Order **2** (1985), 145–164. 0167–8094/85.15.
© 1985 *by D. Reidel Publishing Company.*

Inequalities for the Greedy Dimensions of Ordered Sets

HENRY A. KIERSTEAD* and WILLIAM T. TROTTER, JR.**
Department of Mathematics and Statistics, University of South Carolina, Columbia, SC 29208, U.S.A.

Communicated by I. Rival

(Received: 11 March 1985; accepted: 3 July 1985)

Abstract. Every linear extension $L : [x_1 < x_2 < \cdots < x_m]$ of an ordered set P on m points arises from the simple algorithm: For each i with $0 \leqslant i < m$, choose x_{i+1} as a minimal element of $P - \{x_j : j \leqslant i\}$. A linear extension is said to be greedy, if we also require that x_{i+1} covers x_i in P whenever possible. The greedy dimension of an ordered set is defined as the minimum number of greedy linear extensions of P whose intersection is P. In this paper, we develop several inequalities bounding the greedy dimension of P as a function of other parameters of P. We show that the greedy dimension of P does not exceed the width of P. If A is an antichain in P and $|P - A| \geqslant 2$, we show that the greedy dimension of P does not exceed $|P - A|$. As a consequence, the greedy dimension of P does not exceed $|P|/2$ when $|P| \geqslant 4$. If the width of $P - A$ is n and $n \geqslant 2$, we show that the greedy dimension of P does not exceed $n^2 + n$. If A is the set of minimal elements of P, then this inequality can be strengthened to $2n - 1$. If A is the set of maximal elements, then the inequality can be further strengthened to $n + 1$. Examples are presented to show that each of these inequalities is best possible.

AMS (MOS) subject classifications (1980). 06A05, 06A10.

Key words. Ordered sets, linear extension, greedy dimensions.

1. Introduction

Let P be an ordered set. Then P can be represented as the intersection of some family of linear orders each of which is a linear extension of P. Such a family is called a *realizer* of P. The *dimension* of an ordered set P is the minimum size of a realizer of P. The concept of dimension has proved to be a useful invariant in the study of ordered sets. A natural extension of this concept arises from requiring the linear orders which form a realizer of P to have certain additional properties.

In this paper we consider one such property which arises in the study of the following scheduling problem which is known as the jump number problem:

> An ordered set P represents a set of tasks to be performed on a single processor. If $x < y$ in P, then x must be performed before y. An admissible schedule is then a

* Research supported in part by the National Science Foundation under ISP-80110451.
** Research supported in part by the National Science Foundation under ISP-80110451 and DMS-8401281.

linear extension of P. Suppose that a set-up cost is paid for each pair $x, y \in P$ with x incomparable to y in P and x and y occurring consecutively in the linear extension. Find a linear extension L of P which minimizes the number of consecutive pairs of L which are incomparable in P.

This scheduling problem is *NP* complete [8]. However, it is natural to search for approximate solutions using the following 'greedy' approach. Form a linear extension L of P by starting with an arbitrary minimal element x_1. Then search for a minimal element x_2 of $P - x_1$ with x_2 covering x_1 in P. If no such point exists, choose x_2 as an arbitrary minimal element of $P - x_1$. This process is repeated in an effort to construct a linear extension of P avoiding consecutive pairs which are incomparable in P.

Any linear extension constructed by this greedy approach is called a *greedy* linear extension. It can be shown [3] that among the linear extensions which provide a solution to the scheduling problem, there is one which is a greedy linear extension.

In [1], Bouchitté, Habib, and Jégou showed that every ordered set can be represented as the intersection of some family of greedy linear extensions. They defined the *greedy dimension* of an ordered set P, denoted $\dim_g(P)$, as the least number t for which there are t greedy linear extensions of P whose intersection is just P, i.e., $\dim_g(P)$ is the minimum size of a greedy realizer of P. In this paper, we develop inequalities which relate the greedy dimension of an ordered set P to both the size and the width of $P - A$ where A is an antichain in P. In order to motivate our results, we pause to summarize some relevant facts from dimension theory. For more detailed information, we refer the reader to the survey article by Keller and Trotter [6].

2. The Dimension of Ordered Sets

Let P be an ordered set. If x and y are incomparable points in P, we write $x \parallel y$ in P. So a collection Σ of linear extensions of P is a realizer of P if and only if for every $x, y \in P$ with $x \parallel y$ in P, there exist $L, L' \in \Sigma$ with $x < y$ in L and $y < x$ in L'. Let X and Y be subsets of P, and let L be a linear extension of P. We write X/Y in L if $x > y$ in L whenever $x \in X$, $y \in Y$ and $x \parallel y$ in P. The following result completely characterizes when such extensions exist.

LEMMA 1 (Rabinovitch [9]). *Let X and Y be subsets of an ordered set P. Then there exists a linear extension L of P with X/Y in L if and only if there do not exist two points $x_1, x_2 \in X$ and two points $y_1, y_2 \in Y$ with $x_1 < y_1$, $x_2 < y_2$, $x_1 \parallel y_2$, and $x_2 \parallel y_1$ in P.* □

The following elementary result follows immediately from Lemma 1.

LEMMA 2 (Hiraguchi [5]). *Let C be a chain in an ordered set P. Then there exist linear extensions L and L' of P so that C/P in L and P/C in L'.* □

Recall that Dilworth's theorem [2] asserts that if the width of an ordered set P is n, then P can be partitioned into n chains. Let $P = C_1 \cup C_2 \cup \cdots \cup C_n$ be such a partition.

For each $i = 1, 2, \ldots, n$, let L_i be a linear extension of P so that C_i/P in L_i. Then clearly $\Sigma = \{L_1, L_2, \ldots, L_n\}$ is a realizer of P. This proves the following well known inequality.

LEMMA 3 (Hiraguchi [5]). *For every ordered set P, $dim(P) \leqslant width(P)$.* □

Dimension is a comparability invariant, i.e., two ordered sets with the same comparability graph have the same dimension [12]. In particular, an ordered set P and its dual P^* have the same dimension. Dimension is monotonic, i.e., if P is a subordered set of Q, then $\dim(P) \leqslant \dim(Q)$. Furthermore, the removal of a point from an ordered set cannot decrease the dimension by an arbitrary amount.

LEMMA 4 (Hiraguchi [5]). *For every ordered set P and for every $x \in P$, $dim(P) \leqslant 1 + dim(P - \{x\})$.* □

Let P be an ordered set and let $\Omega = \{Q_x : x \in P\}$ be a family of ordered sets indexed by the point set of P. Recall that the *lexicographic sum*★ of Ω over P is the ordered set whose point set is $\{(x, y) : x \in P, y \in Q_x\}$ with order given by $(x_1, y_1) < (x_2, y_2)$ if $x_1 < x_2$ in P or if $x_1 = x_2$ and $y_1 < y_2$ in Q_{x_1}.

LEMMA 5 (Hiraguchi [5]). *The dimension of the lexicographic sum of $\Omega = \{Q_x : x \in P\}$ over P is the larger of $dim(P)$ and $max\ \{dim(Q_x) : x \in P\}$.*

In [5], Hiraguchi proved that the dimension of an ordered set P does not exceed $|P|/2$ when $|P| \geqslant 4$. A simple proof of this inequality is obtained by combining Lemma 3 with the following result discovered independently by Kimble [7] and Trotter [10].

LEMMA 6. *Let A be an antichain in an ordered set P with $|P - A| \geqslant 2$. Then $dim(P) \leqslant |P - A|$.* □

The primary difficulty in proving Lemma 6 is the case $|P - A| = 2$. Once this is accomplished, the general result follows by induction on $|P - A|$ using Lemma 4. The argument when $|P - A| = 2$ can be simplified considerably by appealing to Lemma 5 (see [10] for details).

