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DIMENSION THEORY FOR ORDERED SETS

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ABSTRACT

In 1930, E. Szpilrajn proved that any order relation on a set X can be extended to a linear order on X. It also follows that any order relation is the intersection of its linear extensions. B. Dushnik and E.W. Miller later defined the *dimension* of an ordered set $P = \langle X;\leq\rangle$ to be the minimum number of linear extensions whose intersection is the ordering $\leq$.

For a cardinal m, $\underset{\sim}{2}^m$ denotes the subsets of m, ordered by inclusion. As the notation indicates, $\underset{\sim}{2}^m$ is a product of 2-element chains (linearly ordered sets). Any poset $\langle X;\leq\rangle$ with $|X| \leq m$ can be embedded in $\underset{\sim}{2}^m$. O. Ore proved that the dimension of a poset P is the least number of chains whose product contains P as a subposet. He also showed that the product of m nontrivial chains has dimension m. In particular, $\underset{\sim}{2}^m$ has dimension m, a result of H. Komm. Thus, every cardinal is the dimension of some poset.

It is usually very difficult to calculate the dimension of any "standard" poset. However, dimension can be related to other parameters of a poset. For example, the dimension of a finite poset does not exceed the size of any maximal antichain. Also, T. Hiraguchi showed that any poset of dimension $d \geq 3$ has at least $2d$ elements. Moreover, any integer $\geq 2d$ is the size of some poset of dimension d.

I. Rival (ed.), Ordered Sets, 171–211.

Let d be a positive integer. A poset is *d-irreducible* if it has dimension d and removal of any element lowers its dimension. Any poset whose dimension is at least d contains a d-irreducible subposet. Although there is only one 2-irreducible poset, there are infinitely many d-irreducible posets whenever $d \geq 3$. The set of all 3-irreducible posets was independently determined by D. Kelly and W.T. Trotter, Jr. and J.I. Moore, Jr. There is a 3-irreducible poset of any size n not excluded by Hiraguchi; i.e., for any $n \geq 6$. However, R.J. Kimble, Jr. has shown that a d-irreducible poset cannot have size $2d + 1$ when $d \geq 4$. If $d \geq 4$ and $n \geq 2d$ but $n \neq 2d + 1$, then there is a d-irreducible poset of size n.

A finite poset is *planar* if its diagram can be drawn in the plane without any crossing of lines. Planar posets have arbitrary finite dimension. However, K.A. Baker showed that a finite *lattice* is planar exactly when its dimension does not exceed 2. He also showed that the completion of a poset is a lattice that has the same dimension as the poset. Baker's results and three papers of D. Kelly and I. Rival were used to obtain the list of 3-irreducible posets.

The approach of W.T. Trotter and J.I. Moore, Jr. rested on the observation of Dushnik and Miller that a poset has dimension at most 2 if and only if its incomparability graph is a comparability graph. T. Gallai's characterization of comparability graphs in terms of excluded subgraphs was then applied.

Several other connections between dimension theory for posets and graph theory have been established. For example, posets with the same comparability graph have the same dimension. C.R. Platt reduced the planarity of a finite lattice to the planarity of an *undirected* graph obtained by adding an edge to its diagram.

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E. Szpilrajn [1930] showed that any order relation on a set X can be extended to a linear order on X. He also proved that any order relation is the intersection of its linear extensions. If $\mathbb{C}$ is a family of linear orders (chains) whose intersection is the order relation $\leq$, then $\mathbb{C}$ is a *realizer* of $\leq$; we also say that $\mathbb{C}$ *realizes* $\leq$. B. Dushnik and E.W. Miller [1941] defined the *dimension* of an ordered set (poset) $\langle X;\leq\rangle$ to be the minimum cardinality of a realizer of $\leq$.

In this survey, we usually deal with the case of finite dimension. For a positive integer k, any poset of dimension at least k contains a finite subposet of dimension k. (This compactness property follows from the compactness theorem of first-order logic; see Harzheim [1970] and the review by K.A. Baker.) Moreover, any finite poset has finite dimension. Consequently, most of the posets that we consider will be finite.

1. INTRODUCTION

In this section, we shall introduce the many definitions and constructions that are required to study dimension. The 16 posets in the first figure are meant to provide the reader with some realistic examples in order to better appreciate the definitions and constructions we give. We want the reader to get a feeling for the main concepts before we get involved in difficult dimension calculations. No subsequent section requires any background except this introduction, and possibly, Section 3.

If a poset is denoted by P, then its underlying set will usually also be denoted by P, and its order relation by $\leq$ or $\leq_P$. An extension of P means an extension of $\leq_P$. If P is a subposet of Q, then $\dim P \leq \dim Q$ since any linear extension of Q restricts to a linear extension of P. T. Hiraguchi [1951] showed that the dimension increases by at most one when a single point is added. Equivalently, the "one-point removal theorem" is valid; that is, removing one point from a poset decreases its dimension by at most one. We shall consider this and other removal theorems in Sections 5 and 6.

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The converse of a binary relation R is denoted by R^d. Clearly, a poset P and its dual P^d have the same dimension. A poset is *d-irreducible* if it has dimension $d \geq 2$ and the removal of any element lowers its dimension. An *irreducible* poset is d-irreducible for some $d \geq 2$. By the compactness property, every irreducible poset is finite. By the one-point removal theorem, a poset is d-irreducible when $\dim P \geq d$ and $\dim(P - \{x\}) < d$ for every $x \in P$. The only 2-irreducible poset is the 2-element antichain (two incomparable elements). Figure 1 shows all 3-irreducible posets with at most 8 elements (up to isomorphism and duality). (We are using the notation of Kelly [1977].) Two of these posets have 6 elements, three have 8 elements, and the remainder have 7 elements. In Section 3, we show how to check whether a poset has dimension at most two.

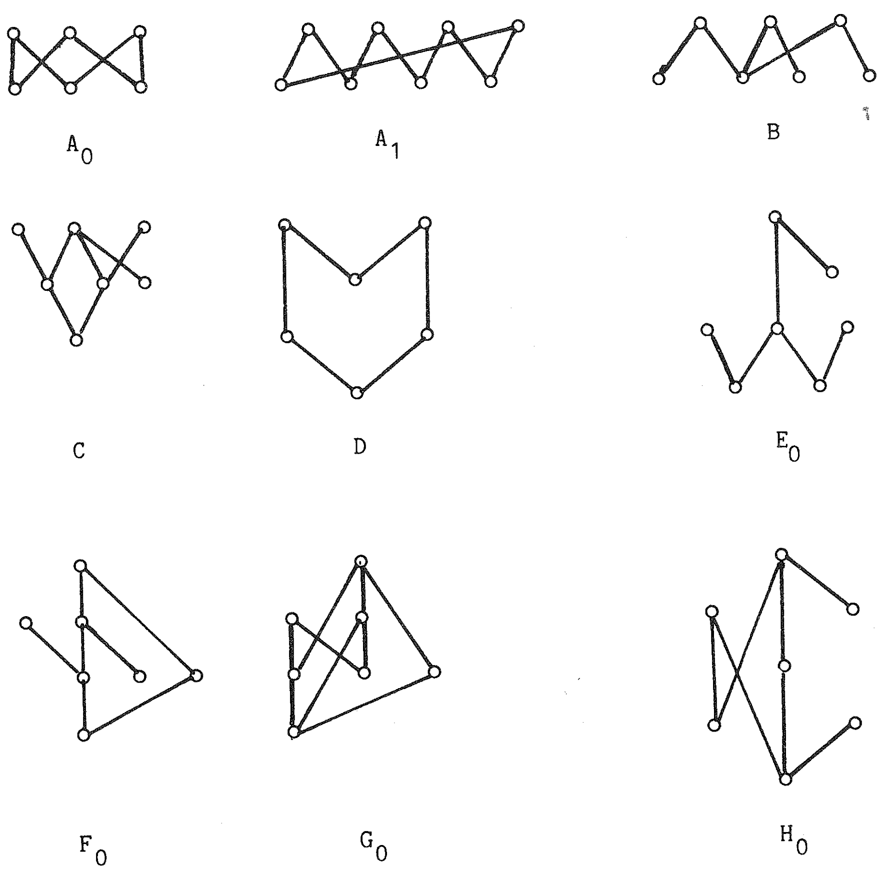
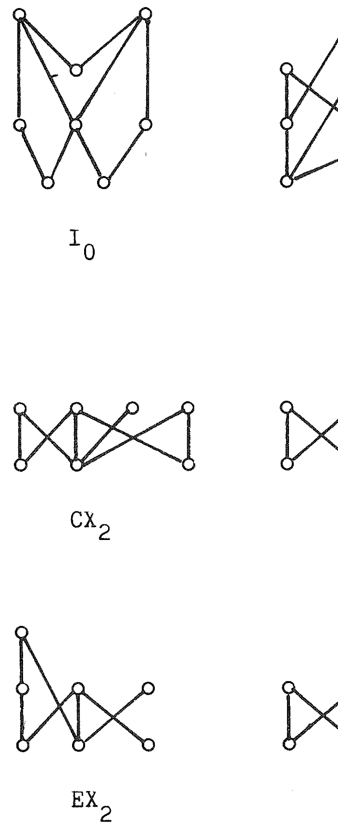


FIGURE 1. The 3-irreducible posets with at most 8 elements.



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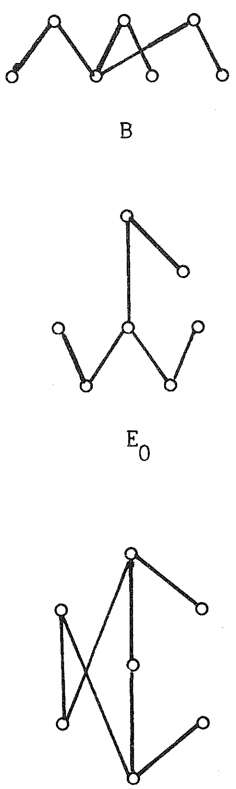

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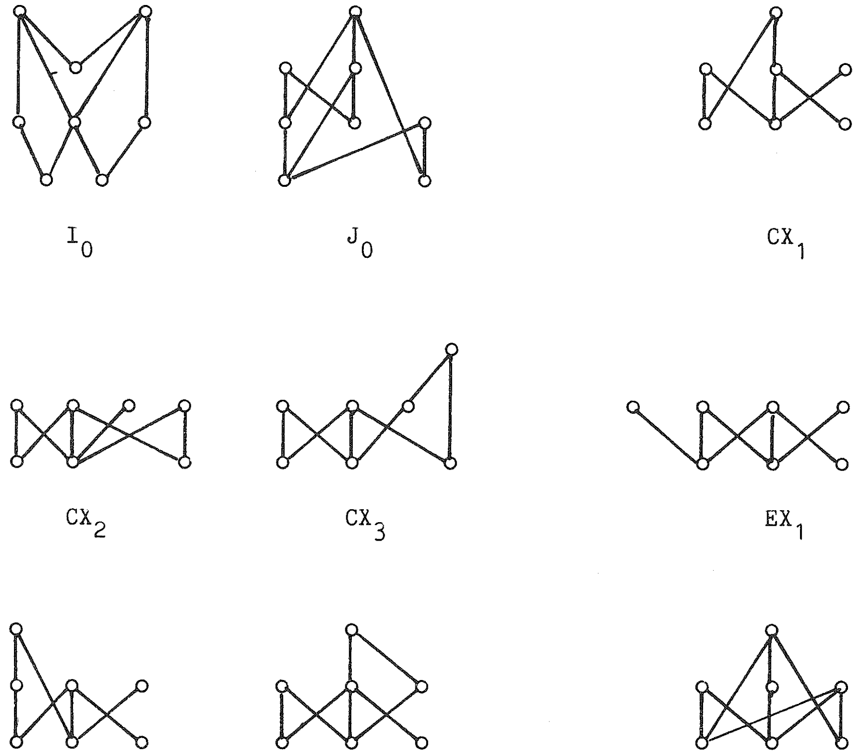

FIGURE 1. (concluded)

By Figure 1, any poset of dimension 3 must have at least 6 elements. In fact, this is a special case of the following inequality of Hiraguchi [1951], valid for $|P| \geq 4$:

$$\dim P \leq |P|/2.$$

This result is proved in Section 5.

