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Angle Orders

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Abstract. A finite poset is an angle order if its points can be mapped into angular regions in the plane so that x precedes y in the poset precisely when the region for x is properly included in the region for y. We show that all posets of dimension four or less are angle orders, all interval orders are angle orders, and that some angle orders must have an angular region less than 180° (or more than 180°). The latter result is used to prove that there are posets that are not angle orders.

The smallest verified poset that is not an angle order has 198 points. We suspect that the minimum is around 30 points. Other open problems are noted, including whether there are dimension-5 posets that are not angle orders.

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1. Introduction

This paper investigates an unusually rich and fascinating class of finite partially ordered sets. Members of this class, referred to as angle orders, are representable under proper inclusion by angular regions in the plane. Our results are summarized later in this introduction and verified in ensuing sections. The paper concludes with some interesting open problems.

Throughout, a *poset* is a pair $(X, \prec)$ in which $\prec$ is an asymmetric and transitive binary relation on a nonempty finite set X. An *angular region* is a closed region A of $\mathbb{R}^2$ bounded by a pair (r_1, r_2) of distinct rays emanating from a point v that contains all points swept out by rays from v in the clockwise direction from r_1 to r_2. The *vertex* v of A is unique unless the angle from r_1 to r_2 is 180°. We refer to an angular region A as *little* if its described angle is less than 180° and as *big* if its angle exceeds 180°. Only A that are little or are half planes are convex: see Figure 1. The set of all angular regions in $\mathbb{R}^2$ is denoted by $\mathscr{A}$.

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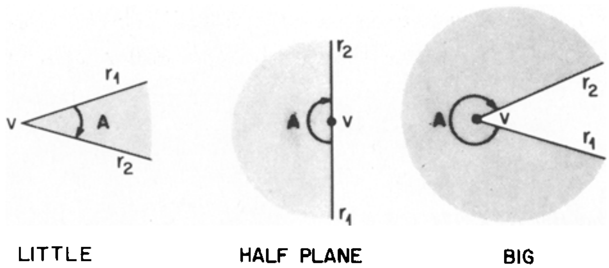

Fig. 1. Angular regions.

DEFINITION. A poset $(X, \prec)$ is an *angle order* if there exists a mapping $f\colon X \to \mathscr{A}$ such that, for all $x, y \in X$,

$$x \prec y \Leftrightarrow f(x) \subset f(y).$$

We refer to such an f as a *representation* of $(X, \prec)$. The great variety of representations of an angle order is suggested in part by Figure 2, which shows six different ways for pairs of little angular regions that $\{f(x) \not\subset f(y), f(y) \not\subset f(x)\}$ can be realized when $\{x, y\}$ is an incomparable pair. This variety contributes to the challenge of analyzing angle orders.

For every $f\colon X \to \mathscr{A}$ there is a *dual* mapping $f^*\colon X \to \mathscr{A}$ defined by $f^*(x) = [f(x)]^*$, where A^* denotes the *closure of the complement* of $A \in \mathscr{A}$. It is easily seen that $A^* \in \mathscr{A}$, $(A^*)^* = A$, $A \cap A^* = r_1 \cup r_2$, $A \cup A^* = \mathbb{R}^2$, and $A \subset B \Leftrightarrow B^* \subset A^*$, for all $A, B \in \mathscr{A}$. Clearly, A^* is little [big] if A is big [little].

The *dual* $(X, \prec')$ of a poset $(X, \prec)$ has $x \prec' y \Leftrightarrow y \prec x$. Since $f^*(x) \subset f^*(y) \Leftrightarrow f(y) \subset f(x)$, it follows that the class of angle orders is closed under duality. We shall use this fact later.