3. The Greedy Dimension of Ordered Sets

Bouchitté, Habib, and Jégou [1] showed that for an ordered set P, $\dim(P) = \dim_g(P)$ if P is of dimension at most two or if P is a distributive lattice or if the diagram for P is N-free. These results were communicated to Trotter who observed that in fact the greedy dimension of P never exceeds the width of P. This inequality follows immediately from the following lemma, which should be compared with Lemma 2.

LEMMA 7. *Let C be a chain in an ordered set P. Then there exists a greedy linear extension L of P with C/P in L.* □

★ Unfortunately, the notation here is far from standard.

The inequality $\dim_g(P) \leqslant \text{width}(P)$ and the proof of Lemma 7 were obtained independently by Bouchitté, Habib, and Jégou [1]. We do not include a proof of Lemma 7 in this section since the result is a special case of both Lemmas 10 and 11 which we prove in Section 5. However, we will pause here to construct a family $\{P_n : n \geqslant 2\}$ which shows that the inequality $\dim_g(P) \leqslant \text{width}(P)$ is best possible even when $\dim(P) = 3$. The ordered set P_n is shown in Figure 1.

$P_n; n \geqslant 2$

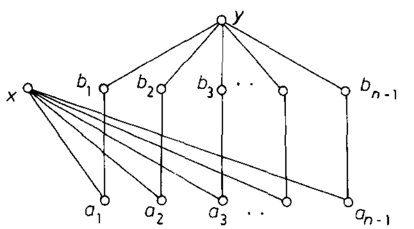

Fig. 1.

When $n = 2$, it is obvious that $\dim(P_n) = 2$. For each $n \geqslant 3$, $\dim(P_n) \geqslant 3$ since the subordered set generated by $\{x, y, a_1, b_1, a_2, b_2\}$ is three-dimensional. On the other hand, $\dim(P_n) \leqslant 3$ since it is obvious that $\dim(P_n - \{x\}) = 2$. However, apart from a permutation of subscripts, P_n admits only two essentially different greedy linear extensions.

Type 1: $[a_1 < b_1 < a_2 < b_2 < \cdots < a_{n-1} < b_{n-1} < y < x]$

Type 2: $[a_1 < b_1 < a_2 < b_2 < \cdots < a_{n-1} < x < b_{n-1} < y]$.

The fact that $\dim_g(P) \geqslant n - 1$ follows easily from the observation that $x \parallel b_i$ for $i = 1, 2, \ldots, n-1$, but in any greedy linear extension L of P_{n-1} there is at most one value of i for which $x < b_i$ in L. So if $\dim_g(P_n) = n - 1$ and $\Sigma = \{L_1, L_2, \ldots, L_{n-1}\}$ is a family of greedy linear extensions of P_n whose intersection is just P_n, then each L_i is a Type 2 extension. This requires $x < y$ in L_i for $i = 1, 2, \ldots, n-1$. The contradiction shows that $\dim_g(P_n) \geqslant n = \text{width}(P_n)$.

This same family also illustrates some of the pathological properties of greedy dimension. For the chain $C = \{x\}$, there is no greedy linear extension L of P_n so that P/C in L. For each $n \geqslant 3$, $\dim_g(P_n^*) = 3$ while $\dim_g(P_n) = n$ so that greedy dimension is not a comparability invariant. For each $n \geqslant 3$, $\dim_g(P_n - x) = 2$, and thus the removal of a point may decrease the greedy dimension of an ordered set by an arbitrary amount. Furthermore, if we take a chain D so that $P_n \subset D \times D \times D$, then $\dim_g(P_n) = n$ but $\dim_g(D \times D \times D) = 3$. Thus, greedy dimension is not monotonic.

Greedy dimension behaves in a peculiar fashion with respect to lexicographic sums.

LEMMA 8. *Let P be an ordered set and let R be the lexicographic sum of the family $\Omega = \{Q_x : x \in P\}$ over P. Also let $t = \max\{\dim_g(Q_x) : x \in P\}$. Then the greedy dimension of R satisfies:*

$$\max\{\dim(P), t\} \leqslant \dim_g(R) \leqslant \max\{\dim_g(P), t\}.$$

Proof. Suppose first that $s = \max\{\dim_g(P), t\}$, and let Σ be a greedy realizer of P with $|\Sigma| = s$.

Also, for each $x \in P$, let Σ_x be a greedy realizer of Q_x with $|\Sigma_x| = s$. Now let $L \in \Sigma$ and for each $x \in P$, let $L_x \in \Sigma_x$. Then it is straightforward to verify that the linear extension M of R defined by taking M as the lexicographic sum of $\{L_x : x \in P\}$ over L is a greedy linear extension of R. Thus, $\dim_g(R) \leqslant s$.

On the other hand, let M be any greedy linear extension of R. Then for each $x \in P$, the restriction of M to the copy of Q_x induced by the points in $Q'_x = \{(x, y) : y \in Q_x\}$ is easily seen to be a greedy linear extension of Q'_x. Thus $\dim_g(R) \geqslant t$. The inequality $\dim_g(R) \geqslant \dim(P)$ is trivial since P is isomorphic to a subordered set of R. □

To see why we cannot strengthen the preceding lemma, consider the ordered set $P = P_n$ shown in Figure 1 and the ordered set R obtained from P by replacing each b_i by a two-element antichain while all other points are replaced by one point ordered sets. (See Figure 2). The following three linear orders form a greedy realizer of R; thus $\dim_g(R) = 3$ when $n \geqslant 3$.

$$L_1 : [a_1 < b_1 < a_2 < b_2 < \cdots < a_{n-1} < b_{n-1} < x$$
$$< c_{n-1} < \cdots < c_2 < c_1 < y]$$
$$L_2 : [a_1 < c_1 < a_2 < c_2 < \cdots < a_{n-1} < c_{n-1} < x$$
$$< b_{n-1} < \cdots < b_2 < b_1 < y]$$
$$L_3 : [a_{n-1} < b_{n-1} < c_{n-1} < a_{n-2} < b_{n-2} < c_{n-2} < \cdots$$
$$< a_2 < b_2 < c_2 < a_1 < b_1 < c_1 < y < x].$$

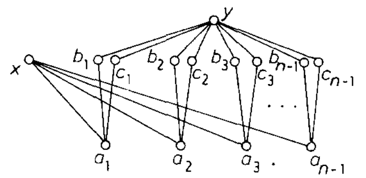

Fig. 2.

Let $C = \{x_1 < x_2 < \cdots < x_k\}$ be a chain in an ordered set P. We say that C is an *initial chain* of P when $C = \{y \in P : y \leqslant x_k \text{ in } P\}$. We call C a *maximal initial chain* when there is no point $x_{k+1} \in P - C$ so that $\{x_1 < x_2 < \cdots < x_k < x_{k+1}\}$ is an initial chain. Although the removal of a point can collapse the greedy dimension of an ordered set P by an arbitrary amount, this is not the case for maximal initial chains.

LEMMA 9. *Let C be a maximal initial chain in an ordered set P. Then $\dim_g(P) \leqslant 1 + \dim_g(P - C)$.*

Proof. Let $\dim_g(P - C) = t$ and let $\Sigma = \{L_1, L_2, \ldots, L_t\}$ be a greedy realizer of $P - C$. For each $i = 1, 2, \ldots, t$, let M_i be the linear extension of P obtained from L_i by adding the points of C at the bottom of L_i. Clearly, each M_i is a greedy linear extension of P. Now apply Lemma 7 to obtain a greedy linear extension M_{t+1} of P so that C/P in M_{t+1}. Since P/C in M_i for each $i = 1, 2, \ldots, t$, it follows that $\{M_1, M_2, \ldots, M_{t+1}\}$ is a greedy realizer of P. Thus, $\dim_g(P) \leqslant t + 1 = 1 + \dim_g(P - C)$. □

4. The Principal Theorems

The primary purpose of this paper will be to develop inequalities for the greedy dimension of an ordered set P in terms of the cardinality and width of $P - A$ where A is an antichain in P with $P - A \neq \emptyset$.

Our first major result will be to show that Lemma 6 also holds for greedy dimension.