The posets A_0 and A_1 of Figure 1 are the two smallest members of an infinite family A_n $(n \geq 0)$ of 3-irreducible posets. These posets, called *crowns*, appeared in Harzheim [1970] and Baker, Fishburn and Roberts [1971]. D. Kelly and I.

Rival [1975b] discovered infinitely many more 3-irreducible posets. They presented the four infinite families E_n, F_n, G_n, H_n ($n \geq 0$), whose smallest members are shown in Figure 1. The list of all 3-irreducible posets was completed independently by D. Kelly [1977], and W.T. Trotter and J.I. Moore [1976a]. Up to duality, this list consists of the seven infinite families A_n, E_n, F_n, G_n, H_n, I_n, J_n ($n \geq 0$), whose smallest members are shown in Figure 1, together with the remaining posets of that figure. We shall return to the topic of irreducible posets in Section 7.

The following alternative definition is often credited to O. Ore [1962], but appeared earlier in Hiraguchi [1955] : The dimension of a poset P is the minimum size of a family $(C_i \mid i \in I)$ of chains such that P is isomorphic to a subposet of the direct product $\Pi(C_i \mid i \in I)$. If $(C_i \mid i \in I)$ is a family of linear extensions realizing P, then the diagonal mapping ϕ embeds P into the direct product of these chains ($\phi(x)$ is the constant function with value x for each x in P). To complete the proof of the equivalence of the two definitions, let P be the direct product of the family $(C_i \mid i \in I)$ of chains. For $i \in I$, we define the extension E_i of P by $\langle x, y\rangle \in E_i$ iff $\pi_i(x) < \pi_i(y)$, where π_i denotes the i-th projection function. If E_i' is a linear extension of E_i for each $i \in I$, then $(E_i' \mid i \in I)$ realizes P. H. Komm [1948] proved that $\dim \underset{\sim}{2}^{\mathfrak{m}} = \mathfrak{m}$, where $\underset{\sim}{2}$ denotes the 2-element chain and $\mathfrak{m}$ denotes a possibly infinite cardinal. It follows that the product of $\mathfrak{m}$ nontrivial chains has dimension $\mathfrak{m}$.

The Ore definition of dimension is often appropriate when no explicit calculation of dimension is required. When the dimension must be calculated, the original Dushnik-Miller definition is used. We shall describe a simple technique which reduces the "bookkeeping" involved in calculating the dimension of a finite poset.

A (linear) extension of a subposet of P will be called a *partial (linear) extension* of P. $\mathrm{Tr}(R)$ denotes the transitive closure of a binary relation R. For any partial extension E, $\mathrm{Tr}(\leq_P \cup E)$ is an extension of P. For a poset $P, \mathcal{I}(P) = \{\langle a,b\rangle \mid a \| b \text{ in } P\}$, the set of *incomparable pairs*. Any $u \in \mathcal{I}(P)$ can be considered as a partial linear extension of P.

For $u, v \in \mathcal{I}(P)$, u *forces* v (in symbols, $\langle u,v\rangle \in F$) if $v \in \mathrm{Tr}(\leq_P \cup \{u\})$. This concept was introduced by I. Rabinovitch and I. Rival [1979], who observed that F is an order relation on $\mathcal{I}(P)$. The incomparable pair $\langle a,b\rangle$ is called a *critical pair* if $x < b$ implies $x < a$, and $x > a$ implies $x > b$. A pair $\langle a,b\rangle \in P^2$ is a *max-min pair* if a is maximal and b is minimal. Observe that every incomparable max-min pair is critical. For a poset P, Crit(P) denotes the set of critical pairs. Crit(P) is the set of minimal elements of $\langle \mathcal{I}(P);F \rangle$. We denote the set of maximal elements of $\langle \mathcal{I}(P);F \rangle$ by $\mathfrak{N}(P)$; these pairs have been called *unforced, nonforcing* and *nonforced*. We shall use the last term. Moreover, $\mathfrak{N}(P)$ is the converse of Crit(P). Thus, critical pairs are also called "reversed nonforced pairs". For any finite poset, every incomparable pair is forced by a critical pair.

For $S \subseteq \mathcal{I}(P)$, an *alternating cycle of length* n for S is a sequence of the form $x_1, y_1, x_2, y_2, \ldots, x_n, y_n$ with $\langle x_i, y_i\rangle \in S$, $1 \leq i \leq n$, and $y_i \leq x_{i+1}$ in P (with subscripts taken modulo n). S is *cycle-free* if it has no alternating cycle. The above cycle is *minimal* if $y_i \leq x_j$ implies $j = i+1 \pmod n$. If S is not cycle-free, then there is a minimal alternating cycle for S. If the above alternating cycle is minimal, then both $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_n\}$ are n-element antichains. For example, $x_1 \leq x_i$ would imply that $y_n \leq x_i$. Thus, the length of a minimal alternating cycle for $\mathcal{I}(P)$ cannot exceed the width of the poset P.

We say that a family $(E_i \mid i \in I)$ of partial extensions *realizes* P (or $\leq_P$) when $(C_i \mid i \in I)$ realizes $\leq_P$ for any choice of linear extensions C_i of P that extend E_i $(i \in I)$. The dimension of P is the minimum (nonzero) size of such a realizer. Let P be a poset in which every incomparable pair is forced by a critical pair. If P is not a chain, then dim P is specified by each of the following three definitions:

(a) The least number of partial extensions of P whose union contains Crit(P).

(b) The least number of partial linear extensions of P whose union contains Crit(P).

(c) The least number of cycle-free sets covering Crit(P).

The set Crit(P) is the smallest subset of $\mathcal{I}(P)$ that could be used in the above definitions; thus, Crit(P) is a "critical" set in dimension calculations. When using definition (a) or (b), we often write a critical pair $\langle a,b\rangle$ as "$a < b$" and call it a *critical inequality*.

Henceforth, any poset is understood to be finite unless the contrary is explicitly stated. P will always denote a poset. A realizer is *irredundant* if no proper subset is a realizer. The dimension is the smallest size of an irredundant realizer consisting of linear extensions, while the *rank* is the largest. Note that realizers used for rank must consist of linear extensions, in contrast to the case for dimension. For example, the realizer consisting of Crit(P), considered as partial linear extensions, is not permitted. Since any proper extension of P satisfies some nonforced pair, I. Rabinovitch and I. Rival [1979] observed that rank(P) $\leq |\mathfrak{n}(P)|$. S.B. Maurer, I. Rabinovitch and W.T. Trotter ([1980a], [1980b], [1980c]) describe how to determine the rank from $\mathfrak{n}(P)$, considered as a directed graph. Since $\mathfrak{n}(P)$ and Crit(P) are converses of each other, any extension satisfying all of $\mathfrak{n}(P)$ will fail every critical inequality. An extension satisfying all of $\mathfrak{n}(P)$ is called a *weak* extension. In Section 5, we show that every irreducible poset, except the 2-element antichain, has a weak extension.

We shall define a subposet $\underset{\sim}{P}(P)$, the set of *irreducible* elements of P, that has the same dimension as P. (The term "irreducible" has a completely different meaning when applied to elements than when applied to posets.) $\underset{\sim}{J}(P)$ is the subposet of *join-irreducible* elements ($a \in \underset{\sim}{J}(P)$ iff $a = \bigvee S$ implies $a \in S$). If P contains a least element 0, it is join-reducible since we allow $S = \emptyset$. $\underset{\sim}{M}(P)$, the subposet of meet-irreducible elements, is defined dually. We define

$$\underset{\sim}{P}(P) = \underset{\sim}{J}(P) \cup \underset{\sim}{M}(P),$$

a subposet of P.

If P is a lattice and $x \in P$, then $x \in \underset{\sim}{J}(P)$ iff x has a unique lower cover. In an arbitrary finite poset P, it is much harder to recognize the join-irreducible elements. This difficulty is overcome by constructing the finite lattice $L = \underset{\sim}{L}(P)$ which contains P as a subposet and satisfies $\underset{\sim}{J}(L) = \underset{\sim}{J}(P)$, $\underset{\sim}{M}(L) = \underset{\sim}{M}(P)$ and $\underset{\sim}{P}(L) = \underset{\sim}{P}(P)$. $\underset{\sim}{L}(P)$, the *completion* of P, is called the "completion by cuts" and defined in Birkhoff [1967]; it is also called the "MacNeille completion". The definition of $\underset{\sim}{L}(P)$ does not require P to be finite. Whenever a complete lattice contains a poset P, it contains the completion of P as a subposet. We shall not define the completion. Given a finite poset P and a lattice L, one can use the following result to determine whether $L = \underset{\sim}{L}(P)$.

LEMMA 1.1. (Banaschewski [1956] or Schmidt [1956]) *If the poset* P *is a subposet of the finite lattice* L, *then* $L = \underset{\sim}{L}(P)$ *iff* $\underset{\sim}{P}(L) \subseteq P$.

In particular, the completion of a finite chain is the chain itself. Using definition (a) or (b) of dimension, the next result shows that $\underset{\sim}{P}(P)$ and P have the same dimension.

LEMMA 1.2. (Kelly [1981]) *For a finite poset* P,

$$\mathrm{Crit}(P) \subseteq \underset{\sim}{M}(P) \times \underset{\sim}{J}(P).$$

Proof. Let $\langle a,b\rangle \in \mathrm{Crit}(P)$ and suppose that $a = \bigwedge S$ with $a \notin S$. For all $x \in S$, $x > a$, and therefore, $x > b$. Consequently, $a = \bigwedge S \geq b$, a contradiction.

PROBLEM 1.3. Let P be an irreducible poset. Since $\underset{\sim}{P}(P)$ has the same dimension as P, $\underset{\sim}{P}(P) = P$. By definition (a), every element of P appears in some critical pair. For $x \in P$, it follows by Lemma 1.2 that $x \in \underset{\sim}{M}(P)-\underset{\sim}{J}(P)$ if and only if x is the first element of some critical pair, but never the second. If x appears in both positions in critical pairs, then x must be doubly irreducible (i.e., $x \in \mathrm{Irr}(P) = \underset{\sim}{J}(P) \cap \underset{\sim}{M}(P)$). We do not know whether the converse holds.

