RECENT PROGRESS IN EXTREMAL PROBLEMS IN DISCRETE GEOMETRY

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1. Introduction.

Extremal problems in discrete geometry constitute a fascinating and surprisingly difficult area of research in combinatorial mathematics. The reader is encouraged to consult P. Erdös' survey papers [5], [6], [7], [8] and W. Moser's summary [11] of recent progress on a wide range of problems in this area. In this paper, we concentrate on extremal problems in the Euclidean plane and announce several new results which have been obtained by the authors and other researchers within the last year. Proofs will appear elsewhere.

The starting point is the authors' covering lemma [12] which, despite its somewhat technical flavor, may be of independent interest. Suppose that a pair of coordinate axes has been chosen. When we use the term square, it will also be assumed that the sides are parallel to the coordinate axes. The following lemma asserts the existence of a family of squares covering a positive fraction of a set of n points with an additional restriction on the number of points contained in each square.

Lemma (Szemerédi and Trotter [12]): Let r_1, r_2 be integers with $r_2 \geq 256\, r_1$, and let $\mathcal{P}$ be any set of n points in the plane. Then there exists a family $\mathcal{Q}$ of squares so that:

1. No point in the plane is in the interior of more than one square.
2. Each square contains at least r_1 but no more than r_2 points of $\mathcal{P}$.
3. At least n/16 of the points in $\mathcal{P}$ are covered by the squares in $\mathcal{Q}$.

□

2. Inequalities involving points and lines.

The covering lemma is used to localize the intersection patterns of lines. Linear transformations are applied to a configuration of points and lines to rotate the configuration and increase or decrease the slope of the lines as required. The principal result established with the assistance of this lemma is the following theorem limiting the number of incidences between a set of n points and a family of t lines in the Euclidean plane.

*Research supported in part by NSF Grants ISP-8011451 and MCS-8202172.

Theorem 1 (Szemerédi and Trotter [13]): There exists a constant c_1 so that if $\mathcal{P}$ is a set of n points and $\mathcal{L}$ is a family of t lines in the Euclidean plane, then the number of incidences between points in $\mathcal{P}$ and lines in $\mathcal{L}$ is at most $c_1\, n^{2/3}t^{2/3}$ whenever $\sqrt{n} \leq t \leq \binom{n}{2}$. □

P. Erdös [5] had conjectured the conclusion of Theorem 1 in the special case t = n. From Theorem 1, the following theorem, settling in the affirmative a conjecture of Erdös and Purdy [4], follows as an immediate corollary.

Theorem 2 (Szemerédi and Trotter [13]): There exists a constant c_2 so that if $\mathcal{P}$ is a set of n points and $\mathcal{L}$ is a family of t lines each containing at least k points from $\mathcal{P}$ where $k \leq \sqrt{n}$, then $t < c_2\, n^2/k^3$. □

A second corollary is the following result which is a partial solution to Dirac's conjecture [8].

Theorem 3 (Szemerédi and Trotter [12], Beck [1]): There exists a constant $c_3 > 0$ so that if $\mathcal{P}$ is a set of n points, not all on the same line, and $\mathcal{L}$ is the family of all lines determined by $\mathcal{P}$, then there exists at least one point in $\mathcal{P}$ which belongs to more than $c_3 n$ lines from $\mathcal{L}$. □

Let $\mathcal{P}$ be a set of n points and let $\mathcal{L} = \{\ell_1, \ell_2, \ldots, \ell_t\}$ be a family of lines each containing at least two points from $\mathcal{P}$. For each $j = 1, 2, \ldots, t$, let y_j count the number of points from $\mathcal{P}$ which are on line ℓ_j. Assume that the lines have been labelled so that $y_1 \leq y_2 \leq \ldots \leq y_t$. Then $\Sigma_{i=1}^{t} \binom{y_i}{} \leq \binom{n}{2}$. However, this condition is not sufficient to insure that an arbitrary nondecreasing sequence arises in this fashion. Let $\mathcal{E}(n)$ count the number of distinct nondecreasing sequences $y_1 \leq y_2 \leq \ldots \leq y_t$ determined by all possible configurations of n points in the plane. Theorem 1 can be used to prove the following result which also settles in the affirmative a conjecture of P. Erdös:

Theorem 4 (Szemerédi and Trotter [13]): There exists a constant c_4 so that $\mathcal{E}(n) < 2^{c_4\sqrt{n}}$ for all $n \geq 1$. □

3. Inequalities involving distance.

Two of the oldest problems is discrete geometry are:

a. Find the maximum number $f(n)$ of pairs of points in the Euclidean plane which can be at unit distance apart.

b. Find the minimum number $g(n)$ of distinct distances determined by n points in the Euclidean plane.

P. Erdős conjectures that $f(n) < n^{1 + \frac{1}{\log \log n}}$ and $g(n) > \frac{cn}{\sqrt{\log n}}$ and has offered substantial sums for the solutions to these problems. Until recently progress has been slow. Szemerédi [8] proved $f(n) = o(n^{3/2})$. Beck and Spencer [2] proved $f(n) < n^{1.499}$. Recently, Spencer, Szemerédi and Trotter used the technique employed in proving the theorems in Section 1 to improve this to $f(n) < c\, n^{4/3}$.

For the other problem Erdős [4] proved $g(n) > n^{1/2}$ and Moser [10] improved this to $g(n) > n^{2/3}$. Recently, Fan Chung [3] proved that $g(n) > n^{5/7}$ and it appears that further improvements can be obtained by utilizing the incidence inequalities. In any case, it is clear that these problems and similar problems in discrete geometry will continue to attract the attention of researchers in future years.

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