THEOREM 1. *Let A be an antichain in an ordered set P with $|P - A| \geqslant 2$. Then $\dim_g(P) \leqslant |P - A|$.* □

Theorem 1 is trivially true when $|P - A| = 2$ since an ordered set P with $\dim(P) \leqslant 2$ satisfies $\dim_g(P) = \dim(P)$. However, the general result will require a fundamentally different approach than the one used to prove Lemma 6 since the removal of a point can collapse the greedy dimension of an ordered set by an arbitrary amount.

In [10], Trotter proved the following inequalities for ordinary dimension.

THEOREM 2. *Let A be an antichain in an ordered set P with $\mathrm{width}(P - A) = n \geqslant 1$. Then the following inequalities hold.*

(a) *$\dim(P) \leqslant 2n + 1$.*

(b) *If A is the set of minimal elements of P, then $\dim(P) \leqslant n + 1$.*

(c) *If A is the set of maximal elements of P, then $\dim(P) \leqslant n + 1$.* □

Examples constructed in [10] and [11] show that each of these inequalities is best possible. Of course, 1b and 1c are equivalent since an ordered set and its dual have the same dimension. Our second major result is the following theorem providing parallel inequalities for greedy dimension.

THEOREM 3. *Let A be an antichain in an ordered set P with $P - A \neq \emptyset$ and let $n = \mathrm{width}(P - A)$. Then the following inequalities hold:*

(a) *$\dim_g(P) \leqslant n^2 + n$ when $n \geqslant 2$ and $\dim_g(P) \leqslant 3$ when $n = 1$.*

(b) *If A is the set of minimal elements of P, then $\dim_g(P) \leqslant 2n - 1$ when $n \geqslant 2$ and $\dim_g(P) \leqslant 2$ when $n = 1$.*

(c) *If A is the set of maximal elements of P, then $\dim_g(P) \leqslant n + 1$.* □

In the next section, we will establish two technical lemmas which yield greedy linear extensions satisfying special properties. We use these lemmas in Section 6 to obtain the upper bounds of Theorem 1 and 3. In Section 7, we will construct examples to show that each of these inequalities is best possible.

5. Algorithmic Constructions for Greedy Linear Extensions

Theorems 1 and 3 assert the existence of certain greedy linear extensions of the ordered set P. In this section we develop a general algorithmic construction for greedy linear extensions from which we obtain two classes of greedy linear extensions satisfying special technical conditions. All the greedy linear extensions needed to prove Theorems 1 and 3 will be members of these classes.

Consider a linear order L on P constructed in the following recursive manner. Suppose that $|P| = m$. At stage 0, let $P^0 = P$. Suppose that at each stage $i < m$, P^i has been constructed. At stage $i + 1$ choose $x_{i+1} \in P^i$ and let $P^{i+1} = P^i - \{x_{i+1}\}$. Finally set $x_i < x_j$ in L iff $i < j$. For any subset S of P, let $\mathrm{MIN}(S)$ denote the antichain consisting of the minimal elements of S. Observe that: (1) if $x_{i+1} \in \mathrm{MIN}(P^i)$ for all i, then L is a linear extension of P; and furthermore (2) if $x_{i+1} \in \mathrm{MIN}(P^i)$ and $x_{i+1} > x_i$ in P whenever there exists $y \in \mathrm{MIN}(P^i)$ such that $y > x_i$ in P, then L is a greedy linear extension of P. With these observations in mind, let $S_i = \{x \in \mathrm{MIN}(P^i) : x_i < x\}$, $G_i = S_i$ if $S_i \neq \emptyset$, and $G_i = \mathrm{MIN}(P^i)$ if $S_i = \emptyset$. In the algorithms which follow, we will use the method and notation introduced above. We will assure that L is a greedy linear extension by always choosing $x_{i+1} \in G_i$. We will provide that L has certain additional properties by placing further restrictions on the choice of x_{i+1}.

The following notation and terminology will be useful. Letters A and B will denote antichains, while the letters C, E, and F denote chains. Linear orders will be denoted by L, M, and N. Recall that for $x, y \in P$, x *covers* y in P, which we denote by $x :> y$ in P, if $x > y$ in P and there is no z with $x > z > y$ in P. We say that a subset S of P is *rooted* in a chain C if for every $x \in S - \mathrm{MIN}(P)$, there exists $y \in C$ such that $x :> y$ in P. Let L be a linear extension of P constructed as above. We write I_i to denote the chain $\{x_j, x_{j+1}, \ldots, x_i\}$ in L where j is the least positive integer such that $\{x_j, x_{j+1}, \ldots, x_i\}$ is also a chain in P. For a subset S of P, we let $U(S) = \{x \in P : x > y$ in P for some $y \in S\}$, $D(S) = \{x \in P : x < y$ in P for some $y \in S\}$, $U[S] = U(S) \cup S$, and $D[S] = D(S) \cup S$. If $S = \{x\}$ we may write x instead of $\{x\}$ in this notation. Let Q be a subordered set of P and let C be a chain in Q. We say C is *maximal in* Q when there is no point $q \in Q - C$ so that $C \cup \{q\}$ is a chain. We denote the *dual* of an ordered set Q by Q^*, i.e., $x < y$ in Q iff $y < x$ in Q^*. For a subset S of P we let S^i denote $S \cap P^i$. For a subset S of P let $\mathrm{MAX}(S)$ denote the set of maximal elements of P. If S is a chain, let $\min(S)$ denote the least element of S and $\max(S)$ denote the largest element of S.

We now present the two technical Lemmas that are needed to prove Theorems 1 and 3. The reader may choose to see how these Lemmas are used in Section 7 before reading their proofs.

LEMMA 10. *Let E be a chain in P and let B be an antichain in P with $B \cap E = \emptyset$. Suppose that $\{B_1, B_2\}$ is a partition of B such that B_1 is rooted in E. Let N be a linear order on B_1 such that the N-least element $\hat{b}$ of B_1 satisfies* (*) $D(b) \cap E \subset D(\hat{b}) \cap E$ *for all $b \in B_1$. Then there exists a greedy linear extension $L = L(E, B_1, B_2, N)$ such that:*

(a) $E - U(B_2)/P - U[B_1]$ *in* L;

(b) $B_1/P - U[B_1]$ *in* L;

(c) *L restricted to B_1 is N;*

(d) *for each $u \in B_2 - \mathrm{MIN}(P)$, there exists an element $x \in P$ such that $u :> x$ in P and there does not exist $y \in D(B)$ with $x < y < u$ in L.*

Proof. We will use the method described above to construct L. Thus we must provide a scheme for choosing x_{i+1} from G_i.

Case 1: $B_2 \cap G_i \neq \emptyset$. Choose $x_{i+1} \in B_2 \cap G_i$ such that

$$|U(x_{i+1})| \leqslant |U(y)| \quad \text{for all } y \in B_2 \cap G_i.$$

Case 2: $B_2 \cap G_i = \emptyset$ but $G_i - B \neq \emptyset$. Choose $x_{i+1} \in G_i - B$ such that

$$|U[x_{i+1}] \cap E| \leqslant |U[y] \cap E| \text{ for all } y \in G_i - B.$$

Case 3: $G_i \subset B_1$. Choose $x_{i+1} \in G_i$ such that

$$x_{i+1} \leqslant y \text{ in } N \quad \text{for all } y \in G_i.$$

Next we show that the greedy linear extension L constructed using this preference scheme satisfies (a) – (d).

Proof of (a): Suppose that L does not satisfy (a). Choose the least i such that $x_{i+1} \in E - U(B_2)$, but there exists $y \in P - U[B_1]$ with $x_{i+1} \| y$ in P and $x_{i+1} < y$ in L. Then $y \in P^i$. Select $z \in \mathrm{MIN}(P^i)$ such that $z \leqslant y$ in P. Then $z \in P - U[B_1], x_{i+1} \| z$ in P, and $x_{i+1} < z$ in L.