PROBLEM 1.4. From the Ore definition,

$$\dim(P \times Q) \leq \dim P + \dim Q,$$

for any posets P and Q. In general, this inequality can be strict. For example, the product of two nontrivial antichains is an antichain and thus has dimension two, although the sum of the dimensions is four. A poset is bounded if it has a least element (zero) and a greatest element (one). K.A. Baker [1961] showed that equality holds above if P and Q are bounded posets. Let us indicate a proof. Suppose the linear extensions $C_1, C_2, \ldots, C_{m+n}$ realize $P \times Q$, $\langle 0,1\rangle < \langle 1,0\rangle$ holds in $C_1, \ldots, C_m$, and $\langle 1,0\rangle < \langle 0,1\rangle$ holds in $C_{m+1}, \ldots, C_{m+n}$. If $\langle a,b\rangle \in \mathcal{I}(P)$, then $\langle a,1\rangle < \langle b,0\rangle$ must hold in C_i for some $1 \leq i \leq m$. Consequently, $m \geq \dim P$ and, similarly, $n \geq \dim Q$. Therefore, $\dim(P \times Q) \geq \dim P + \dim Q$. Thus, dimension is additive for direct products of bounded posets. Let $b(P, Q) \in \{0,1,2\}$ count how many of P and Q are not bounded. We *conjecture* that

$$\dim(P \times Q) \geq \dim P + \dim Q - b(P, Q) .$$

In the $b = 2$ case, it would suffice to prove (or disprove) this conjecture for irreducible posets P and Q.

K.A. Baker [1961] proved that $\underset{\sim}{L}(P)$ has the same dimension as P for any (possibly infinite) poset P. The completion of a chain is a complete chain. A product of complete chains is a complete lattice, and therefore contains, as a subposet, the completion of each of its subposets. Using the Ore definition of dimension, Baker's result now follows easily. (A proof using the Dushnik-Miller definition is given in Kelly [1981].) Recall that $\underset{\sim}{P}(P) = \underset{\sim}{P}(\underset{\sim}{L}(P))$. If P is finite, then

$$\dim P = \dim \underset{\sim}{P}(P) = \dim \underset{\sim}{P}(\underset{\sim}{L}(P)) = \dim \underset{\sim}{L}(P),$$

because the operator $\underset{\sim}{P}$ preserves dimension in the first and last equalities. This proves Baker's result for finite P.

For $n \geq 3$, let us define the poset

$$S_n = \underset{\sim}{P}(\underset{\sim}{2}^n).$$

S_n has $2n$ elements and dimension n. We call S_n the *standard* poset of dimension n. Let $S_n = \{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n\}$, where a_i is minimal and $a_i \| b_i$ for $1 \leq i \leq n$. The following $n-1$ partial linear extensions realize $S_n - \{b_1\}$:

$$a_3, a_4, \ldots, a_n, a_1, b_2, a_2$$
$$a_2, a_1, b_3, a_3$$
$$b_i, a_i \qquad (4 \leq i \leq n).$$

Consequently, S_n is n-irreducible. Observe that S_3 is the poset A_0 of Figure 1. For an infinite cardinal $\mathfrak{m}$, the poset $S_{\mathfrak{m}}$, consisting of one-element subsets of $\mathfrak{m}$ and their complements, is the *standard* poset of dimension $\mathfrak{m}$.

We can associate a hypergraph to a finite poset P so that the chromatic number of the hypergraph is the dimension of the poset. The hypergraph $\mathcal{H}(P)$ is defined as follows:

The vertex set of $\mathcal{H}(P)$ is $\mathrm{Crit}(P)$.

An edge of $\mathcal{H}($

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Schönheim and Woodrow [1
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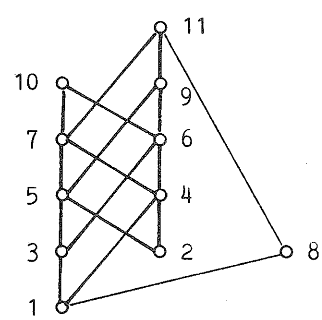

G_2

FIGURE 2.

An edge of $\mathcal{H}(P)$ is any set of the form

$$\{\langle x_i, y_i\rangle \mid 1 \le i \le n\}$$

where $x_1, y_1, x_2, y_2, \ldots, x_n, y_n$ is a minimal alternating cycle for $\mathrm{Crit}(P)$.

By definition (c), P has dimension at most k if and only if the hypergraph $\mathcal{H}(P)$ is (vertex) k-colorable. By Kelly, Schönheim and Woodrow [1981], a graph can be constructed that is k-colorable if and only if $\mathcal{H}(P)$ is.

Calculating the chromatic number of $\mathcal{H}(P)$ is not usually the easiest way to determine the dimension of a poset P. Even the hypergraph itself may be difficult to calculate. For example, the only posets P of Figure 1 for which $\mathcal{H}(P)$ is a graph are A_0, A_1, C, D, and F_θ. However, $\mathcal{H}(G_n)$ is a graph whenever $n \ge 1$. The simple structure of this graph will be exploited in Section 7. In Figure 2, we show G_2 and its hypergraph; the labelling indicates a weak linear extension (so that $i > j$ as integers whenever $\langle i, j\rangle$ is critical).

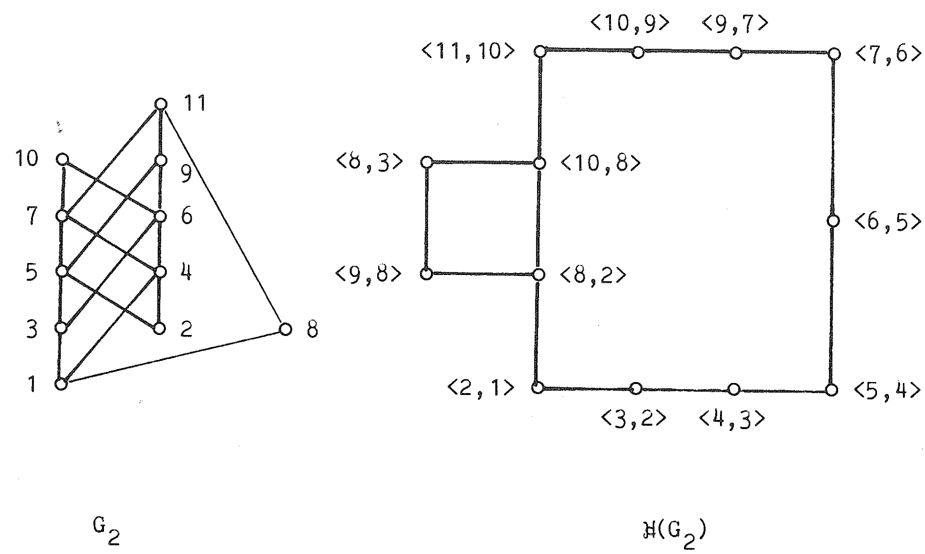

FIGURE 2. A poset and its hypergraph

The connection between dimension and graph (or hypergraph) coloring has been exploited to determine the computational complexity of dimension. Section 4 is devoted to this topic.

2. TWO INTERPRETATIONS OF DIMENSION

THEOREM 2.1. (Trotter and Moore [1976b]) *Let G be a possibly infinite connected graph. Let P be all connected induced subgraphs of G, ordered by inclusion. (P includes the empty set.) The dimension of P is the number of noncut vertices of G. (A vertex v of G is a noncut vertex when the removal of v leaves a connected subgraph.)*

Proof. A critical pair for P is $\langle A,\{u\}\rangle$ where A is a connected component of $G-\{u\}$. Since every incomparable pair is forced by a critical pair, we can use definition (c) of dimension.

Let v_i, $0 \le i < \mathfrak{m}$, be the noncut vertices of G. The subposet

$$\{\, \{v_i\} \mid 0 \le i < \mathfrak{m} \,\} \cup \{\, G - \{v_i\} \mid 0 \le i < \mathfrak{m} \,\}$$

is isomorphic to the standard $\mathfrak{m}$-dimensional poset. Thus, dim $P \ge \mathfrak{m}$. Next, we define $\mathfrak{m}$ cycle-free sets covering Crit(P). For $0 \le i < \mathfrak{m}$, S_i consists of the critical pairs $\langle A,\{u\}\rangle$ with $d(x,v_i) > d(u,v_i)$ for all $x \in A$. Let $\langle A,\{u\}\rangle$ be a critical pair. If $u = v_i$, then $\langle A,\{u\}\rangle$ occurs in S_i. Otherwise, we can assume that u is a cut vertex. Choose a noncut vertex v_i in a connected component of $G-\{u\}$ different from A. For $x \in A$, $d(x,v_i) = d(x,u) + d(u,v_i) > d(u,v_i)$. Consequently, $\langle A,\{u\}\rangle$ occurs in S_i. Since each S_i is cycle-free, dim $P = \mathfrak{m}$.

THEOREM 2.2. (Wille [1975]) *Let L be a finite distributive lattice and $P = \underset{\sim}{J}(L)$, the subposet of join-irreducible elements.* dim P *is the least number of chains whose union generates L.*

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Proof. If P is
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conclude that P is
chains. This completes

3. DIMENSION AT MOST TW

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Remark. Applying the decomposition theorem of Dilworth [1950], the above statement means that dim P is the minimum width of a generating set of L.

Proof. If P is embedded in the direct product of finite chains $C_1, C_2, \ldots, C_n$, then applying the usual duality theory for finite distributive lattices (see, for example, Birkhoff [1967]), L is a homomorphic image of the free $\{0,1\}$-distributive product of $\{0,1\}$-chains $D_1, D_2, \ldots, D_n$, where $|D_i| = |C_i| + 1$ for $1 \le i \le n$. For $1 \le i \le n$, D'_i, the image of D_i in L, is a chain. Clearly, $D'_1 \cup D'_2 \cup \ldots \cup D'_n$ generates L.

If L is generated by the $\{0,1\}$-chains D_i, $1 \le i \le n$, then L is a homomorphic image of the free $\{0,1\}$-distributive product of $D_1, D_2, \ldots, D_n$. Reversing the previous argument, we conclude that P is embedded in the direct product of n chains. This completes the proof.

3. DIMENSION AT MOST TWO

As usual, all posets are finite. For a poset P, $C(P)$ will denote its covering digraph; $\langle a, b\rangle$ is in $C(P)$ exactly when $a \prec b$ in P. The usual diagram ("Hasse diagram") of P is a *representation* of $C(P)$ in the plane where each edge $\langle a,b\rangle$ is represented by a straight line segment with a lower than b. The poset P is *planar* if $C(P)$ can be represented in the plane with no crossing of edges. A bounded planar poset is a lattice. A lattice is planar iff its dimension does not exceed two. If a poset is not a lattice we could take its completion, and determine whether that is planar. In fact, to prove that a poset has dimension at most two, it suffices to embed it in any planar lattice. If a planar poset P has a zero, its dimension does not exceed three (Trotter and Moore [1977]). However, planar posets have arbitrary finite dimension (Kelly [1981]).