Our study of angle orders is related to other investigations of posets with representa-

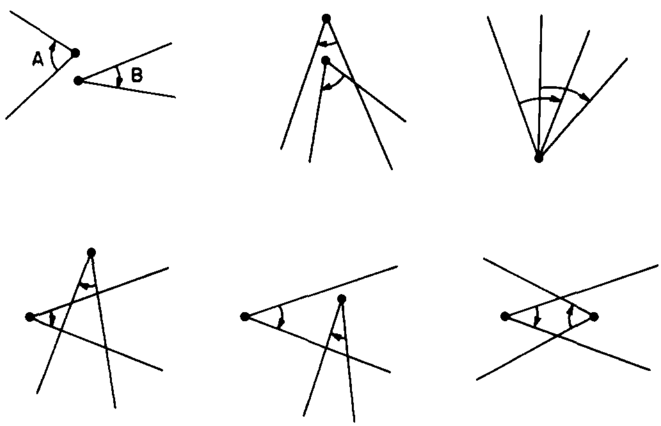

Fig. 2. $A \not\subset B$ and $B \not\subset A$.

tions based on closed bodies (usually assumed to be convex) in finite-dimensional Euclidean spaces [4, 6]. We give two examples for the representing family $\mathscr{I}$ of nondegenerate and bounded closed intervals in $\mathbb{R}$.

First, consider the class of posets for which there is a $g: X \to \mathscr{I}$ such that, for all $x, y \in X$,

$$x \prec y \Leftrightarrow g(x) \subset g(y).$$

It is easily seen that this is the class of all posets with dimension 2 or less. The *dimension* $D(X, \prec)$ of a poset [2] is the least number of linear orders (chains) on X whose intersection is $(X, \prec)$. Consequently, $D(X, \prec) \leqslant n$ if and only if there are $g_i: X \to \mathbb{R}$ with $x \neq y \Rightarrow g_i(x) \neq g_i(y)$ for $i = 1, \ldots, n$ such that, for all $x, y \in X$, $x \prec y \Leftrightarrow g_i(x) < g_i(y)$ for $i = 1, \ldots, n$.

Second, consider the class of posets for which there is a $g: X \to \mathscr{I}$ such that, for all $x, y \in X$,

$$x \prec y \Leftrightarrow \sup g(x) < \inf g(y).$$

This is the class of *interval orders*. Since the poset $(\{a, b, x, y\}, \{(a, b), (x, y)\})$ has $D = 2$ but is not an interval order, and since there are interval orders with arbitrarily large dimension [1], neither of our two $\mathscr{I}$ classes is included in the other. However, both are included in the class of angle orders.

The next few paragraphs describe the main results of our study.

Small dimensional posets are discussed in Section 2. We prove that every poset with $D \leqslant 4$ is an angle order. Special representations are used to characterize $D \leqslant 2$ and $D \leqslant 3$ precisely. It is noted also that there are angle orders of arbitrarily large dimension that have representations which use only two angles, both of which can be less than 180°.

The $D \leqslant 4$ proof in Section 2 uses representations whose angles are all less than 90°. The same type of representation is then used in Section 3 to prove that all interval orders are angle orders even though their dimensions are unbounded.

Section 4 considers two n-dimensional subposets [2, 5] of the lattice of subsets of $\{1, 2, \ldots, n\} = \mathbf{n}$ ordered by proper inclusion. The first, L_n, takes X as the subsets with cardinalities in $\{0, 1, 2, n-1\}$. Then second, B_n, takes X as the subsets of $\mathbf{n}$ with cardinalities in $\{1, n-2, n-1, n\}$. We show that each L_n and B_n is an angle order. Moreover, for all large n, every representation of L_n has a big angular region and, dually, every representation of B_n has a little angular region.

These results are combined in Section 5 with the fact that a big angular region cannot be included in a little angular region to prove that there are posets that are not angle orders. We have found it surprisingly difficult to construct a small poset that is not an angle order. Our best to date has 198 points and dimension 7.

However, we suspect that the lattice $(2^5, \subset)$ with 32 points and dimension 5 is not an angle order. This and other open questions are discussed briefly in Section 6.

2. Small Dimensions

Here, and later, we shall let $v(x)$, $r_1(x)$, and $r_2(x)$ denote the parameters of an angular region $f(x)$ as described earlier. The vertex of $f(x)$ is $v(x)$ with $\{v(x)\} = r_1(x) \cap r_2(x)$.