Let j be the least integer such that $x_j \in I_{i+1}$ and $x_j \| z$ in P. We claim that z should have been preferred to x_j at stage j. Clearly $z \in \mathrm{MIN}(P^{j-1})$ since $P^i - P^{j-1} = \{x_j, x_{j+1}, \ldots, x_{i+1}\}$. If $x_{j-1} \| x_j$ in P, then $G_{j-1} = \mathrm{MIN}(P^{j-1})$; otherwise $x_{j-1} \in I_{i+1}$ and, by the choice of j, $x_{j-1} < z$ in P. Either way $z \in G_{j-1}$.

Since $x_j \leqslant x_{i+1}$ in P and $x_{i+1} \in E - U(B_2), x_j \notin B_2$. Thus, Case 1 does not hold at stage j. In particular, $z \notin B_2$. Since $z \leqslant y$ in P and $y \notin U[B_1], z \notin B_1$. Since $z \in G_i - B$, we note that Case 2 holds at stage j. Since $x_{i+1} \in E$, $z \| x_{i+1}$ in P, and $x_j \leqslant x_{i+1}$ in P,

$$|U[z] \cap E| < |U[x_j] \cap E|,$$

which confirms the claim that z is preferred to x_j at stage j, and thus proves (a).

Proof of (b): Suppose that L does not satisfy (b). Choose the least i such that $x_{i+1} \in B_1$ but there exists $y \in P - U[B_1]$ with $x_{i+1} \| y$ in P and $x_{i+1} < y$ in L. Then $y \in P^i$. Choose $z \in \mathrm{MIN}(P^i)$ with $z \leqslant y$ in P. Clearly $z \in P - U[B_1], x_{i+1} \| z$ in P, and $x_{i+1} < z$ in L.

Since $x_{i+1} \in B_1$, Case 3 holds at stage $i + 1$. Thus $z \notin G_i$. Thus $x_{i+1} :> x_i$ in P, but $z \| x_i$ in P. Since $x_{i+1} \in B_1$, $x_{i+1} \notin \mathrm{MIN}(P)$, and B_1 is rooted in E, there exists an element $e \in E$ such that $x_{i+1} :> e$ in P. If $x_i \neq e$, then $e \| x_i$ in P, $e \in E \cap D(B_1)$, and $x_i \in P - U[B_1]$, but $e < x_i$ in L, which violates (a). On the other hand, if $x_i = e$, then $x_i \| z$ in P, $x_i \in E \cap D(B_1)$, and $z \in P - U[B_1]$, but $x_i < x_{i+1} < z$ in L, which again violates (a). This contradiction completes the proof of (b).

Proof of (c): Suppose that L does not satisfy (c). Choose the least i such that $x_{i+1} \in B_1$ but there exists $y \in B_1$ with $y < x_{i+1}$ in N but $x_{i+1} < y$ in L. Then Case 3 holds at stage $i+1$. By (b) $y \in \mathrm{MIN}(P^i)$. Since we preferred x_{i+1} to y at stage $i+1$, $y \notin G_i$. Thus, $x_i < x_{i+1}$ in P. Since $x_{i+1} \notin \mathrm{MIN}(P)$, there exists $e \in E$ such that $x_{i+1} :> e$ in P. By (a) we conclude that $x_i = e$. By (*), $x_i < \hat{b}$ in P. Using (b) we see that $\hat{b} :> e$, and thus by (a), $\hat{b} \in G_i$. It follows that $x_{i+1} = \hat{b}$, which contradicts $y < x_{i+1}$ in N and completes the proof of (c).

Proof of (d): Suppose that $u \in B_2 - \mathrm{MIN}(P)$. Let $B_u = \{x \in P : u :> x \text{ in } P\}$. Then B_u is a nonempty antichain. Let k be the largest integer such that $x_k \in B_u$. We claim that if $x_k < y < u$ in L, then $y \in U[B_2]$. In particular, $y \notin D(B)$. Let $y = x_i$ and $x_j = \min(I_i \cap \{x_{k+1}, x_{k+2}, \ldots, x_i\}$. Then $u \in G_{j-1}$. Since $u \in B_2$, Case 1 holds at stage j. Then $x_j \in B_2$ and $y = x_i \in U[B_2]$. Thus x_k witnesses that (d) holds for u. This completes the proof of (d) and Lemma 10. □

LEMMA 11. *Let E and F be disjoint chains in an ordered set P. Suppose further that $E = \emptyset$ or E is maximal in $E \cup F$. Let Be be an antichain rooted in E. Then there exists a greedy linear extension $M = M(E, F, B)$ of P such that:*

(a) *$E/(P - U[B]) \cup F$ in M;*
(b) *B/F in M;*
(c) *$E - D(F)/P$ in M; and*
(d) *For all $a, b \in MIN(P)$, if*
  (i) *$a \in B$ if and only if $b \in B$;*
  (ii) *$U(a) \cap F = U(b) \cap F$; and*
  (iii) *$U(a) \subsetneq U(b)$;*
  then $a < b$ in M.

Proof. We shall construct M according to our general method. Thus we must provide a scheme for choosing x_{i+1} from G_i. Below we give four preference tests T_1, T_2, T_3, and T_4 for choosing $x_{i+1} \in G_i$ so that if $y \in G_i$ and y is preferred to x_{i+1} by T_j, then there exists $k < j$ such that x_{i+1} is preferred to y by T_k.

T_1: Let $S_i = \{b \in B \cap G_i : b \parallel f \text{ in } P \text{ for some } f \in F^i\}$. Prefer y to z if $y \notin S_i$ and $z \in S_i$.

T_2: Prefer y to z if $|\{e \in E : y \leqslant e \text{ in } P\}| < |\{e \in E : z \leqslant e \text{ in } P\}|$.

T_3: Prefer y to z if $|\{f \in F : f \leqslant y \text{ in } P\}| > |\{f \in F : f \leqslant z \text{ in } P\}|$.

T_4: Prefer y to z if $U(y) \subsetneq U(z)$.

We now show that the greedy linear extension M constructed using this preference scheme satisfies (a), (b), (c), and (d).

The following claim will be used in the verification of (a), (b), and (c).

CLAIM: *If $x_n \in E$ and $x_k \in I_n \cap F$, then there exists $e \in E$ such that $e \parallel x_k$ in P and $e < x_k$ in M.*

Proof. Let l be the least integer such that $k < l \leqslant n$ and $x_l \in E$. Then $x_l \in I_n$ and thus $x_k < x_l$ in P. Since $E \neq \emptyset$, E is maximal in $E \cup F$. Thus x_l is not the least element of E. Let $e \in E$ be the unique element immediately below x_l in E. By the choice of l, $e \leqslant x_k$ in M. Since $E \cap F = \emptyset$, $e \neq x_k$ and thus $e < x_k$ in M. If $e < x_k$ in P, then $E \cup$

$\{x_k\}$ is a chain, contradicting the hypothesis that E is maximal in $E \cup F$. The contradiction requires $e \parallel x_k$ in P. This concludes the proof of the claim.

The reader should note that once (a) has been established, then we may assume thereafter that the hypothesis to this claim is never satisfied. This remark follows from the observation that the conclusion of the claim contradicts E/F in M and thus contradicts (a).

Proof of (a): Suppose that (*a*) is false and choose the least i such that $x_{i+1} \in E$ but there exists $y \in (P - U[B]) \cup F$ with $x_{i+1} \parallel y$ in P and $x_{i+1} < y$ in M. Then $y \in P^i$. If $y \in F$ we may assume without loss of generality that $y = \min(F^i)$. Choose $z \in \mathrm{MIN}(P^i)$ with $z \leqslant y$ in P. Then choose the least integer j such that $x_j \in I_{i+1}$ and $x_j \parallel z$ in P. It follows that $z \in G_{j-1}$. Since $x_j \leqslant x_{i+1}$ in P, $z \parallel x_{i+1}$, and $x_{i+1} \in E$, z is preferred to x_j by T_2. Thus, x_j must be preferred to z by T_1. Thus $z \in S_{j-1}$, i.e., $z \in B$ and $z \parallel f$ in P, where $f = \min(F^{j-1})$.