Let $\phi: L \to C_1 \times C_2$ be a (poset) embedding of the lattice L into the direct product of the two chains C_1 and C_2. We identify each chain C_i with the subposet $\{1,2,\ldots,|C_i|\}$ of the reals. We shall write $\langle a_1, a_2\rangle$ for $\phi(a)$. As in

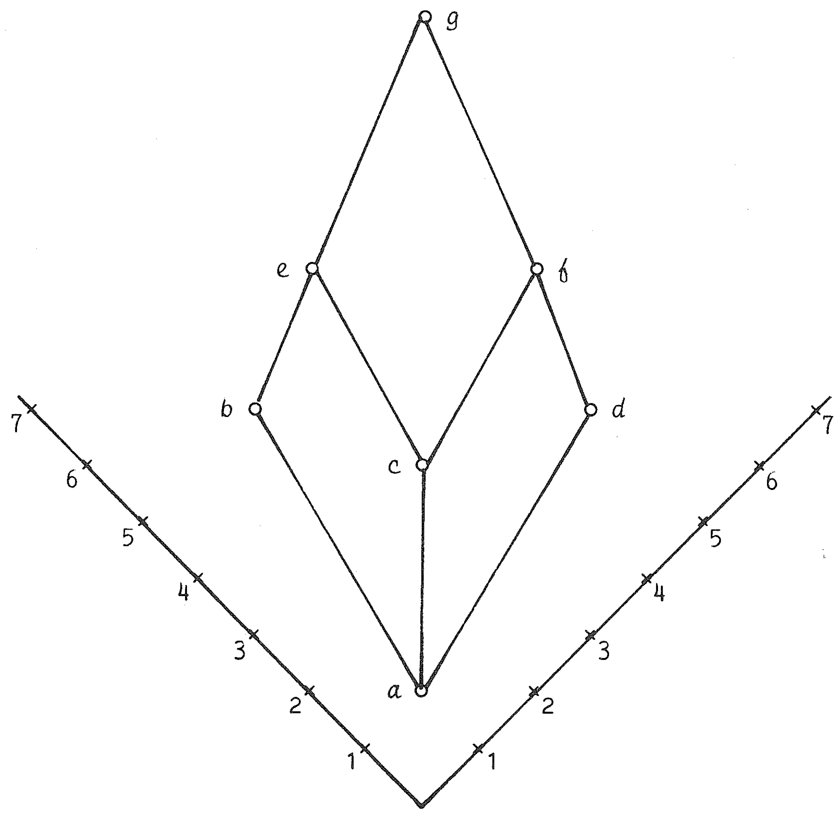
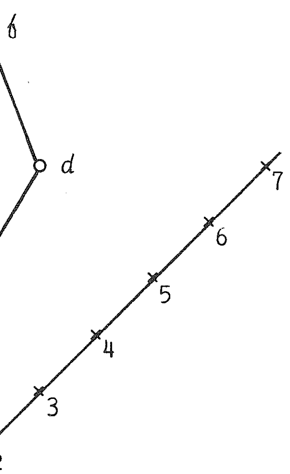


FIGURE 3. A planar representation

Figure 3, we draw axes at 45° to the horizontal so that the line $x_1 + x_2 = 0$ is horizontal and the line $x_1 - x_2 = 0$ is vertical. Each $a \in L$ is plotted at the point with coordinates $\langle a_1, a_2 \rangle$. Joining covering pairs by straight lines, we obtain a planar representation of L. Certainly, a is plotted below b whenever $a < b$ in L. Let $a \prec b$ and $c \prec d$ in L with $|\{a, b, c, d\}| = 4$, and suppose that the corresponding edges cross. It would then follow that $\langle a_1, a_2 \rangle$ and $\langle c_1, c_2 \rangle$ are both less than $\langle b_1, b_2 \rangle$ and $\langle d_1, d_2 \rangle$, and therefore, that $a \vee c \leq b \wedge d$ holds in L. Since $a \prec b$, we conclude from $a \leq a \vee c \leq b$ that $a \vee c \in \{a, b\}$. If $a \vee c = b$, then $c < b \leq b \wedge d \leq d$ would imply $b = d$, a contradiction. Therefore, $a \vee c = a$. By symmetry, $a \vee c = c$, implying $a = c$, a contradiction.



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The planar representation of the lattice L in Figure 3 is derived from the diagonal mapping and the following two linear extensions of L:

$$C_1 = a, b, c, e, d, f, g$$

$$C_2 = a, d, c, f, b, e, g$$

Let G be the *incomparability graph* of P, the graph with vertex set P and edge set $\mathcal{I}(P)$. In Section 8, we show that G determines the dimension of P. This result is usually stated using the complement of G, which is called the *comparability graph* of P and denoted by Comp(P). We shall also write $\mathcal{I}(P)$ for the incomparability graph of G. A *transitive orientation* of a graph G is a strict order relation λ on the vertex set of G such that x and y are comparable with respect to λ iff xy is an edge of G. Clearly, a graph having a transitive orientation is the same thing as a comparability graph. A transitive orientation λ of $\mathcal{I}(P)$ is called a *conjugate* order for $\leq_P$ (or for P). B. Dushnik and E.W. Miller [1941] showed that dim $P \leq 2$ iff P has a conjugate order. K.A. Baker [1961] combined this result with a result of J. Zilber (ex. 7(c), p.32 of Birkhoff [1967]) to show the equivalence for lattices between planarity and dimension at most two.

For a poset P embedded in the product $C_1 \times C_2$ of two chains, the *standard* conjugate order λ is defined by:

$$x \,\lambda\, y \text{ iff } \pi_1(x) < \pi_1(y) \quad (\text{for } x \| y \text{ in } P).$$

If λ is a conjugate order for $\leq_P$, then $\leq_P \cup \lambda$ and $\leq_P \cup \lambda^d$ are two linear orders that realize P. Consequently, given any conjugate order λ for a poset P, there is an embedding of P into the product of two chains for which λ is standard.

A conjugate order λ for a lattice L has a geometric interpretation: for $x \| y$ in L, $x \,\lambda\, y$ iff x is "to the left of" y in some planar representation of L. Given a planar representation of a lattice L, the set of lower covers of any element has an obvious left-to-right ordering λ. If $x \| y$ in L, then let x' and y' be the lower covers of $x \vee y$ that are comparable with x and y respectively. We define $x \,\lambda\, y$ if $x' \,\lambda\, y'$, and $y \,\lambda\, x$ if $y' \,\lambda\, x'$; λ is the left-to-right ordering of L with respect to the planar representation. For example, in Figure 3, b is to the left of f. (For more details, see Kelly and Rival [1975a].)

If the planar representation of a lattice L is obtained from an embedding into the product of chains C_1 and C_2 by the procedure we gave above, observe that the left-to-right ordering coincides with the standard conjugate order.

Let λ be a conjugate order for a poset P. We can assume that P is a subposet of a product $C_1 \times C_2$ of chains and that λ is standard. Since $C_1 \times C_2$ is a lattice it contains $\underset{\sim}{L}(P)$ as a subposet. Clearly, the standard conjugate order for $\underset{\sim}{L}(P)$ extends λ. We have shown that any conjugate order for a poset P can be extended to a conjugate order for $\underset{\sim}{L}(P)$. (In fact, the extension is unique.) Thus, any conjugate order λ for an arbitrary poset P has the following interpretation: λ is the restriction to P of the left-to-right ordering of $\underset{\sim}{L}(P)$ with respect to some planar representation.

M.C. Golumbic ([1977a], [1977b], [1980]) has developed an algorithm to decide whether a graph is a comparability graph. Since this algorithm has complexity $O(n^3)$, where n is the number of vertices, deciding whether $\dim(P) \leq 2$ has complexity $O(|P|^3)$ by the Dushnik-Miller result.

D. Kelly and I. Rival [1975a] determined the minimum list $\mathcal{L}$ of lattices such that every nonplanar lattice contains a lattice in $\mathcal{L}$ as a subposet. In particular, each lattice in $\mathcal{L}$ is nonplanar. In contrast to K. Kuratowski's [1930] characterization of planar graphs, $\mathcal{L}$ is infinite. $\mathcal{P} = \{ \underset{\sim}{P}(L) \mid L \in \mathcal{L} \}$ is an infinite set of 3-irreducible posets because, for each $L \in \mathcal{L}$, the *subposet* $L - \{x\}$ is a planar lattice for each $x \in \underset{\sim}{P}(L)$ (Kelly and Rival [1975b]). $\mathcal{P}$ contains the first nine posets of Figure 1.

Let $G(P)$ be the ordinary graph corresponding to the covering digraph $C(P)$, and let $G_{01}(P)$ be the graph obtained from $G(P)$ by adding an extra edge from 0 to 1. C.R. Platt [1976] showed that a lattice L is planar iff $G_{01}(L)$ is planar. If P is an n-element poset with dimension at most 2, then $|\underset{\sim}{L}(P)| \leq n^2$. Clearly, one cannot form the completion in any polynomial algorithm to check whether $\dim(P) \leq 2$. If a subposet Q of $\underset{\sim}{L}(P)$ is not a lattice, one can add a point x to Q so that $Q \cup \{x\}$ is a subposet of $\underset{\sim}{L}(P)$. Starting with P, this procedure is repeated until a lattice L is obtained or Q has more that n^2 elements. If L is obtained, $G_{01}(L)$ is formed and Platt's result is applied. Thus, we have indicated another proof that dimension at most 2 is polynomial, but with a much less efficient algorithm than Golumbic's.

4. COMPUTATIONAL COMPLE

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4. COMPUTATIONAL COMPLEXITY OF DIMENSION

M. Yannakakis [1981] has shown that it is NP-complete to determine if the dimension of a finite poset is at most 3. The reader is referred to the book of M.R. Garey and D.S. Johnson [1979] for the precise definitions and background on computational complexity. The intuitive idea (which more than suffices for us) is that any algorithm for deciding an NP-complete problem will use an excessive amount of time, even for small examples. M. Yannakakis associated to each finite graph G a poset $Y(G)$ such that G is 3-colorable iff $\dim(Y(G)) \le 3$. We shall present his construction of $Y(G)$. Since 3-colorability is NP-complete (Garey, Johnson and Stockmeyer [1976]), the above equivalence proves that the dimension at most 3 problem is also NP-complete. Since the dimension ≤ 2 problem is polynomial (see Section 3), it follows that the dimension equal to 3 problem is NP-complete. Consequently, dimension equal to k is NP-complete for each fixed $k \ge 3$. By similar methods, M. Yannakakis [1981] also showed that the dimension ≤ 4 problem for posets of unit length is NP-complete, but he left open the complexity of the dimension ≤ 3 problem for posets of unit length. Using a reduction to hypergraph 2-colorability, which is NP-complete by Lovász [1973], E.L. Lawler and O. Vornberger [1981] proved the weaker result that the dimension problem is NP-complete.