Any one of three simple representations can be used to characterize all posets with $D \leqslant 2$. The first uses the same $v(x)$ for all $f(x)$; the second uses the same 90° angle with fixed orientation for all $f(x)$; the third has all $v(x)$ on the same line, which also includes all r_1 rays.

To illustrate these, suppose $D(X, \prec) \leqslant 2$ and let $g_1, g_2 : X \to \mathbb{R}$ satisfy $x \prec y \Leftrightarrow g_i(x) < g_i(y)$ for $i = 1, 2$, along with $g_i(x) \neq g_i(y)$ whenever $x \neq y$. Assume with no loss in generality that the g_i are scaled so that $0 < g_i(x) < \pi/2$ for all x. The representations are as follows:

(1) $v(x) = (0, 0)$ for all x. $r_1(x)$ lies at an angle of $g_1(x)$ radians above the positive abscissa; $r_2(x)$ lies at an angle of $g_2(x)$ radians below the positive abscissa;

(2) $v(x) = (g_1(x), g_2(x))$. The angle for $f(x)$ is the 90° angle southwest of $v(x)$;

(3) $v(x) = (g_1(x), 0)$. $r_1(x)$ extends left from $v(x)$ along the abscissa; $r_2(x)$ lies above the abscissa in the negative direction at an angle of $g_2(x)$ radians.

These are illustrated on Figure 3. For (1), both g_i are used for the angle; for (2), both g_i are used for the vertex; for (3), one g_i is used for the vertex and the other is used for the angle. In each case $f(x) \subset f(y)$ if and only if $g_1(x) < g_1(y)$ and $g_2(x) < g_2(y)$, so $f(x) \subset f(y) \Leftrightarrow x \prec y$.

Converse assertions hold. Suppose f is a representation of $(X, \prec)$. If $v(x) = v(y)$ for all $x, y \in X$, then $D(X, \prec) \leqslant 2$; if all $f(x)$ have 90° angles with a common orientation, then $D(X, \prec) \leqslant 2$; if $\cap_X r_1(x)$ includes an infinite half line, then $D(X, \prec) \leqslant 2$. We omit the simple proofs.

The ideas behind the preceding representations for $D \leqslant 2$ combine to characterize $D \leqslant 3$. With $x \prec y \Leftrightarrow g_i(x) < g_i(y)$ for $i = 1, 2, 3$, we use one of the g_i to position vertices along a straight line and use the other two to position the angles for r_1 and r_2. Details are left to the reader.

Before proving that all posets with $D \leqslant 4$ are angle orders, we note that there are angle orders of arbitrarily large dimension that can be represented with just two angles. Two representations of this type for the n-dimensional poset [2] formed by the 1-sets and $(n-1)$-sets in $\mathbf{n} = \{1, 2, \ldots, n\}$ ordered by $\subset$ are shown in Figure 4. The upper represen-

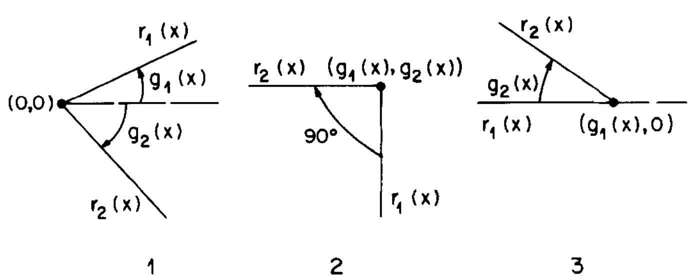

Fig. 3. Three representations for $D \leqslant 2$.