Since $z \in B$, $y \in (P - U[B]) \cup F$, and $z \leqslant y$ in P, $y \in F$ and, thus, $f < y$ in P. Since $y = \min(F^i)$, we conclude that $f = x_k$ for some k such that $j \leqslant k < i + 1$. Thus $x_k \in I_{i+1}$. Now by the claim, with $n = i + 1$, there exists $e \in E$ such that $e \parallel x_k$ and $e < x_k$ in M. This contradicts the choice of i, and thus completes the proof of (a).

Proof of (b): Suppose that (b) is false and choose the least i such that $x_{i+1} \in B$, but there exists $f \in F$ with $x_{i+1} \parallel f$ and $x_{i+1} < f$ in M. Thus $f \in P^i$. Without loss of generality, assume that $f = \min(F^i)$. Clearly $x_{i+1} \in S_i$. Thus by T_1, $G_i \subset B$ and $b \parallel f$ in P for every $b \in G_i$. Choose $z \in \mathrm{MIN}(P^i)$ such that $z \leqslant f$ in P. Clearly $z \notin S_i$. Thus, by T_1, $z \notin G_i$. So $i > 0$, $z \parallel x_i$ in P, and $x_{i+1} :> x_i$ in P. Since $x_{i+1} \notin \mathrm{MIN}(P)$ and B is rooted in E, there exists $e \in E$ such that $x_{i+1} :> e$. Since $E/P - U[B]$ in M and $x_i \in P - U[B]$, $e = x_i$.

Now let j be the least integer such that $x_j \in I_i$ and $x_j \parallel z$ in P. Then $z \in G_{j-1}$ and z is preferred to x_j by T_2. Thus x_j is preferred to z by T_1. In particular, $z \in S_{j-1}$ and, thus, $z \in B$ and $z \parallel f'$ in P, where $f' = \min(F^{j-1})$. Then $f' < f$. So $f' = x_k$ for some k such that $j \leqslant k < i$. Thus $x_k \in I_i$ and we observe that the hypothesis to our claim is satisfied. By our previous remarks, this constitutes a contradiction of (a). Thus (b) holds.

Proof of (c): Suppose that (c) is false and choose the least i so that $x_{i+1} \in E - D(F)$, but there exists $y \in P$ with $x_{i+1} \parallel y$ in P and $x_{i+1} < y$ in M. Choose $z \in \mathrm{MIN}(P^i)$ with $z \leqslant y$ in P. Then $x_{i+1} \parallel z$ in P and $x_{i+1} < z$ in M. In view of (a), we may assume that $z \in U[B] - F$.

Now choose the least integer k so that $x_k \in I_{i+1}$ and $x_k \parallel z$ in P. It follows that $z \in G_{k-1}$.

Since z is preferred to x_k by T_2, it must be the case that z loses to x_k on T_1. This requires that $z \in B \cap G_{k-1}$ and that there exists a point $f \in F^{k-1}$ with $z \parallel f$ in P. Since $x_{i+1} \in E - D(F)$, we cannot have $x_{i+1} < f$ in P. If $f = x_l$ for some l with $k \leqslant l \leqslant i$, then the hypothesis of the claim is satisfied. We conclude that $f \neq x_l$ for all l with $k \leqslant l \leqslant i$. However, this in turn implies that $x_{i+1} < f$ in L and $x_{i+1} \parallel f$, which again contradicts (a). This completes the proof of (c).

Proof of (d): Suppose that a, $b \in \mathrm{MIN}(P)$, (i) $a \in B$ iff $b \in B$, (ii) $U(a) \cap F = U(b) \cap F$ and (iii) $U(a) \nsubseteq U(b)$. Let i be the least integer such that $x_{i+1} = a$ or $x_{i+1} = b$. Then

$G_i = \mathrm{MIN}(P^i)$. Using (i) and (ii) we see that $a \in S_i$ iff $b \in S_i$. Thus, a and b tie on T_1. By (iii) b cannot be preferred to a by T_2. So suppose they tie on T_2. Then they also tie on T_3. But a is preferred to b by T_4. Then $a < b$ in M. This completes the proof of (d) and Lemma 11. □

We close this section with the remark that Lemma 7 is easily seen to be a special case of Lemma 10 as well as Lemma 11. To see this, let C be any chain in P. Then C/P in $L(C, \emptyset, \emptyset, \emptyset)$ and C/P in $M(C, \emptyset, \emptyset)$.

6. Proofs of the Principal Theorems

In this section, we present the proofs of Theorem 1 and 3. Actually, Theorem 1 will be obtained as an easy corollary to the following technical result.

LEMMA 12. *Let P be an ordered set and let A denote the set of minimal elements of P. Suppose that $P - A \neq \emptyset$ and*

$$P - A = C_1 \cup C_2 \cup \cdots \cup C_t \cup A_1 \cup A_2 \cup \cdots \cup A_s$$

is a partition where each C_i is a chain and each A_j is a nontrivial antichain. Then

$$\dim_g(P) \leqslant 2t + |A_1| + |A_2| + \cdots + |A_s|.$$

Proof. We proceed by induction on $|P|$. In view of Lemma 8, it is clear that we may assume that P is indecomposable with respect to lexicographic sums. In particular, we may assume that there do not exist distinct points $a_1, a_2 \in A$ with $U(a_1) = U(a_2)$.

We now proceed to form a family Σ of greedy linear extensions of P by the following rules. If $t > 0$, then for each C_i we put two greedy linear extensions of P in Σ:

$$L_{2i-1} = M(C_i, \emptyset, A), \qquad L_{2i} = M(\emptyset, C_i, A).$$

If $s > 0$, then for each $j = 1, 2, \ldots, s$, we label the points in A_j as $\{x_j^k : 1 \leqslant k \leqslant |A_j|\}$ and then define the greedy linear extension L_j^k of P by the rule:

$$L_j^k = M(\{x_j^k\}, \{x_j^{k+1}\}, A).$$

(In this notation, the superscripts are to be interpreted cyclically.)

We claim that Σ is a realizer of P. To verify this claim, we consider an arbitrary pair $x, y \in P$ with $x \parallel y$ in P. We show that there exists $\sigma \in \Sigma$ so that $x < y$ in σ. Once this is accomplished, we may conclude by symmetry that there also exists $\tau \in \Sigma$ for which $y < x$ in τ.

Case 1: $y \in P - A$.
Suppose first that $y \in C_i$ for some i. Then $x < y$ in L_{2i-1}. Now suppose $y \in A_j$ for some j. Choose k so that $y = x_j^k$. Then $x < y$ in L_j^k.

Case 2: $y \in A, x \in P - A$.
Suppose first that $x \in C_i$ for some i. Then $x < y$ in L_{2i}. Now suppose $x \in A_j$ for some j. Choose k so that $x = x_j^k$. Then $x < y$ in L_j^{k-1}.

Case 3: $y \in A, x \in A$.
If there exists a point $z \in P - A$ with $z \in U(x) - U(y)$, then there exists some $\sigma \in \Sigma$ with $z < y$ in σ. Hence, $x < z < y$ in σ. So we may assume that $U(x) \subset U(y)$. Since P is indecomposable, we know that $U(x) \subsetneq U(y)$.

Again, since P is indecomposable, we also know that there exists a point $w \in P - A$ with $w > x$ in P. Then $w > y$ in P. Suppose first that $w \in C_i$ for some i. Then $x < y$ in $L_{2i-1} = M(C_i, \emptyset, A)$. On the other hand, suppose $w \in A_j$ for some j. Choose k so that $w = x_j^k$. Then $U(y) \cap \{x_j^k\} = U(x) \cap \{x_j^k\}$ so $x < y$ in M_j^{k-1}. This completes the proof of Lemma 12. □

COROLLARY (Theorem 1). *Let A be an antichain in an ordered set P with $|P - A| \geq 2$. Then $\dim_g(P) \leq |P - A|$.*

Proof. We proceed by induction on $|P|$. In view of Lemma 8, we may assume that P is indecomposable with respect to lexicographic sum. Also, we may assume A is a maximal antichain in P.

