it would be of great interest to complete the cycle by explaining how Gallai's [4] characterization of comparability graphs can be derived from independent work on posets and interval graphs.

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