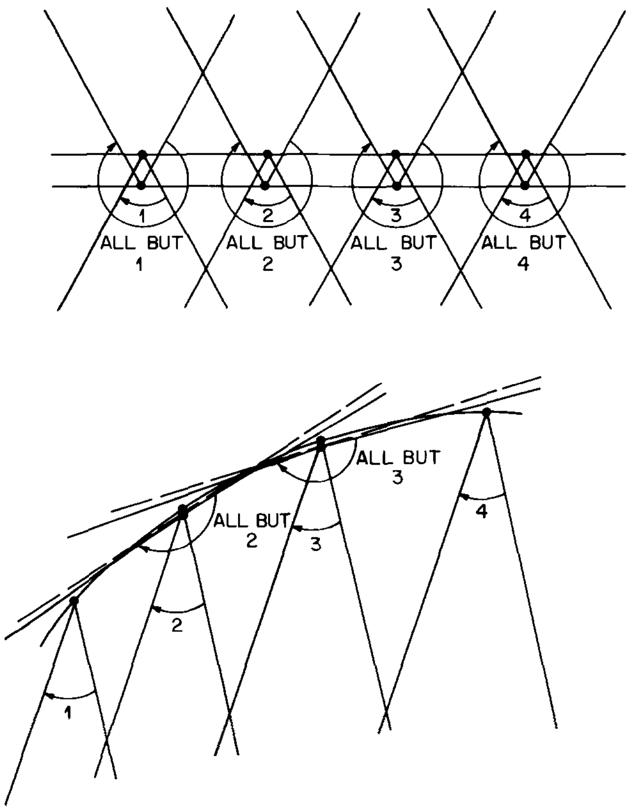

Fig. 4. Representations with two angles.

tation has all $v(\{i\})$ on the same straight line, all $v(\mathbf{n}\backslash\{i\})$ on another parallel straight line, and every angle in each of the two classes has the same orientation. The lower representation has all $v(\{i\})$ on an arc, and all its angular regions are little. The orientations of the larger angle vary in this case.

Our main result for small D is

THEOREM 1. *Every poset with dimension* 4 *or less is an angle order.*

Proof. Given $D(X,\prec) \leqslant 4$, let $g_i : X \to \mathbb{R}$ for $i = 1, 2, 3, 4$ be such that $g_i(x) \neq g_i(y)$ whenever $x \neq y$, with $x \prec y \Leftrightarrow g_i(x) < g_i(y)$ for all $i \in \{1, \ldots, 4\}$. We use the first two g_i for vertices:

$$v(x) = (g_1(x), g_2(x)).$$

Rescale g_3 and g_4, preserving their orders, so that, for all $x \in X$ and $i = 3, 4, \pi/4 - \delta < g_i(x) < \pi/4$, where δ is positive but small. Then let $r_1(x)$ be at an angle of $g_3(x)$ radians counterclockwise from the line in the southwest direction from $v(x)$, and let $r_2(x)$ be at an angle of $g_4(x)$ radians clockwise from the same line: see Figure 5.

Suppose $g_i(x) < g_i(y)$ for $i = 1, 2, 3, 4$. Then, with δ small, $f(x) \subset f(y)$. Conversely,

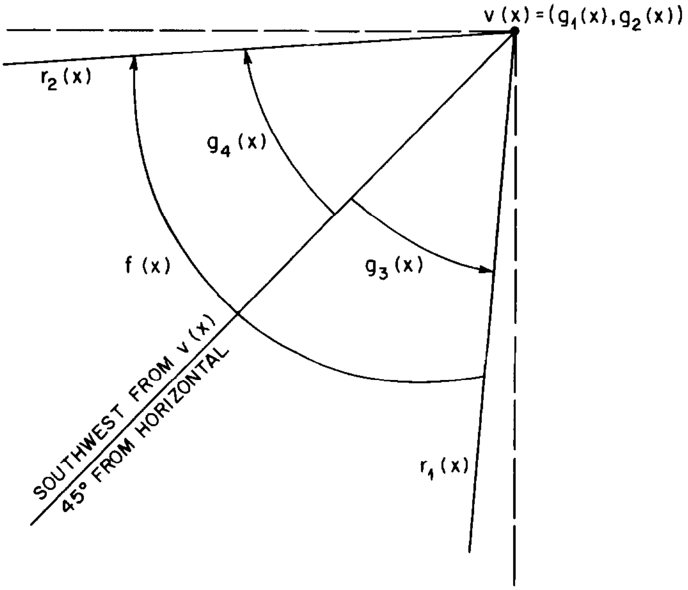

Fig. 5. Angular region near 90°.

if both $g_1(x) < g_1(y)$ and $g_2(y) < g_2(x)$, or if $v(x)$ is southwest of $v(y)$ and either $g_3(y) < g_3(x)$ or $g_4(y) < g_4(x)$, we get $f(x) \not\subset f(y)$ and $f(y) \not\subset f(x)$. □

We do not presently know whether every five-dimensional poset is an angle order. See Sections 5 and 6.