Let $|P - A| = n$. If $n = 2$, then $\dim(P) \leq 2$ and $\dim_g(P) = \dim(P) \leq 2$. So we may assume $n \geq 3$.

Now suppose P contains a maximal initial chain C so that $C \cap (P - A) \neq \emptyset$. Let $Q = P - C$ and $B = A - C$. Then $|Q - B| \leq n - 1$. If $|Q - B| \leq 2$, then $\dim_g(Q) \leq 2$ so $\dim_g(P) \leq 1 + \dim_g(Q) \leq 3 \leq n$. If $2 < |Q - B| \leq n - 1$, then $\dim_g(P) \leq 1 + \dim_g(Q) \leq n$.

So we may therefore assume that $C \cap (P - A) = \emptyset$ for every maximal initial chain C in P. Thus, A is the set of minimal points in P, and every maximal initial is a singleton.

Suppose that n is even, say $n = 2m$. Partition $P - A$ into m subsets containing exactly two elements of $P - A$. If we denote this partition as $P - A = C_1 \cup C_2 \cup \cdots \cup C_t \cup A_1 \cup A_2 \cup \cdots \cup A_s$, then $2t + |A_1| + |A_2| + \cdots + |A_s| = n$ so that $\dim_g(P) \leq n$ by Lemma 12.

So we may assume that n is odd.

If $P - A$ contains a three-element chain C_1, then we may partition $P - A - C_1$ into two-element subsets and use Lemma 12 to conclude that $\dim_g(P) \leq n - 1$. So we may assume that $P - A$ does not contain a three-element chain.

If $P - A$ contains a three-element antichain A_1, then we may partition $P - A - A_1$ into two-element subsets and use Lemma 12 to conclude that $\dim_g(P) \leq n$. So we may assume $P - A$ does not contain a three-element antichain.

Now every ordered set containing five or more points contains a three-element chain or a three-element antichain, so it remains only to consider the case $n = 3$. Furthermore, since P is indecomposable, it is easy to see that $P - A$ must be the union of two chains $C_1 = \{x < y\}$ and $C_2 = \{z\}$ with $x \parallel z$ and $y \parallel z$ in P. In this case, we observe that $\{M(C_1, C_2, A), M(C_2, C_1, A), M(\{z\}, \{y\}, A)\}$ is a greedy realizer of P. □

As previously noted, the family $\{P_n;\ n \geq 3\}$ illustrated in Figure 1 shows that the inequality in Theorem 1 is best possible even in the class of three-dimensional ordered sets. Of course, since $\dim_g(P_n) = \text{width}(P_n) = n$, we observe that the inequality $\dim_g(P) \leq |P|/2$ when $|P| \geq 4$ is also best possible even in the class of three-dimensional ordered sets.

We now proceed to prove Theorem 3. In order to facilitate the exposition, we prove the five inequalities in a different order. In each case, we proceed by induction on $|P|$, so we will assume that P is indecomposable with respect to lexicographic sums. We also assume A is maximal.

PART I. *If A is the set of maximal elements of P, $P - A \neq \emptyset$, and $n = width(P - A)$, then $dim_g(P) \leqslant n + 1$.*

Proof. Since P is indecomposable, $A \cap \mathrm{MIN}(P) = \emptyset$. Partition $P - A$ into chains $C_1, C_2, \ldots, C_n$. Let $A_1 = \{a \in A : a :> c$ in P for some $c \in C_1\}$. Without loss of generality we may assume that $A_1 \neq \emptyset$ and C_1 is maximal in $P - A$. Observe that A_1 is rooted in C_1. Let $M = M(C_1, \emptyset, \emptyset)$. Then C_1/P in M. Let N_1 be the dual of M restricted to A_1. Notice that the N_1-least element a_1 of A_1 satisfies $D(a) \cap C_1 \subset D(a_1) \cap C_1$, for every $a \in A_1$. Let $L_1 = L(C_1, A_1, A - A_1, N_1)$. Let $A_j = \{a \in A - A_1 :$ there exists $c \in C_j$ so that $a :> c$ in P and there does not exist $y \in D(A)$ such that $c < y < a$ in $L\}$. By (d) of Lemma 10, $\{A_2, A_3, \ldots, A_n\}$ is a partition of $A - A_1$. Clearly C_j/A_j in L_1, for $j = 2, 3, \ldots, n$.

For each $j = 2, 3, \ldots, n$, let N_j be the dual of the restriction of L_1 to A_j. Then again the N_j-least element a_j of A_j satisfies $D(a) \cap C_j \subset D(a_j) \cap C_j$ for every $a \in A_j$. Let $L_j = L(C_j, A_j, A - A_j, N_j)$.

We claim that $\Sigma = \{M, L_1, \ldots, L_n\}$ is a greedy realizer of P. To verify this, we must show that if $x \parallel y$ in P, then there exist $\sigma, \tau \in \Sigma$ such that $x < y$ in σ and $y < x$ in τ.

Case 1. $x, y \in P - A$.
Choose j and k such that $y \in C_j$ and $x \in C_k$. Then $x < y$ in L_j and $y < x$ in L_k.

Case 2. $x \in P - A$ and $y \in A_1$.
Then $x < y$ in L_1. If $x \in C_1$, then $y < x$ in M. Otherwise $x \in C_k$ for some $k \geqslant 2$ and $y < x$ in L_k.

Case 3. $x \in P - A$ and $y \in A - A_1$.
If $x \in C_j$ and $y \in A_j$ for some $j \geqslant 2$, then $y < x$ in L_1 and $x < y$ in L_j. Otherwise $x \in C_k$ and $y \in A_j$ for k and j such that $k \neq j$ and $j \geqslant 2$. Then $y < x$ in L_k and $x < y$ in L_j.

Case 4. $x, y \in A_1$.
Then by the choice of N_1, $x < y$ in M iff $y < x$ in L_1.

Case 5. $x, y \in A_j$ for some $j \geqslant 2$.
Then by the choice of N_j, $x < y$ in L_j iff $y < x$ in L_1.

Case 6. $x \in A_j$ and $y \in A_k$ with $j \neq k$.
Then $y < x$ in L_j and $x < y$ in L_k.

This completes the proof of Part I. □

PART II. *If A is the set of minimal elements of P and $width(P - A) = n \geqslant 2$ then $dim_g(P) \leqslant 2n - 1$.*

Proof. Partition $P - A$ into chains $C_1, C_2, \ldots, C_n$.

Let $\bar{C}_1, \bar{C}_2, \ldots, \bar{C}_n$ be maximal chains in P such that $C_i \subset \bar{C}_i$ for all $i \leqslant n$. Let $c_i = \min \bar{C}_i$. Since A is a maximal antichain, each $c_i \in A$. For $i = 1, 2, \ldots, n-1$, let $M_i = M(\bar{C}_n, C_i - \bar{C}_n, A)$ and $M_i' = M(\bar{C}_i, \emptyset, \emptyset)$. Let $M_n = M(\emptyset, C_n, A)$.

We claim that $\Sigma = \{M_i : 1 \leqslant i \leqslant n\} \cup \{M_i' : 1 \leqslant i \leqslant n-1\}$ is a greedy realizer of P. To verify this claim we consider any $x \parallel y$ in P.

Case 1. $x, y \in P - A$.
Suppose that $x \in \bar{C}_i$, for some $i < n$. Then $y < x$ in M_i'. If $x \in \bar{C}_n$ then there exists $j < n$ such that $y \in C_j - \bar{C}_n$; so $y < x$ in M_j. By symmetry we can also find $\sigma \in \Sigma$ such that $x < y$ in σ.

