COMBINATORICA 4 (2—3) (1984) 183—185

ON COLORING GRAPHS WITH LOCALLY SMALL CHROMATIC NUMBER

H. A. KIERSTEAD[1], E. SZEMERÉDI[1] and W. T. TROTTER, Jr.[1,2]

Received 19 September 1983

In 1973, P. Erdős conjectured that for each $k \geqq 2$, there exists a constant c_k so that if G is a graph on n vertices and G has no odd cycle with length less than $c_k n^{1/k}$, then the chromatic number of G is at most $k+1$. Constructions due to Lovász and Schriver show that c_k, if it exists, must be at least 1. In this paper we settle Erdős' conjecture in the affirmative. We actually prove a stronger result which provides an upper bound on the chromatic number of a graph in which we have a bound on the chromatic number of subgraphs with small diameter.

0. Introduction

P. Erdős conjectured that for every positive integer k, there exists a constant c_k such that if G is a graph on n vertices with no odd cycle of length less than $c_k n^{1/k}$ then the chromatic number of G is at most $k+1$. In this paper we prove the following theorem of which Erdős' conjecture is the special case $c=2$, with $c_k=4k$.

Theorem 1. *For each pair of positive integers c and k, if G is a graph on n vertices with no subgraph H, whose chromatic number is greater than c and whose radius in G is at most $2kn^{1/k}$, then the chromatic number of G is at most $k(c-1)+1$.*

We refer the reader to Erdős' paper [1] for a discussion of the background of his conjecture. In section 2, we discuss two constructions. The first construction shows that our theorem is essentially best possible when $c=2$. The second shows that our theorem is essentially best possible for any rational number $k \leqq 2$.

Expressions of the form $n^{m/k}$ will always mean $[n^{1/k}]^m$. If x and y are vertices of a graph G, the distance from x to y, denoted by $d_G(x, y)$, is the number of edges in the shortest path in G from x to y. If S is a set of vertices in G the distance from x to S, denoted by $d_G(x, S)$, is $d_G(x, S)=\min_{y\in S} d_G(x, y)$. If H is a subgraph of G then the radius of H in G, denoted by $R_G(H)$, is $R_G(H)=\min_{x\in H}\big(\max_{y\in H} d_G(x, y)\big)$. When G is clear from the context we will omit the subscript in the above notations. The chromatic number of a graph G is denoted by $\chi(G)$.

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[1] Research supported by NSF Grant ISP—8 011 451.
[2] Research supported by NSF Grant MCS—8 202 172.
AMS subject classification (1980): 05 C 15

1. Proof of the Principal Theorem

We begin with a definition.

Definition. An (α, β)-*obstruction* in a graph G is a subgraph Q of G such that $|Q| \geqq \alpha$ and $R(Q) \leqq \beta$.

In order to make our inductive argument we shall prove the following more general lemma which yields the theorem in the case $k=l$, $W=G$, and $|I|=1$, since if $\chi(G)>k(c-1)+1$, there are not enough vertices in $G-I$ for the required obstruction.

Lemma 2. *Let G be a graph on n vertices such that for any subgraph H of G, if $R_G(H) \leqq 2kn^{1/k}$, then $\chi(H) \leqq c$. If $0 \leqq l \leqq k$, $W \subset G$, $\chi(W)>l(c-1)+1$, and $I \subset W$ is independent, then $W-I$ contains an $(n^{l/k}, 2ln^{1/k})$-obstruction.*

Proof. We shall argue by induction on l. First suppose that $l=0$. Then $\chi(W) \geqq 2$, so there exist two adjacent vertices u and v in W. One of these, say u, is not in I, so $\{u\}$ is a $(1, 0)$-obstruction in $W-I$.

Now assume the result for $l=m$ and consider the case $l=m+1 \leqq k$. Let H be a c-partite subgraph of W, with c-partition $H=(J_0, \dots, J_{c-1})$, having the maximum number of vertices. Consider the subgraph $W'=W-\bigcup_{i<c-1} J_i$. Clearly $\chi(W')>m(c-1)+1$, so by the inductive hypothesis $W'-J_{c-1}$ contains an $(n^{m/k}, 2mn^{1/k})$-obstruction P_0. Note that $H \cap P_0 = \emptyset$. For each $j>0$, let $P_j=\{x \in H\colon d(x, P_0)=j\}$. Let $Q=\bigcup_{j \leqq 2n^{1/k}} P_j$. Clearly $R(Q) \leqq 2n^{1/k}+R(P_0) \leqq 2ln^{1/k}$.

Claim. For any $j<2n^{1/k}$, $|(P_j \cup P_{j+1})-I| \geqq n^{m/k}$. If not, then the cardinality of $H' =(H-((P_j \cup P_{j+1})-I) \cup P_0)$ is greater than the cardinality of H. We shall obtain a contradiction by showing that $\chi(H') \leqq c$. Partition H' into (H_0, H_1) where $H_0 =(P_j \cap I) \cup \bigcup_{i<j} P_i$ and $H_1=(P_{j+1} \cap I) \cup \bigcup_{i>j+1} P_i$. Since $R(H_0) \leqq 2kn^{1/k}$, $\chi(H_0) \leqq c$. Since $H_1 \subset H$, $\chi(H_1) \leqq c$. Since I is independent, there are no edges between H_0 and H_1. Thus $\chi(H') \leqq c$.

By the claim, it is clear that $|Q-I| \geqq 1/2 \cdot 2n^{1/k} \cdot n^{m/k}$. Thus $Q-I$ is an $(n^{l/k}, 2ln^{1/k})$-obstruction. ∎

2. Examples

In this section, we consider whether the bound on $\chi(G)$ of Theorem 1 can be improved for integer values of k and also whether we gain anything by allowing k to be any rational number. We shall not be interested in lowering the constant in the bound on $R(H)$. The first example, due to Gallai in the case $k=2$, Lovász [2] in the general case, and independently to Schrijver [4], shows that when $c=2$, our bound is best possible, regardless of whether k is rational.

Example 3. *For every positive rational k, there exists a graph G on n vertices such that if H is a subgraph of G with $R(H)<1/2 \cdot n^{1/k}$, then $\chi(H) \leqq 2$, but $\chi(G)=k+2$.* ∎

The second example shows that regardless of c, if $1<k\leqq 2$ then our result cannot be improved. It was constructed by Schmerl [3] to prove a result in recursive combinatorics.

Example 4. *For any rational number k and any positive integer c such that $1<k<2$, there exists a graph G on n vertices such that if H is a subgraph of G with $R(H)<1/2n^{1/k}$, then $\chi(H)\leqq c$, but $\chi(G)=2(c-1)+1$.* ∎

The question of whether Theorem 1 is best possible when both c and k are greater than 2 is still open.

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H. A. Kierstead, E. Szemerédi and W. T. Trotter, Jr.

Department of Mathematics and Statistics
University of South Carolina
Columbia, S. C. 29208, U.S.A.