3. Interval Orders

The configuration of angular regions used in the proof of Theorem 1 also suffices to represent some posets whose dimensions exceed 4, as we now show in proving

THEOREM 2. *Every interval order is an angle order.*

Proof. Let $I_n = \{ [i, j] : 1 \leqslant i \leqslant j \leqslant n,\ i \text{ and } j \text{ integers}\}$. It is well known that if $(X, \prec)$ is a finite interval order then there is an n for which the points in X can be mapped into the intervals in I_n so that, for all $x, y \in X$, $x \prec y$ if and only if x's interval lies wholly to the left of y's interval on the line. Since each I_n ordered in this way is an interval order, it suffices to show that each of these is an angle order.

Our proof of this is pictured in Figure 6. The vertices $v([i, j])$ are ordered from left to right along the top left arc of a circle lexicographically with $v([i, j])$ before $v([k, l])$ if and only if $j < l$ or $(j = l,\ i < k)$. We label $v([i, j])$ as ij on the figure. We also identify a point just before the cluster of vertices that end in j. It is depicted as a little open circle and labeled j. For each x in I_n, $r_1(x)$ can be taken vertically downward from $v(x)$ [or tilted a little to the left of vertical to conform to the configuration used in the preceding proof], and $r_2([i, j])$ is the ray emanating from $v([i, j])$ that goes through the point labeled i (little open circle). It follows by construction and inspection that $f([i, j]) \subset f([k, l]) \Leftrightarrow j < k$. □

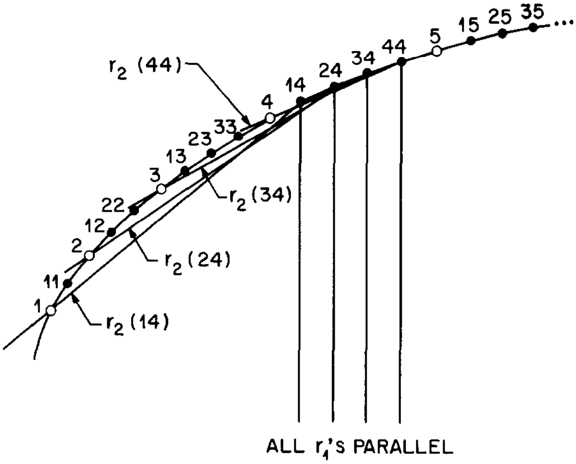

Fig. 6. Representation for interval order.

4. Big and Little Regions

We now prove that there are angle orders which must have a big (little) angular region in every representation. Two subposets of $(2^n, \subset)$ will be used in this:

$$L_n = (\{x \in 2^n : |x| \in \{0, 1, 2, n-1\}\}, \subset),$$
$$B_n = (\{x \in 2^n : |x| \in \{1, n-2, n-1, n\}\}, \subset).$$

THEOREM 3. *Every L_n is an angle order. There is an integer n_0 such that if $n \geqslant n_0$ then every representation of L_n has a big angular region.*

REMARKS. Our proof uses $n_0 = 21$ although some smaller n_0 might suffice. We note also that if the $(n-1)$-sets are deleted from L_n, then the remainder is representable by little angular regions: see Figure 7. The figure depicts $f(\emptyset)$, the $f(\{i\})$, and $f(\{3, 6\})$.