Case 2. $x, y \in A$.
Since P is indecomposable we may assume without loss of generality that there exists $z \in U(y) - U(x)$. Suppose that $z \in C_i - \bar{C}_n$ for some $i < n$. Then $y < z < x$ in M_i. Otherwise $z \in \bar{C}_n$ and $y < z < x$ in M_n.

Next we try to put y over x. If there exists $z \in U(x) - U(y)$ we are done by symmetry, so we may suppose that $U(x) \subsetneq U(y)$. By (d) of Lemma 11, $x < y$ in M_i' for each $i = 1, 2, \ldots, n-1$.

Case 3. $x \in P - A$ and $y \in A$.
Let $x \in C_i$. If $i < n$ and $x \in C_i - \bar{C}_n$, then $x < y$ in M_i. Otherwise $x \in \bar{C}_n$ and $x < y$ in M_n. By Case 2 there exists $\sigma \in \Sigma$ such that $y < c_i < x$ in σ.

This completes the proof of Part II. □

PART III. *If $A = \mathrm{MIN}(P)$ and $P - A$ is a chain, then $\dim_g(P) \leqslant 2$.*

Proof. From Theorem 2, we know $\dim(P) \leqslant 2$. Thus $\dim_g(P) \leqslant 2$. □

PART IV. *If A is an antichain in P and width$(P - A) = n \geqslant 2$, then* $\dim_g(P) \leqslant n^2 + n$.

Proof. Partition $P - A$ into the chains $C_1, C_2, \ldots, C_n$. For each $i = 1, 2, \ldots, n$, let $\bar{C}_i$ be a maximal chain in P with $C_i \subset \bar{C}_i$. Then let $\mathbf{A} = P - (\bar{C}_1 \cup \bar{C}_2 \cup \cdots \cup \bar{C}_n)$. For each $i = 1, 2, \ldots, n$ let $A_i = [\mathbf{A} \cap (\mathrm{MIN}(P)] \cup \{a \in A : a :> c$ in P for some $c \in \bar{C}_i\})$. Let $E_i = \bar{C}_i \cap D(A_i)$ and $F_i = \bar{C}_i - E_i$. Let $M_i = M(E_i, F_i, A_i)$ for each $i = 1, 2, \ldots, n$.

For each i, define a partial order Q_i on A_i by setting $a < b$ in Q_i iff $U(a) \subset U(b)$ and $D(b) \subset D(a)$. Note that Q_i is antisymmetric because P is indecomposable. If every element of A_i belongs to $\mathrm{MIN}(P)$, let N_i be an arbitrary linear extension of Q_i and let a_i be the N_i-least element of A_i. Otherwise, let x_i be the largest element of E_i. Choose a_i from A_i so that a_i is a minimal element in Q_i and $a_i :> x_i$ in P. Then let N_i be any linear extension of Q_i for which a_i is the least element in N_i. Let $L_i = L(\bar{C}_i, A_i, \mathbf{A} - A_i, N_i)$.

Consider integers i, j with $1 \leqslant i, j \leqslant n$ and $i \neq j$. Let $A_{ij} = \{a \in \mathbf{A} - A_i :$ there exists $c \in C_j - \bar{C}_i$ such that $a :> c$ in P and there does not exist $x \in D(\mathbf{A})$ such that $c < x < a$ in $L_i\}$. Observe that any element of $\mathbf{A}$ which is minimal in P or which covers a point of $\bar{C}_i$ belongs to A_i. By (d) of Lemma 10, for each $a \in \mathbf{A} - A_i$, there exists $x \in P - \bar{C}_i$ such that $a :> x$ in P and there is no point of $D(\mathbf{A})$ between x and a in L_i. Thus for each i, $\{A_i\} \cup \{A_{ij} : j \neq i\}$ is a partition of $\mathbf{A}$. For all $1 \leqslant i, j \leqslant n$ with $i \neq j$, let $M_{ij} = M(\bar{C}_j, \bar{C}_i - \bar{C}_j, A_{ij})$.

We claim that the n^2+n greedy linear extensions in $\Sigma=\{L_i : 1 \leqslant i \leqslant n\} \cup \{M_i : 1 \leqslant i \leqslant n\} \cup \{M_{ij} : 1 \leqslant i, j \leqslant n$ with $i \neq j\}$ form a greedy realizer for P. Consider an arbitrary pair $x \parallel y$ in P. We must demonstrate that there exist $\sigma, \tau \in \Sigma$ such that $x<y$ in σ and $y<x$ in τ.

Case 1. $x, y \in P - \mathbf{A}$.
Suppose $x \in \bar{C}_i$ and $y \in \bar{C}_j$. Then $x<y$ in M_{ij} and $y<x$ in M_{ji}.

Case 2. $x \in P - \mathbf{A}$ and $y \in \mathbf{A}$.
We first find a greedy linear extension in Σ which puts x over y. Let $x \in \bar{C}_j$. If $y \in A_i$ for some $i \neq j$ then $y<x$ in M_{ij} since $y \notin U[A_{ij}]$. Otherwise $y \in A_{ij}$ for all $i \neq j$. Then using (d) of Lemma 10, $y<x$ in L_i for all $i \neq j$.

Now we show that there is a greedy linear extension in Σ which puts y over x. Again let $x \in \bar{C}_j$. If $x \in E_j$ and $y \in A_i$, for some i, then $x<y$ in L_i. So suppose that $x \in F_j$. If there exists k such that $x \in F_k$ and $y \in A_k$, then $x<y$ in M_k. Otherwise there exists $k \neq j$ with $y \in A_{jk}$ and $x \notin \bar{C}_k$. Then $x<y$ in M_{jk}.

Case 3. $x, y \in \mathbf{A}$.
If $x \in A_i - A_j$ and $y \in A_j - A_i$, then $x<y$ in L_j and $y<x$ in L_i. Otherwise $x, y \in A_i$ for some i. First suppose that $x<y$ in Q_i. Then $x<y$ in L_i. Also $U(x) \subsetneq U(y)$ or $D(y) \subsetneq D(x)$. In the first case there exists $u \in P - \mathbf{A}$ such that $y<u$ in P and $u \parallel x$ in P. By Case 2 there is a greedy linear extension $\sigma \in \Sigma$ such that $y<u<x$ in σ. A similar proof works if $D(y) \subsetneq D(x)$. Now suppose that x is incomparable to y in Q_i. Then arguing as above we can use Case 2 to show that there exist $\sigma, \tau \in \Sigma$ such $x<y$ in σ and $y<x$ in τ. □

PART V. *Let A be an antichain in P and suppose that $P-A$ is a chain in P. Then $dim_g(P) \leqslant 3$.*

Proof. Without loss of generality C is a maximal chain. Let $M_1=M(C, \emptyset, \emptyset)$. Then C/A in M_1. Also the M_1-largest element $\hat{a}$ of A satisfies $D(a) \cap C \subset D(\hat{a}) \cap C$ for all $a \in A$. Let N be the dual of the restriction of M_1 to A. Then let $L_1=L(C, A, \emptyset, N)$. Then $A/D(A)$ in L_1 and $x<y$ in M_1 iff $y<x$ in L_1 for all $x, y \in A$. Finally, let $M_2=M(D(A), C-D(A), A)$. Then $A/C-D(A)$ in M_2. This completes the proof of Part V and Theorem 3. □

7. The Examples

In this section, we construct examples of ordered sets to show that each of the inequalities in Theorem 3 is best possible. We follow the same format as in the preceding section.