Let G be a finite graph with its vertices labelled $1, 2, \ldots, n$ that has m edges $E_1, E_2, \ldots, E_m$. The underlying set of the poset $Y(G)$ consists of a_i, b_i, a_i', b_i' $(1 \le i \le n)$ and c_k, d_k, c_k', d_k' $(1 \le k \le m)$. The ordering of $Y(G)$ is indicated in Figure 4. The comparabilities in the left-hand diagram are the same for every graph with n vertices and m edges. For $i \in E_k$, let

$$u_{ik} = \begin{cases} c_k & \text{, if } i = \min E_k\text{;} \\ d_k & \text{, if } i = \max E_k\text{.} \end{cases}$$

and

$$v_{ik} = \begin{cases} d_k & \text{, if } i = \min E_k\text{;} \\ c_k & \text{, if } i = \max E_k\text{.} \end{cases}$$

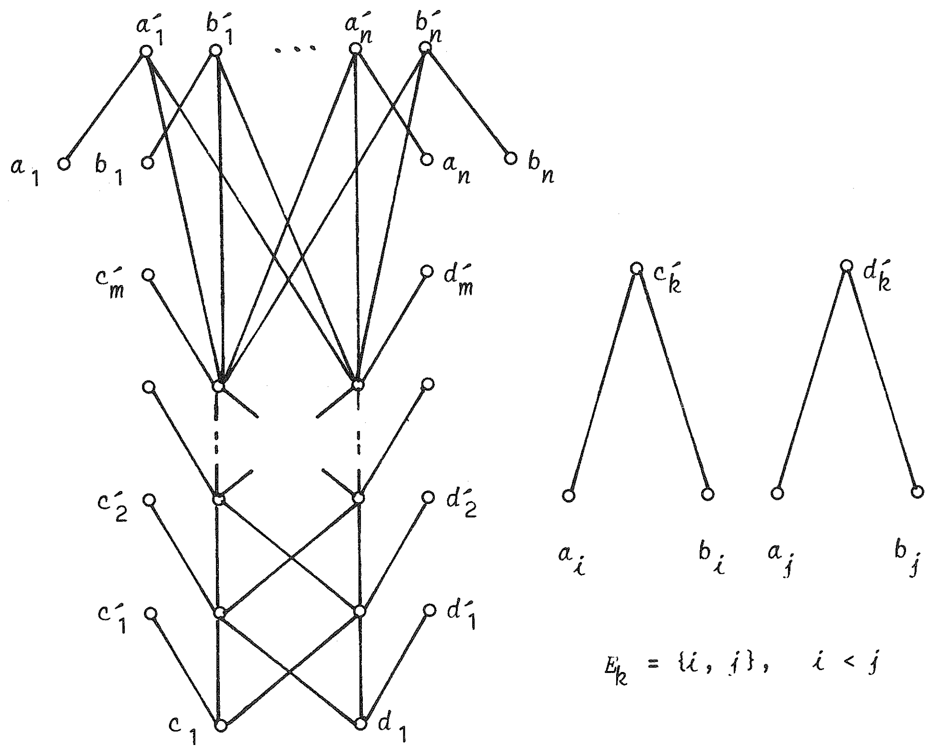

FIGURE 4. The associated poset $Y(G)$

As indicated in Figure 4, the following comparabilities also hold in $Y(G)$, where $i \in E_k$:

$$a_i < u'_{ik} \quad , \quad b_i < u'_{ik}.$$

Let $1 \le i, j \le n$ and $1 \le k \le m$. The critical inequalities for $Y(G)$ are:

$$a'_i < b_j \quad , \quad b'_i < a_j$$

$$a'_i < a_j \quad , \quad b'_i < b_j \qquad (i \ne j)$$

$$c'_k < d_k \quad , \quad d'_k < c_k$$

$$a_i < c_1 \; , \; b_i < c_1 \; , \; a_i < d_1 \; , \; b_i < d_1$$

$$v'_{ik} < a_i \quad , \quad v'_{ik} < b_i \qquad (i \in E_k) .$$

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5. DIMENSION, WIDTH AND

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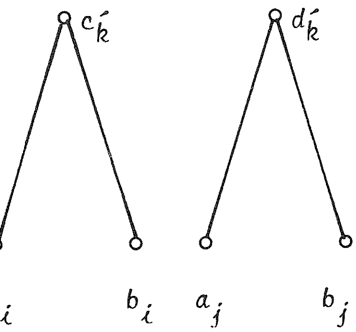

$E_k = \{i, j\}$, $i < j$

poset $\Upsilon(G)$

comparabilities also hold

$\leq k \leq m$. The critical

$\neq j)$

, $b_i < d_1$

$\in E_k)$.

We first assume that $\dim(\Upsilon(G)) \leq 3$, and let L_0, L_1 and L_2 be three linear extensions realizing $\Upsilon(G)$. Any isolated vertex is colored arbitrarily. If $i \in E_k$, then the following three critical inequalities must hold in three different linear extensions:

$$a'_i < b_i \quad , \quad b'_i < a_i \quad , \quad u'_{ik} < v_{ik} \ .$$

If the last one holds in L_α, then we set $\mathrm{col}(i) = \alpha$. If $E_k = \{i, j\}$ and i and j are both assigned the color α, then $c'_k < d_k$ and $d'_k < c_k$ would both hold in L_α, a contradiction. Thus, G is 3-colorable.

Let G be a graph that is colorable with the three colors 0, 1, 2. We shall define three partial linear extensions that realize $\Upsilon(G)$. If $i \notin E_k$, then we define u'_{ik} to be the empty sequence. The range of i is 1, 2, ..., n, in this order when we write $i\uparrow$, and in the opposite order when we write $i\downarrow$. When no arrow appears, any linear order is suitable. Also, k ranges over the set $\{1, 2, \ldots, m\}$. For $\alpha = 0, 1, 2$, L_α is the following linear extension of $\Upsilon(G)$:

$$\begin{aligned}
&(a_i\, b_i \mid \mathrm{col}(i) = \alpha) \oplus (u'_{ik}\, c_k\, d_k \mid \mathrm{col}(i) = \alpha,\ k\uparrow)\\
&\oplus (a'_i\, b'_i \mid \mathrm{col}(i) = \alpha)\\
&\oplus (a_i\, a'_i\, b_i\, b'_i\, u'_{ik} \mid \mathrm{col}(i) = \alpha + 1\ ,\ \ i\uparrow)\\
&\oplus (b_i\, b'_i\, a_i\, a'_i\, u'_{ik} \mid \mathrm{col}(i) = \alpha + 2\ ,\ \ i\downarrow)\ .
\end{aligned}$$

(The ordinal sum, indicated by $\oplus$, is explained in the next section; addition is modulo 3.) Every critical inequality holds in one of these extensions. Therefore, $\dim(\Upsilon(G)) \leq 3$.

5. DIMENSION, WIDTH AND CARDINALITY

In this section, we shall derive some upper bounds on the dimension of a poset, in terms of parameters such as width and cardinality. Essential use will be made of R.P. Dilworth's decomposition theorem (Dilworth [1950]). All posets in this section are finite.

Let P be a poset and $(Q_x \mid x \in P)$ be a family of posets indexed by P. The *ordinal sum* of this family is the poset $\langle R;\leq\rangle$, where R consists of the pairs $\langle x,y\rangle$ with $x \in P$ and $y \in Q_x$, and $\langle x_1,y_1\rangle \leq \langle x_2,y_2\rangle$ iff $x_1 < x_2$ or $x_1 = x_2 = x$ and $y_1 \leq y_2$ in Q_x. The ordinal sum is *improper* if the base poset P is trivial or if each Q_x is trivial. If the base poset is the chain $1 < 2 < \ldots < n$, we write the ordinal sum as $Q_1 \oplus Q_2 \oplus \cdots \oplus Q_n$. A poset is *decomposable* if it can be written as a proper ordinal sum. In particular, an indecomposable poset with more than two elements is *connected*; that is, it cannot be written as a disjoint union $P + Q$ of two posets.

A subset S of a poset P is *partitive* when both the following conditions hold whenever $x, x' \in S$, $y \notin S$:

$$x < y \text{ iff } x' < y ;$$

$$x > y \text{ iff } x' > y .$$

A partitive subset S is *proper* when $1 < |S| < |P|$. If a poset has a proper partitive set, it is decomposable, and conversely.

THEOREM 5.1. (Hiraguchi [1951]) *If $\langle R;\leq\rangle$ is the ordinal sum of the family $(Q_x \mid x \in P)$ over P, then* dim R *is the maximum of the dimensions of P and the posets Q_x, $x \in P$.*

To prove Theorem 5.1, a realizer of R is constructed from a realizer of P by replacing each $x \in P$ by a linear extension of Q_x. This result also appears in Novák [1961]. Theorem 5.1 implies that any irreducible poset is indecomposable. Recall that a *weak* extension of a poset P satisfies every nonforced pair of P. The next result shows that every irreducible poset except $\underset{\sim}{1} + \underset{\sim}{1}$, the 2-element antichain, has a weak extension.

LEMMA 5.2. *Each nontrivial poset contains $\underset{\sim}{1} + \underset{\sim}{1}$ or has a weak extension.*

Proof. We write X for the underlying set of P. Observe that $n(P)$ is a transitive relation on X. If $n(P)$ is not antisymmetric, then $n(P)$ contains both $\langle x, y\rangle$ and $\langle y, x\rangle$ for some x, y in X. The antichain $\{x, y\}$ is then a partitive subset of P. We can now assume that $n(P)$ is a strict order relation on X. Since $\langle x, y\rangle \in n(P)$ and $y < z$ implies that $x < z$, it follows that $\leq_P \cup\, n(P)$ is an order relation on X.

For a nonempty subset S of P, the *down-set* for S, denoted by $D(S)$, is the set of all $y \in P$ such that $y < x$ for some $x \in S$. The *up-set* $U(S)$ is defined dually. The down-set for $\{x\}$ is written as $D(x)$. Observe that the principal ideal $(x] = \{x\} \cup D(x)$ and the principal filter $[x) = \{x\} \cup U(x)$. If A and B are disjoint subsets of a poset P, then an extension E of P is said to *put* A *under* B when $a < b$ in E for each incomparable pair $\langle a, b\rangle$ of P with $a \in A$ and $b \in B$.

THEOREM 5.3. (Rabinovitch [1973]) *Let A and B be disjoint subsets of a poset P. There does not exist an extension of P putting A under B iff there are $a_1, a_2 \in A$ and $b_1, b_2 \in B$ with $b_1 < a_1$, $b_2 < a_2$, $b_1 \| a_2$ and $b_2 \| a_1$ (so that these four elements form $\underset{\sim}{2} + \underset{\sim}{2}$).*

Proof. Let $S = (A \times B) \cap \mathcal{I}(P)$. If S is cycle-free, then clearly the desired extension exists. If, on the other hand, S contains a minimal alternating cycle $a_1, b_1, \ldots, a_n, b_n$, then a_1, a_2, b_n, b_1 form the required copy of $\underset{\sim}{2} + \underset{\sim}{2}$.

If C is a chain in P, then an extension putting C under $P - C$ is called an *upper extension* of C. Lower extensions are defined dually. T. Hiraguchi [1951] first established the following result, which we obtain as an immediate corollary of Theorem 5.3. In fact, if $c_1 < c_2 < \ldots < c_n$ is a chain in P, then the least upper extension of this chain is:

$$(c_1] \oplus ((c_2] - (c_1]) \oplus \ldots \oplus ((c_n] - (c_{n-1}]) \oplus (P - (c_n]).$$

COROLLARY 5.4. *Any chain in a poset has an upper extension.*

THEOREM 5.5. (Hiraguchi [1955]) *The dimension of a poset does not exceed its width.*

Proof. Let P be a poset of width n. By the theorem of Dilworth [1950], P can be covered by chains C_i, $1 \le i \le n$. For each i, let E_i be an upper extension of C_i, whose existence is guaranteed by Corollary 5.4. If $a \in C_i$ and $a \| b$ in P, then $a < b$ in E_i. Therefore, $(E_i \mid 1 \le i \le n)$ realizes P, so that $\dim P \le n$.