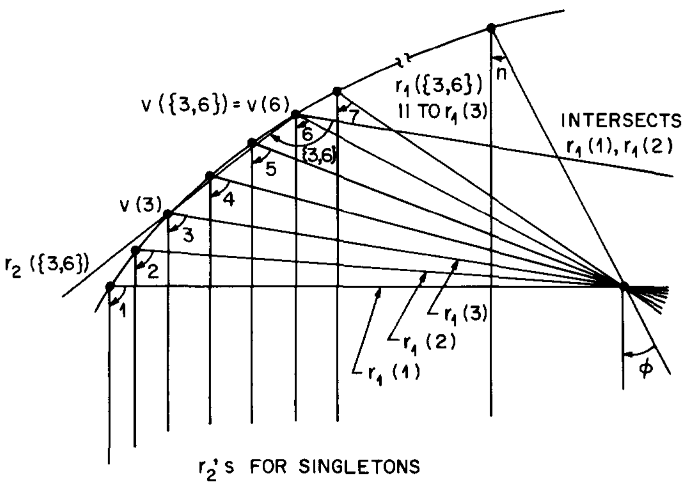

Fig. 7. Representation with little regions.

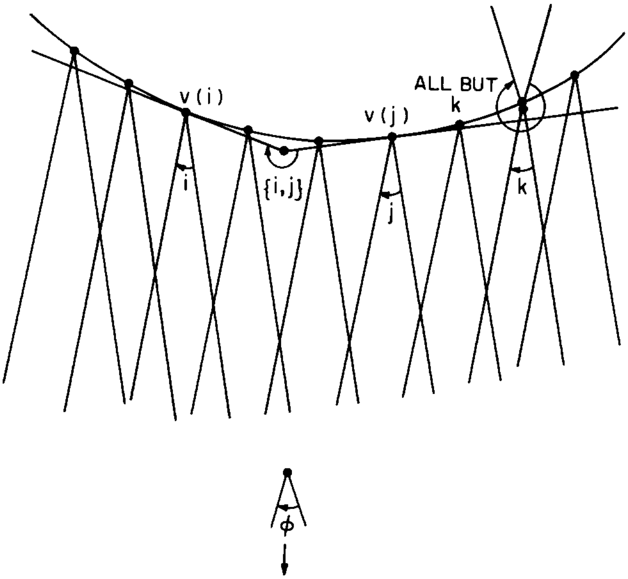

Fig. 8. Representation of L_n.

Given $i < j$, the vertex of $f(\{i, j\})$ is $v(\{j\})$, $r_1(\{i, j\})$ is parallel to $r_1(\{i\})$, and $r_2(\{i, j\})$ goes through $v(\{i\})$.

Proof. Figure 8 illustrates a representation of L_n. The singleton vertices lie along the lower arc of a circle. Their angles open downward and include $f(\emptyset)$. The big angular region for doubleton $\{i, j\}$ has rays through $v(\{i\})$ and $v(\{j\})$. It includes $f(\{i\})$ and $f(\{j\})$ but cuts every other $f(\{k\})$ below its vertex for incomparability. The vertex of $f(\mathbf{n}\backslash\{k\})$ is just beneath the vertex of $f(\{k\})$, and its angle is near $360°$.

We now show that every representation of L_n for $n \geqslant 21$ must have a big angular region. Assume to the contrary that $n \geqslant 21$ and that f is a representation of L_n with no big angular region. We shall obtain a contradiction.

With no loss in generality, assume that $f(\emptyset)$ includes the positive abscissa. Then every $f(x)$ includes the positive abscissa. For each singleton in L_n, let $\alpha(i)$ and $\beta(i)$ be respectively the angle in radians between $r_1(\{i\})$ and the positive abscissa and the angle between $r_2(\{i\})$ and the positive abscissa. Let α^* be the largest $\alpha(i)$ and β^* the largest $\beta(i)$, and $\alpha(a) = \alpha^*$ and $\beta(b) = \beta^*$ for $a, b \in \mathbf{n}$. Since all angles for f are $180°$ or less, consideration of $f(\{a, b\})$ shows that $\alpha^* + \beta^* \leqslant \pi$.

Let $N = \mathbf{n}\backslash\{a, b\}$. By considering the $(n - 1)$-sets in $2^{\mathbf{n}}$ that omit a point in N and therefore contain both a and b, it is easily seen that the vertices of the angular regions for the singletons in N must be distinct and lie in a 'convex' pattern with respect to the vertex of $f(\emptyset)$: see Figure 9. There is a line through each such vertex that has all other singleton vertices as well as $f(\emptyset)$ to its right.