Since the standard n-dimensional poset has width n, the inequality in Theorem 5.5 is the best possible. We give another example of an irreducible poset with equal width and dimension.

EXAMPLE 5.6. For each $n \geq 4$, the poset P_n of Figure 5 has dimension n. (For clarity, we have only indicated the incomparable max-min pairs of S_{n-1}.)

Proof. By Theorem 5.5, $\dim(P_n) \leq n$. Suppose that the linear extensions $C_1, C_2, \ldots, C_{n-1}$ realize P_n. We can assume that $c_i < a_i$ holds in C_i for $1 \leq i \leq n-1$. Suppose $b_i < u$ in C_j. It then follows that $b_i < a_j$ in C_j, from which we conclude that $i = j$. Thus, $b_i < u < c_i < a_i$ holds in C_i for each i. In a similar manner, we deduce that $v < a_i$ in C_i for each i. Consequently, $v < d$ in each C_i and, therefore, in P_n. This contradiction shows that P_n has dimension n.

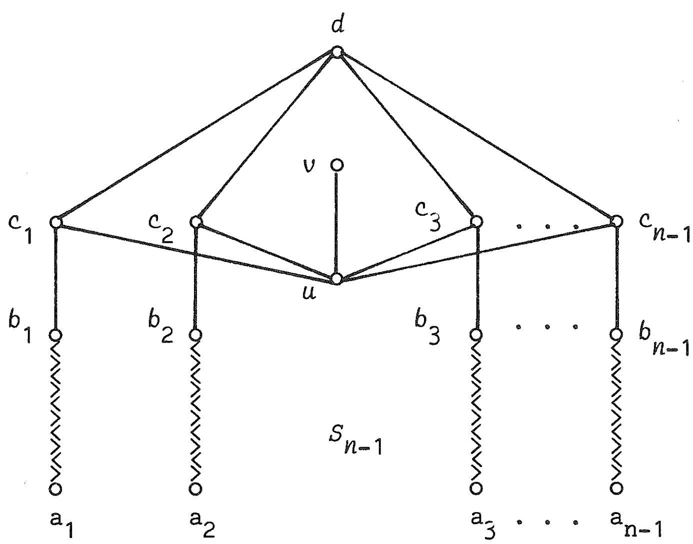

FIGURE 5. The poset P_n

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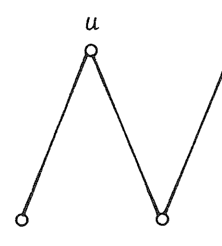

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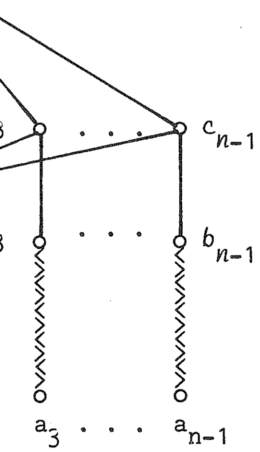

P_n

For $x \in P$, $I(x)$, or $I_P(x)$, will denote the set of all $y \in P$ with $x || y$. Observe that the sets $D(x)$, $U(x)$, $I(x)$, $\{x\}$ form a partition of P. If S is a subset of P and E is an extension of P, then $E(S)$ denotes the restriction of E to S. Clearly, $E(S)$ is a partial extension of P. We can now state and prove the one-point removal theorem.

THEOREM 5.7. (Hiraguchi [1951]) *Removing one point from a poset decreases its dimension by at most one.*

Proof. For a poset P, let $E=E_1, E_2, \ldots, E_n$ be extensions realizing $Q = P - \{x\}$. We define two extensions of P:

$$M_1 = E(D(x) \cup I(x)) \oplus \{x\} \oplus E(U(x)),$$

$$M_2 = E(D(x)) \oplus \{x\} \oplus E(U(x) \cup I(x)).$$

Any incomparable pair of P involving x is satisfied in M_1 or M_2. Let $\langle a,b\rangle \in \mathcal{I}(Q)$, and assume that $b < a$ in $E_2, E_3, \ldots, E_n$. It follows that $a < b$ in E_1. Suppose that $a \nless b$ in both M_1 and M_2. This can only happen if $b \in D(x)$ and $a \in U(x)$. This implies $b < a$ in P, a contradiction. Therefore, the partial extensions $M_1, M_2, E_2, E_3, \ldots, E_n$ realize P, so that $\dim P \le \dim Q + 1$.

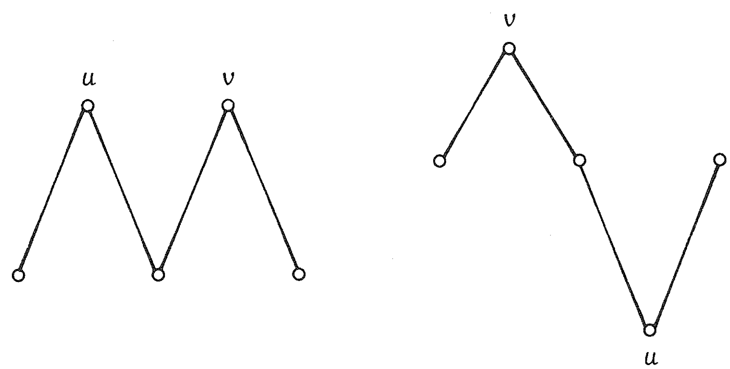

FIGURE 6. A pair of 2-dimensional posets

THEOREM 5.8. (Kimble [1973], Trotter [1975b]) *If* A *is an antichain in a poset* P , *then* $\dim P \le \max(2, |P-A|)$.

Proof. Let A be an antichain in a poset P with $|P-A| \le 2$. We shall show that $\dim P \le 2$. The general result will then follow by Theorem 5.7. By induction on $|P|$, we can assume that P is connected. We can also assume that A is maximal. Let P^* denote the poset obtained from P by collapsing each partitive antichain to a single point. By Theorem 5.1, $\dim P \le \max(2, \dim P^*)$. If $|P-A|$ is at most one, then $\dim P$ is at most two because $|P^*| \le 2$ in this case. Let $P-A = \{u,v\}$. If $u \| v$, then P^* is a subposet of the left poset of Figure 6, or its dual. If $u < v$, then P^* is a subposet of the right poset of Figure 6. Since both posets of Figure 6 are 2-dimensional, the proof is complete.

We now combine the preceding results to obtain Hiraguchi's inequality.

THEOREM 5.9. (Hiraguchi [1951]) *If* P *is a poset with at least 4 elements, then* $\dim P \le |P|/2$.

Proof. Let A be an antichain in P of maximum size. If $|A| \le |P|/2$, then $\dim P \le |P|/2$ by Theorem 5.5. Otherwise, $|P-A| \le |P|/2$, and the result follows by Theorem 5.8.

We conclude this section with two additional inequalities whose proofs utilize Dilworth's decomposition theorem.

THEOREM 5.10. (Trotter [1975b]) *If* M *is the set of maximal elements of a poset* P, *then* $\dim P \le 1 + \text{width}(P - M)$.

Proof. Let $n = \text{width}(P - M)$ and let

$$P - M = C_1 \cup C_2 \cup \dots \cup C_n$$

be a partition into chains. Let E_i be a lower extension of C_i for $1 \le i \le n$, with E_n linear. Also, let

$$E_{n+1} = E_n(P - M) \oplus (E_n)^d(M),$$

a linear extension of P. Since $(E_1, E_2, \dots, E_n, E_{n+1})$ is a realizer of P, the result follows.

THEOREM 5.11. (Trotter
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6. MORE REMOVAL THEOREMS

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THEOREM 5.11. (Trotter [1974b]) *If A is an antichain in a poset P and $A \neq P$, then* $\dim P \le 1 + 2\,\text{width}(P - A)$.

Proof. Let $n = \text{width}(P - A)$ and let

$$P - A = C_1 \cup C_2 \cup \ldots \cup C_n$$

be a partition into chains. As above, let E_i be a lower extension of C_i for each i, with E_n linear. Also, let F_i be an upper extension of C_i for each i. These $2n$ extensions, together with $(E_n)^d(A)$, realize P.

Equality holds in Theorem 5.10 for the poset P_n of Example 5.6. The inequality in Theorem 5.11 is shown to be the best possible in Trotter [1974b]. If A and B are antichains in a poset P, then $\langle A, B\rangle$ is a *double antichain* iff $a \le b$ whenever $a \in A$ and $b \in B$. The *double width* of P is the maximum value of $|A| + |B|$ for a double antichain $\langle A, B\rangle$. O. Pretzel [1977] has proved an analog of Dilworth's decomposition theorem for double width, and has generalized the inequality of Theorem 5.11 by replacing twice the width by the double width.

6. MORE REMOVAL THEOREMS

As usual, all posets are finite. Removing an antichain can lower the dimension drastically. If all the maximal elements are removed from a standard n-dimensional poset, then one is left with an antichain, a poset of dimension 2. In fact, if all but two maximal elements are removed from S_n, the dimension becomes two. For example, if $n=7$, the remaining poset can be embedded in the planar lattice of Figure 7. However, the dimension drops by at most two when a chain is removed from a poset.

THEOREM 6.1. (Hiraguchi [1951]) *If C is a chain in a poset P, then* $\dim P \le 2 + \dim(P - C)$.

Proof. Add a lower and an upper extension of C to a realizer of P-C to obtain a realizer for P (consisting of partial extensions).

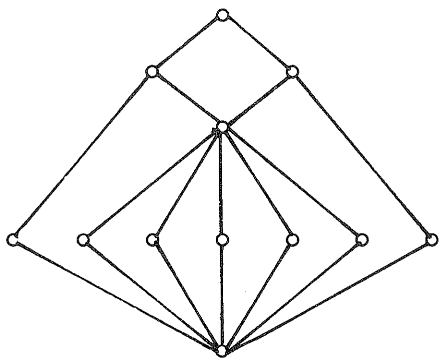

FIGURE 7. A planar lattice

If $n \geq 4$ and a 2-element chain is removed from S_n, then the dimension drops by 2. Since the remaining poset has width $n - 1$, this may have been unexpected. Hiraguchi's [1951] original proof of his inequality (Theorem 5.9) depended on many removal theorems, as did the later proof of Bogart [1973]. The maximum decrease in dimension is one for the two-point removal theorems and two for the four-point ones. In both cases, at least one four-point removal theorem was required. Chain removal theorems give conditions under which removal of a chain decreases the dimension by at most one.

THEOREM 6.2. (Bogart [1973]) *If* C *is a chain in* P *and each point in* $P - C$ *is incomparable with at most one point in* C, *then* $\dim P \leq 1 + \dim(P - C)$.

Proof. Let C be $c_1 < c_2 < \ldots < c_n$. Let $E = E_1, E_2, \ldots, E_t$ be extensions realizing $P - C$. For $1 \leq i \leq n$, let $D_i = D(c_i) - C$ and $U_i = U(c_i) - C$. Also, let $D_0 = \emptyset = U_{n+1}$ and $D_{n+1} = P = U_0$. The assumptions imply that each $x \in P - C$ is in $(D_{i+1} - D_i) \cap (U_i - U_{i+1})$ for some (unique) value of i in $\{0,1,\ldots,n\}$. It is then immediate that P is realized by $E_2, E_3, \ldots, E_t$ and the following two extensions of P:

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$$E(D_1) \oplus c_1 \oplus E(D_2 - D_1) \oplus c_2 \oplus \ldots \oplus c_n \oplus E(P - D_n)$$

$$E(P - U_1) \oplus c_1 \oplus E(U_1 - U_2) \oplus c_2 \oplus \ldots \oplus c_n \oplus E(U_n)$$

This completes the proof.

PROBLEM 6.3. A pair $\langle a, b\rangle$ of distinct elements in a poset P is called *removable* if $\dim(P - \{a,b\}) \ge \dim P - 1$. If a poset with at least three elements is not irreducible, then it has a removable pair. (First, choose an element whose removal does not lower the dimension; the second element can be chosen arbitrarily.) We believe that every d-irreducible poset ($d \ge 3$) contains a removable pair. In fact, we *conjecture* that every critical pair of such a poset is removable.

The *rank* of a cover $a \prec b$ is the number of incomparable pairs $\langle x,y\rangle$ with $a < x$ and $y < b$. T. Hiraguchi [1951] showed that a cover of rank at most one is removable. This result is generalized in the following theorem.

THEOREM 6.4. *If the cover* $a \prec b$ *in* P *has rank at most* $(\dim P) - 3$, *then it is removable.*

Proof. Let $L = L_1, L_2, \ldots, L_n$ be linear extensions realizing $P - \{a,b\}$. By the one-point removal theorem, $n \ge (\dim P) - 2$. Thus, there are at most $n-1$ incomparable pairs $\langle x,y\rangle$ such that $a < x$ and $y < b$. If $\langle x,y\rangle$ is any of these pairs, we can assume that $y < x$ holds in L_i for some $2 \le i \le n$. P is realized by $L_2, L_3, \ldots, L_n$ and the following two linear extensions:

$$L(D(a)) \oplus a \oplus L(D(b) - D(a)) \oplus b \oplus L(P - D(b))$$

$$L(P - U(a) \oplus a \oplus L(U(a) - U(b)) \oplus b \oplus L(U(b))$$

Thus, $\dim P \le 1 + \dim (P - \{a,b\})$.

Except in the $\dim P = 3$ case, Theorem 6.4 yields Hiraguchi's result that covers $a \prec b$ of rank at most one are removable. However, if $\dim P = 3$, it is clear that $P - \{a,b\}$ is not a chain. (Otherwise, P would have width at most 2.) Thus, $P - \{a,b\}$ has dimension at least 2, and we can reason as in the proof of Theorem 6.4. Note that each chain in the standard

n-dimensional poset $(n \geq 4)$ is a cover of rank $n - 2$ whose removal decreases the dimension by two.

The *rank* of an incomparable pair $\langle a,b \rangle$ is the number of incomparable pairs $\langle x,y \rangle$ where x is comparable to both a and b, and y is incomparable to both a and b. An incomparable pair of rank at most one is removable. We shall present a more general result.

THEOREM 6.5. *An incomparable pair* $\langle a,b \rangle$ *of rank at most* $(\dim P) - 3$ *is removable.*

Proof. Let $L = L_1, L_2, \ldots, L_n$ be linear extensions realizing $P - \{a,b\}$. There are at most $n-1$ incomparable pairs $\langle x,y \rangle$ for which x is comparable with both a and b, and y is incomparable with both a and b. For each of these pairs $\langle x,y \rangle$ we can assume that, if $x > a$, then $x < y$ holds in L_i for some $i \geq 2$, and if $x < a$, then $x > y$ holds in L_i for some $i \geq 2$. P is realized by $L_2, L_3, \ldots, L_n$ and the following two linear extensions:

$$L(\mathrm{D}(a)) \oplus a \oplus L(P - \mathrm{D}(a)) - \mathrm{U}(b)) \oplus b \oplus L(\mathrm{U}(b))$$

$$L(\mathrm{D}(b)) \oplus b + L(P - \mathrm{D}(b)) - \mathrm{U}(a)) \oplus a \oplus L(\mathrm{U}(a))$$

This completes the proof.

Since a pair of maximal elements in S_n $(n \geq 4)$ is not removable, Theorem 6.5 is the best possible result. In particular, any incomparable max-min pair is removable. An incomparable pair $\langle a,b \rangle$ is a DU-*pair* if $\mathrm{D}(a) \subseteq \mathrm{D}(b)$ and $\mathrm{U}(a) \subseteq \mathrm{U}(b)$.

THEOREM 6.6. *Any* DU-*pair* $\langle a,b \rangle$ *is removable.*

Proof. Let $E = E_1, E_2, \ldots, E_n$ be extensions realizing $P - \{a,b\}$. P is realized by $E_2, E_3, \ldots, E_n$ and the following two extensions:

$$E(\mathrm{D}(a)) \oplus a \oplus E(\mathrm{D}(b) - \mathrm{D}(a)) \oplus b \oplus E(P - \mathrm{D}(b))$$

$$E(P - \mathrm{U}(b)) \oplus b \oplus E(\mathrm{U}(b) - \mathrm{U}(a)) \oplus a \oplus E(\mathrm{U}(a))$$

7. IRREDUCIBLE POSETS

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7. IRREDUCIBLE POSETS

In Section 1, we mentioned that the list of all 3-irreducible posets was discovered independently by D. Kelly [1977], and W.T. Trotter and J.I. Moore [1976a]. We shall discuss the two solutions. The different approaches originate from the two characterizations of a poset P of dimension at most 2: the completion of P is planar and $\mathfrak{I}(P)$ is a comparability graph.

Using arguments with a geometric flavor, D. Kelly and I. Rival [1975a] characterized planar lattices by the exclusion *as subposets* of a family $\mathcal{L}$ of nonplanar lattices; moreover, $\mathcal{L}$ is the minimum such collection. If $\mathcal{P} = \{\underset{\sim}{P}(L) \mid L \in \mathcal{L}\}$, then a lattice L is planar iff it does not contain any poset in $\mathcal{P}$. As we mentioned in Section 3, every poset in $\mathcal{P}$ is 3-irreducible (Kelly and Rival [1975b]) and $\mathcal{P}$ includes the first nine posets of Figure 1. The determination of $\mathcal{L}$ was based on the characterization of dismantlable lattices in Kelly and Rival [1975a] or M. Ajtai [1973]. A finite lattice is *dismantlable* (Rival [1974]) if every sublattice with at least 3 elements contains a doubly irreducible element. K.A. Baker, P.C. Fishburn and F.S. Roberts [1971] showed that every planar lattice is dismantlable. Let $\mathfrak{R}$ be a candidate for the set of all 3-irreducible posets. Certainly, $\mathfrak{R}$ must contain $\mathcal{P}$. If a poset P has dimension greater than 2, then $L = \underset{\sim}{L}(P)$ is nonplanar, and therefore, contains a poset in $\mathcal{P}$. It remains to show that P contains a poset in $\mathfrak{R}$. The first step is to handle the crowns. We can then assume that P does not contain any crowns. By the characterization of dismantlable lattices, this means that L is dismantlable. This has two consequences: we can assume that $L - \{x\}$ is a planar lattice for a doubly irreducible element x of L; any element of $L - P$ is simultaneously the join of two elements of P and the meet of two elements of P. The remaining details appear in Kelly [1977].

A poset P is 3-irreducible iff $G = \mathfrak{I}(P)$ is not a comparability graph but every proper induced subgraph of G is a comparability graph. T. Gallai [1967] provided a forbidden subgraph characterization of comparability graphs by discovering the minimum collection $\mathbb{C}$ of graphs so that a graph fails to be a comparability graph iff it contains a graph from $\mathbb{C}$ as an induced subgraph. It follows that the incomparability graph of a 3-irreducible poset belongs to $\mathbb{C}$. From this observation, it follows that we need only examine graphs from the family $\mathbb{C}$ and determine those whose complement is a comparability graph. Any transitive orientation yields a 3-irreducible poset. In fact, up to duality, there is no choice of orientation because the comparability graph of an irreducible poset is uniquely orderable (see Section 8). The details are given in Trotter and Moore [1976a].

A problem for a class of posets is to determine a formula for the dimension and to specify the irreducible ones. We shall discuss the family S_n^k of W.T. Trotter [1974a]. Other examples appear in Kelly [1981]. Of course, one may only be able to express the dimension in terms of other combinatorial parameters as in Dushnik [1950] (the parameters defined there were investigated further in Spencer [1971] and Trotter [1976b]). For integers n, k with $n \geq 0$, the *general crown* S_n^k is the poset of unit length with $n + k$ maximal elements $a_1, a_2, \ldots, a_{n+k}$ and $n + k$ minimal elements $b_1, b_2, \ldots, b_{n+k}$. The order on S_n^k is defined by $b_i < a_j$ iff $j \in \{i, i+1, \ldots, i+k\}$, where addition is modulo $n+k$. Note that S_n^k has the $(n+k)(k+1)$ nonforced pairs $\langle b_i, a_j \rangle$ where $b_i \| a_j$ in S_n^k. Clearly, $S_n^0 = S_n$.

THEOREM 7.1. (Trotter [1974a]). *For each* n, k *with* $n \geq 3$, $k \geq 0$, *the dimension of* S_n^k *is* $\lceil 2(n+k)/(k+2) \rceil$.

Sketch of Proof. We have already observed that S_n^k has $(n+k)(k+1)$ nonforced pairs. It is relatively easy to show that a linear extension of S_n^k can reverse at most $(k+1)(k+2)/2$ of these nonforced pairs. Therefore,

$$\dim(S_n^k) \geq \lceil 2(n+k)/(k+2) \rceil .$$

We illustrate the construction of a realizer for S_n^k by providing seven partial linear extensions which realize S_{18}^5 .

$$b_7, a_6, b_6, a_5, b_5, a_4, b_4, a_3, b_1, a_7, b_2, a_8, b_3$$

$$b_{10}, a_9, b_9, a_8, a_7, b_4, a_{10}, b_5, a_{11}, b_6, a_{12}, b_7$$

$$b_{14}, a_{13}, a_{12}, b_{12}, a_{11}, b_{11}, a_{10}, b_8, a_{14}, b_9, a_{15}, b_{10}$$

$$b_{17}, a_{16}, b_{16}, a_{15}, b_{15}, a_{14}, b_{11}, a_{17}, b_{12}, a_{18}, b_{13}, a_{19}, b_{14}$$

$$b_{21}, a_{20}, b_{20}, a_{19}, b_{19}, a_{18}, b_{18}, a_{17}, b_{15}, a_{21}, b_{16}, a_{22}, b_{17}$$

$$a_{23}, b_{23}, a_{22}, b_{22}, a_{21}, b_{18}, a_1, b_{19}, a_2, b_{20}, a_3, b_{21}$$

$$a_5, a_4, b_3, a_2, b_2, a_1, b_1, a_{23}, b_{22}, a_5, b_{23}$$

THEOREM 7.2. (Trotter

(a.) *If* $n + k$ =

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(b.) *If* $n + k$

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THEOREM 7.2. (Trotter [1974a]) *Let* $n \geq 3$, $k \geq 0$.