Since $n \geqslant 21$, assume with no loss in generality that $\{1, 2, \ldots, 19\} \subseteq N$ and that $v(\{1\})$ through $v(\{19\})$ go from the bottom up on Figure 9. Then, ignoring 1 and 19, there are integers i, j, k such that

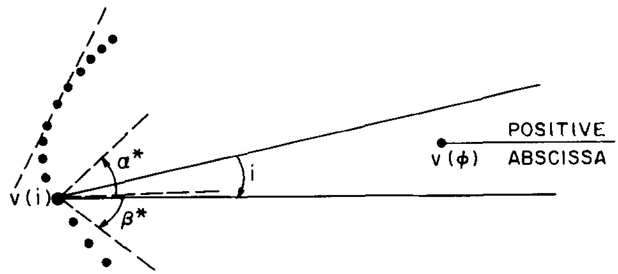

Fig. 9. 'Convex' pattern of $v(i)$ for i in N.

$$2 \leqslant i < j < k \leqslant 18,$$
$$\alpha(i) \leqslant \alpha(j) \leqslant \alpha(k) \quad \text{or} \quad \alpha(i) \geqslant \alpha(j) \geqslant \alpha(k),$$
$$\beta(i) \leqslant \beta(j) \leqslant \beta(k) \quad \text{or} \quad \beta(i) \geqslant \beta(j) \geqslant \beta(k).$$

This follows from the Erdös–Szekeres theorem [3, 7] which says that every linear arrangement of the first $m^2 + 1$ positive integers has either an increasing sequence or a decreasing sequence of at least $m + 1$ integers. The $17 = (4)^2 + 1$ indices in $\{2, 3, \ldots, 18\}$ yield five $\alpha(h)$ that are monotone nondecreasing or monotone nonincreasing in the natural order of their h's; these $5 = (2)^2 + 1$ then have an internal subsequence of at least three $\beta(h)$ that is nondecreasing or nonincreasing.

Figure 10 illustrates the situation established in the preceding paragraph. By the monotonicities of α and β over $\{i, j, k\}$, $v(\{j\})$ must be to the left of $f(\{i, k\})$. This angular region can cut to the right (see dotted lines) at or above $v(\{k\})$, or at or below $v(\{i\})$, but the solid line – which could contain $v(\{i, k\})$ – to the right of $v(\{j\})$ must extend either upward or downward indefinitely.

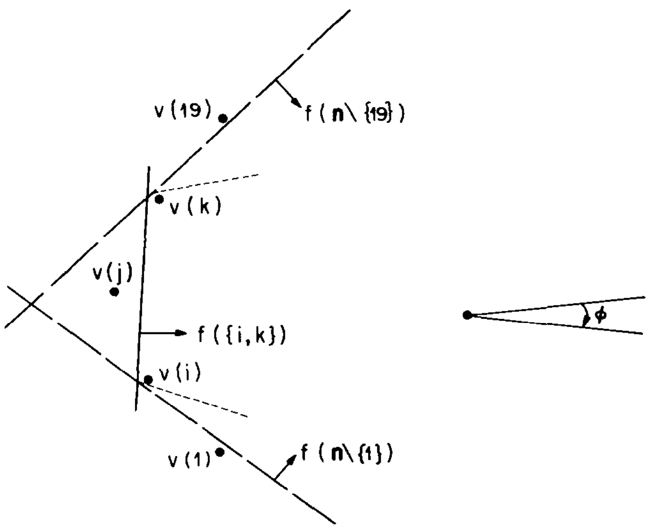

Fig. 10. Set-up for contradiction.