(*a.*) *If* $n + k = q(k + 2) + 1$, *then* S_n^k *is* $(2q + 2)$-*irreducible.*

(*b.*) *If* $n + k = q(k + 2) + \lfloor (k + 2)/2 \rfloor + 1$, *then* S_n^k *is* $(2q + 2)$-*irreducible iff* $k = 0$ *or* k *is odd.*

The study of general crowns is closely related to a graph coloring problem which involves properties of the hypergraph $\mathcal{H}(S_n^k)$. When $n \geq 3$ and $k \geq 1$, $\mathcal{H}(S_n^k)$ is not an ordinary graph. However, there is a family of graphs for which the determination of chromatic number is equivalent to the computation of dimension for general crowns. For $m \geq 1$, $t \geq 0$, define the *web* W_m^t as the graph whose vertex set is $\{w_1, w_2, w_3, \ldots, w_m\}$ with each w_i adjacent to w_{i+t+j} for $j = 1, 2, \ldots, n - 2t - 1$ (cyclically). Note that W_m^t is a regular graph of degree $n - 2t - 1$; also note that W_m^0 is a complete graph on m vertices.

THEOREM 7.3. *For* $m \geq 0$ *and* $t \geq 0$, *the chromatic number of the web* W_m^t *is* $\lceil m/(t + 1) \rceil$.

Proof. Let W be the vertex set of W_m^t. We first show that any independent subset $W' \subseteq W$ has at most $t + 1$ vertices. For each j with $1 \leq j \leq m$, let $I(W_j)$ be the $(2t + 1)$- element subset of W defined by

$$I(w_j) = \{ w_{j-t}, w_{j-t+1}, \ldots, w_j, \ldots, w_{j+t} \}.$$

Observe that $w \in I(w_j)$ if w is not adjacent to w_j. Therefore, $W' \subseteq \bigwedge(I(w) \mid w \in W')$ holds for any independent set of vertices W' in W_m^t. Since each set $I(w)$ is a set of $2t + 1$ consecutive elements of W, it is straightforward to show that if W' is an independent set in W_m^t, then

$$|W'| \leq |\bigwedge(I(w) \mid w \in W')| \leq 2t + 2 - |W'|.$$

Therefore, $|W'| \le t + 1$.

Now suppose that $\chi(W_m^t) = s$ and let $W = W_1 \cup \dots \cup W_s$ be a partition into independent subsets. Then since $|W_i| \le t+1$ for $i = 1,2,\dots,s$, $|W| = m \le s(t+1)$; i.e., $s \ge m/(t+1)$. Since any subset of W consisting of $t + 1$ consecutive elements is independent, W can be partitioned into $\lceil m/(t+1) \rceil$ independent subsets.

When $n \ge 3$, $k \ge 0$ and k is even, say $k = 2\ell$, the hypergraph $\mathcal{H}(S_n^k)$ contains the web W_{n+k}^{ℓ} as an induced subhypergraph. Consider the vertices $\langle a_{i+\ell}, b_i \rangle$ for $1 \le i \le k$. Thus, $\dim(S_n^k) \ge \chi(W_{n+k}^{\ell}) = \lceil (n+k)/(\ell+1) \rceil = \lceil 2(n+k)/(k+2) \rceil$. When $n \ge 3$, $k \ge 0$, and k is odd, say $k = 2\ell + 1$, then $\mathcal{H}(S_n^k)$ contains the web $W_{2(n+k)}^{k+1}$ as an induced subhypergraph. This may be seen by considering the pairs $\{\langle a_j, b_i \rangle \mid j = i + \ell \text{ or } j = i + k + 1\}$ for $i = 1, 2, \dots, n+k$.

Each of the present authors has developed a technique for constructing irreducible posets, both under the name *dimension product*.

If each of P and Q is an irreducible poset satisfying certain extra conditions, or equals $\underset{\sim}{2}$, then the *dimension product* $P \otimes Q$ of P and Q introduced by Kelly [1981] is an irreducible poset of dimension $\dim P + \dim Q$. We shall not give these extra conditions here. We remark that we do not know any irreducible posets which violate any of these conditions. In particular, all 3-irreducible posets satisfy these conditions. As a consequence, for any $n > 3$ every 3-irreducible poset can be embedded in an n-irreducible poset.

In general, the definition of $P \otimes Q$ is quite complicated. It is always a subposet of $R = \underset{\sim}{P}(\underset{\sim}{L}(P) \times \underset{\sim}{L}(Q))$. Although the expression for R may look formidable, R is a subposet of $(P \cup \{0,1\}) \times (Q \cup \{0,1\})$. Note that $\dim R = \dim P + \dim Q$ by the results of Section 1. In fact, for the 3-irreducible posets in $\mathcal{P}$ and for the general crowns, $P \otimes Q = R$.

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Let P be a poset of unit length, and for simplicity, assume no element of P is both minimal and maximal. The *interval dimension* of P, denoted by Idim P, is the least size of a family $\mathfrak{F}$ of linear extensions such that every max-min incomparable pair is satisfied in some member of $\mathfrak{F}$. P is *d-interval irreducible* if Idim P = d $\ge$ 2 and Idim (P-{x}) < d for every $x \in P$. For the family S^n_k of general crowns, interval dimension and dimension coincide. Moreover, being irreducible and being interval irreducible are equivalent for this family. Trotter's *dimension product* of two posets of unit length is a poset of unit length where interval dimension is at least the sum of the interval dimensions of the factors. If P is m-interval irreducible and Q is n-interval irreducible, then Trotter's dimension product is $(m+n)$-interval irreducible, *and* $(m+n)$-irreducible. Trotter's dimension product of P and Q is the disjoint union of P and Q with every minimal element of P put under every maximal element of Q, and every minimal element of Q put under every maximal element of P. Let $d \ge 6$ and $n \ge 2d+2$. Starting with the general crowns and a single 3-interval irreducible poset on 9 points, one can construct d-irreducible posets of unit length on n points using Trotter's dimension product. (The list of all 3-interval irreducible posets of unit length is given in Trotter [1981].)

Recently, W.T. Trotter and J.A. Ross made some striking discoveries about irreducible posets. We have mentioned that Kelly's dimension product, for any 3-irreducible poset and $n > 3$, gives an embedding of P into an n-irreducible poset Q. Both Q and the embedding are given explicitly. Whenever $n > d \ge 3$, Trotter and Ross [1981a] showed that every d-irreducible poset P can be embedded into an n-irreducible poset Q. Rather than trying to construct such a poset Q explicitly, they construct a poset R of dimension n containing P. We can assume that $n = d + 1$. The poset R is constructed so that any subposet of R that does not contain all of P has dimension less than n. The construction uses a weak extension of P in an essential way. There are posets of dimension two (for example, the direct product of a 3-element chain with itself) which are not subposets of any 3-irreducible poset. However, Trotter and Ross [1981b] have shown that every poset of dimension $d \ge 3$ is a subposet of a $(d+1)$-irreducible poset. Their construction uses the dimension product $G_m \otimes S_{d-3}$ for a suitably large value of m. They utilize special properties of $\mathcal{H}(G_m \otimes S_{d-3})$. (Recall that the hypergraph for G_m appears in Figure 2.)

8. COMPARABILITY GRAPHS

We have mentioned comparability graphs in Sections 3 and 7. We refer the reader to the book of Golumbic [1980] for a detailed account of this topic. Since an irreducible poset is indecomposable, it is uniquely orderable. Using this fact, Trotter, Moore and Sumner [1976] showed that the dimension of a poset is determined by its comparability graph. This latter statement is also true for infinite posets, as shown by Arditti and Jung [1980]. Using Golumbic's algorithm for orienting a comparability graph, R. Stanley has observed that the number of linear extensions of a finite poset depends only on its comparability graph.

One might think that the comparability graph also determines the rank. However, the following examples due to Maurer, Rabinovitch and Trotter [1980b] show that posets with the same comparability graph may differ substantially in rank. For $n \geq 2$, let V_n be the poset consisting of an n-element antichain with a zero added. Let $P_n = V_n + V_n$, and $Q_n = V_n + (V_n)^d$. Certainly, these two posets have the same comparability graph. Furthermore, $\dim(P_n) = \dim(Q_n) = 2$, but $\operatorname{rank}(P_n) = 2\lceil (n+1)^2/4 \rceil$ and $\operatorname{rank}(Q_n) = n^2 + 1$.

9. FURTHER TOPICS

A poset P is an *interval order* if we can associate a closed interval I_x of the reals to each $x \in P$ so that $x < y$ in P iff I_x lies entirely to the left of I_y. P.C. Fishburn [1970] showed that a poset is an interval order iff it does not contain $\underset{\sim}{2} \oplus \underset{\sim}{2}$. A *semiorder* is an interval order that does not contain $\underset{\sim}{3} \oplus \underset{\sim}{1}$. I. Rabinovitch ([1973],]1978a], [1978b]) showed that an interval order has arbitrary dimension but is bounded by its height plus one, while the dimension of a semiorder cannot exceed three. (An improved upper bound for the dimension of an interval order is given in Bogart, Rabinovitch and Trotter [1976].)

D. Kelly [1980] constucted finite modular dismantlable lattices of each finite dimension. (The dimension of a distributive dismantlable lattice cannot exceed two.) I. Rival [1976] showed that the dimension of a finite modular dismantlable lattice L is bounded by the width of $\operatorname{Irr}(L)$. There is a finite list of modular lattices such that a modular lattice L has dimension at most two iff no member of this list is a sublattice of L (Kelly [1980]). A version of this result with exclusion as subposets

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was proved previously by R. Wille [1974]; the case of finite L appears in Kelly and Rival [1975a]. The dimension of lattices is also considered in Rival and Sands ([1978],[1979]), Sands [1980], and Babai and Duffus [1981].

The dimension of a poset is the number of order relations of a special type (linear orders) that are needed to represent the ordering of the poset as an intersection. If we replace linear orders by interval orders, then we obtain the *interval dimension* of a poset. (Interval dimension is considered in Trotter and Bogart [1976a] and Trotter and Moore [1976a].) There are analogous dimension concepts for graphs and digraphs (i.e.,binary relations; loops may occur). *Ferrers* relations are defined in Riguet [1951] and *threshhold* graphs are defined in Chvátal and Hammer [1977]. In each of these definitions, a finite family is excluded (as induced relations). The *Ferrers* dimension of a digraph and the *threshhold* dimension of a graph are then defined as indicated above. A. Bouchet [1971] showed that the Ferrers dimension of a (reflexive) order relation is the same as its dimension. He also showed that the Ferrers dimension of a binary relation R equals the dimension of the Galois lattice associated with R. O. Cogis [1980] associated to each binary relation R a graph G such that the Ferrers dimension of R equals the threshhold dimension of G.

For a poset P, let $\mathcal{G}(P)$ be the graph whose edges are the 2-edges of $\mathcal{H}(P)$. O. Cogis [1980] has shown that $\dim P \leq 2$ iff $\mathcal{G}(P)$ is 2-colorable. In fact, he proved the corresponding result for the Ferrers dimension of digraphs. Consequently, deciding whether a digraph has Ferrers dimension at most two has polynomial complexity.

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