Now consider the angular regions $f(\mathbf{n}\setminus\{1\})$ and $f(\mathbf{n}\setminus\{19\})$. These are presumed to be little or to be half planes. With no loss in generality, assume they are half planes. Because of α^* and β^*, the line for $f(\mathbf{n}\setminus\{1\})$ must pass to the right of $v(\{1\})$, and the line for $f(\mathbf{n}\setminus\{19\})$ must pass to the right of $v(\{19\})$. Moreover, for inclusion, each line must have all other singleton vertices on it or to its right: see the dashed lines on Figure 10.

Finally, since f is assumed to be a representation of L_n, we require $f(\{i, k\}) \subset f(\mathbf{n}\setminus\{1\})$ and $f(\{i, k\}) \subset f(\mathbf{n}\setminus\{19\})$. But this is clearly impossible under the conditions established in the preceding two paragraphs. □

Theorem 3 has the following simple corollary.

COROLLARY 1. *Every B_n is an angle order. There is an integer n_0 such that if $n \geqslant n_0$ then every representation of B_n has a little angular region.*

Proof. By the discussion of duality in the introduction, f is a representation of L_n if and only if f^* is a representation of the dual of L_n. Since B_n is order-isomorphic to the dual of L_n under the mapping $x \to \mathbf{n}\setminus x$, Corollary 1 follows from Theorem 3. □

5. Not all Posets are Angle Orders

COROLLARY 2. *There are posets that are not angle orders.*

Proof. Let L'_n and B'_n be disjoint copies of L_n and B_n respectively, and define a poset $(X_n, \prec)$ as follows. The points in X_n are the members of L'_n and B'_n. For all $x, y \in X_n$, define $\prec$ by

$$x \prec y \quad \text{if } (x, y \text{ are in } L'_n \text{ and } x \subset y)$$
$$\text{or } (x, y \text{ are in } B'_n \text{ and } x \subset y)$$
$$\text{or } x \text{ is in } L'_n \text{ and } y \text{ is in } B'_n.$$

When $n \geqslant n_0$ in Theorem 3 and Corollary 1, $(X_n, \prec)$ cannot be an angle order since a big angular region (from L'_n) cannot be included in a little angular region (from B'_n). □

Since the smallest n_0 that suffices for our proof of Theorem 3 is 21, the proof of Corollary 2 shows that there is a poset with $2\,|L_{21}| = 706$ points that is not an angle order.

We can do better. A proof like that used for Theorem 3 shows that *if* $(\{x \in 2^7 : |x| \leqslant 4\}, \subset)$ is an angle order, then every representation must have a big angular region. Since this poset has 99 points, a proof similar to the proof of Corollary 2 yields a poset with 198 points that is not an angle order.

However, 198 is probably much larger than the cardinality of the smallest poset that is not an angle order.

6. Open Questions

We have not been able to determine whether $(2^5, \subset)$ is an angle order. The delicacy of resolving this is suggested by the fact that if any 4-element set is removed from 2^5 then the remainder has dimension 4 and is therefore an angle order by Theorem 1. If $(2^5, \subset)$ is not an angle order, then it seems likely that the smallest poset that is not an angle order has about 30 points.

Two related questions: What is the dimension of the least-dimension poset that is not an angle order? What is the smallest n such that $(2^n, \subset)$ is not an angle order? If $(2^5, \subset)$ can be shown not to be an angle order, then these questions are obviously closed.

A somewhat different concern that is not directly involved with dimensionality is whether it is always possible to add a new least point x_0 ($\prec$ everything in X) to an angle order $(X, \prec)$ so that the augmented poset is also an angle order. Equivalently, if $(X, \prec)$ is an angle order, must it have a representation f such that $\cap f(x)$ includes an angular region? A more demanding question: Does there exist an angle order such that for every representation f there are $x, y \in X$ for which $f(x)$ and $f(y)$ are disjoint?

Finally, we return to the matter of specialized representations. We know that $D \leqslant 2$ and $D \leqslant 3$ are precisely characterized by representations that obey simple restrictions. Is this true also for $D \leqslant 4$? The proofs of Theorems 1 and 2 show that the southwest-angles restriction used in Theorem 1 will also suffice for some angle orders of arbitrarily large dimension. Is there some other way to represent $D \leqslant 4$ that will exclude all angle orders with dimension 5 or more?

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