PART I. *For each $n \geqslant 1$ there exists an ordered set P such that $width(P-MAX(P))=n$ and $dim_g(P) \geqslant n+1$.*

Proof. Let $P=P_{n+1}$ where $\{P_{n+1} : n \geqslant 1\}$ is the family of ordered sets illustrated in Figure 1. The set of maximal elements of P_{n+1} is $\{x, y\}$ and the width of $P_{n+1}-\{x, y\}$ is n. However, $\dim_g(P_{n+1})=n+1$. □

PART II. *For each $n \geqslant 2$, there exists an ordered set Q such that width$(Q - MIN(Q)) = n$ and $dim_g(Q) = 2n - 1$.*

Proof. If $n = 2$, then $Q = P_3^*$ provides the required example. For $n \geqslant 3$ let $Q = Q_n$ where Q_n is defined for $n \geqslant 3$ (and even $n = 2$) as follows. The minimal elements of Q are $\{a_1, a_2, \ldots, a_n\} \cup \{b_1, b_2, \ldots, b_n\}$. The remaining points in Q are $Q - \mathrm{MIN}(Q) = \{x_1, x_2, \ldots, x_n\} \cup \{y_1, y_2, \ldots, y_n\} \cup \{z_1, z_2, \ldots, z_n\}$. For each $i \leqslant n$, $z_i :> y_i :> x_i :> b_i$ in Q. In addition, for each $i, j \leqslant n$ with $i \neq J$, $z_i :> x_j$, $y_i :> b_j$, and $y_i :> a_j$. The ordered set Q_3 is depicted in Figure 3.

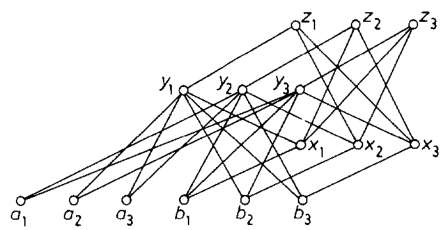

Fig. 3.

Suppose that $\Sigma = \{L_1, L_2, \ldots, L_t\}$ is a greedy realizer of Q. Then in particular Σ must meet the following requirements for all i, j, and k such that $1 \leqslant i, j, k \leqslant n$ and $j \neq k$.

Requirement U_i: There exists $L \in \Sigma$ such that $z_i < a_i$ in L.

Requirement $V_{j,k}$: There exists $L \in \Sigma$ such that $y_j < x_k$ in L.

Next we present a series of easy claims each of which will limit the number of these n^2 requirements that can be met by any one $L \in \Sigma$.

CLAIM 1. *There is at most one i such that L satisfies U_i.*

Proof. If $z_i < a_i$ in L then $a_j < y_i < z_i < a_i < y_j < z_j$ in L for any $j \neq i$. □

CLAIM 2. *If L satisfies U_i and $j \neq i$ then L does not satisfy $V_{j,k}$ for any $k \leqslant n$.*

Proof. If $z_i < a_i$ in L then $x_k < z_i < a_i < y_j$ in L for any $j \neq i$ and any k. □

CLAIM 3. *There exists at most one k for which there exists j such that L satisfies $V_{j,k}$.*

Proof. Let b_k be the L-largest element of $\{b_1, b_2, \ldots, b_n\}$. Then for each $l \leqslant n$ with $l \neq k$, $x_l :> b_l$ in L, since L is a greedy linear extension. Thus, $x_l < b_k < y_j$ in L, for any $l \neq k$ and any j. □

We now complete the proof of Part II by showing that t, the cardinality of Σ, must be at least $2n - 1$. By Claim 1 we may assume that requirement U_i is satisfied by L_i for $1 \leqslant i \leqslant n$. By Claims 2 and 3, at most n of the requirements V_{ij} are satisfied by $L_1, L_2, \ldots, L_n$. By Fact 3, at most $(t - n)(n - 1)$ of the remaining $n(n - 2)$ requirements V_{ij} are satisfied by $L_{n+1}, L_{n+2}, \ldots, L_t$. This requires $(t - n)(n - 1) \geqslant n(n - 2)$, and thus $t \geqslant 2n - 1$, which completes the proof of Part II. □

We note for the record that $\dim_g(Q_2) = 2$.

PART III. *There exists an ordered set Q such that $Q - MIN(Q)$ is a chain and $dim_g(Q) \geqslant 2$.*

Proof. Any nontrivial antichain will do for Q. □

PART IV. *For every $n \geqslant 2$, there exists an ordered set R and an antichain $A \subset R$ such that $width(R - A) = n$ and $dim_g(R) \geqslant n^2 + n$.*

Proof. This construction is considerably more complicated than the others and requires some preliminary development. For an integer m, let $\mathbf{m}$ denote the set $\{1, 2, \ldots, m\}$ while $\mathbf{m}^n$ denotes the Cartesian product of n copies of $\mathbf{m}$. The elements of $\mathbf{m}^n$ are functions from $\mathbf{n}$ to $\mathbf{m}$ so we can lexicographically order $\mathbf{m}^n$ by defining $f < g$ when $f(i) < g(i)$ for the least integer i such that $f(i) \neq g(i)$. We abbreviate this linear order by writing $f < g$ in LEX.

For integers n, m with $n \geqslant 2$ and $m \geqslant 1$, we define an ordered set $R(n, m)$ by the following rules:

(1) The point set of $R(n, m)$ is the union of disjoint sets $A \cup D \cup U$.

(2) A is a maximal antichain of $R(n, m)$.

(3) $D = D(A)$ and $U = U(A)$.

(4) $d < u$ in $R(n, m)$ for every $d \in D$ and $u \in U$.

(5) The points in D are labelled as $\{d(f, j) : f \in \mathbf{m}^n, j \in \mathbf{n}\}$. We set $d(f, j) < d(g, k)$ in $R(n, m)$ iff $f < g$ in LEX. Thus for each $j \in \mathbf{n}$ the elements in $D_j = \{d(f, j) : f \in \mathbf{m}^n\}$ form a chain.

(6) The points in U are labelled as $\{u(i, j) : i \in \mathbf{m}, j \in \mathbf{n}\}$. We set $u(i, j) < u(k, l)$ in $R(n, m)$ iff $i < k$ and $j = l$. Thus for each $j \in \mathbf{n}$ the points in $U_j = \{u(i, j) : i \in \mathbf{m}\}$ form a chain.

(7) The points in A are labelled as $\{a(f, j) : f \in \mathbf{m}^n \text{ and } j \in \mathbf{n}\}$. We set $d(f, j) < a(g, k)$ in $R(n, m)$ iff $f < g$ in LEX, or $f = g$ and $j = k$. Furthermore, $a(f, j) < u(i, k)$ in $R(n, m)$ iff $f(k) \leqslant i$.

We illustrate this definition by providing in Figure 4 a diagram of $R(2, 2)$. To simplify the diagram the coordinates of the various points are printed along the sides of the diagram.

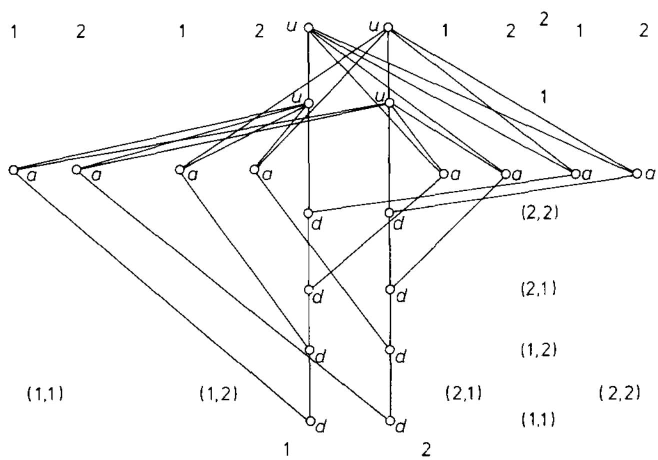

Fig. 4.

The following result is a special case of the 'Product Ramsey Theorem.'

THEOREM. *Let n, p, and r be positive integers. Then there exists an integer m_0 (depending on n, p, and r) so that if $m \geqslant m_0$ and Ψ: $\mathbf{m}^n \to \mathbf{r}$ is any function, then for each $j \in \mathbf{n}$ there exists a p-element subset $H_j \subset \mathbf{m}$ and there exists an integer $\alpha \in \mathbf{r}$ so that $\Psi(f) = \alpha$ for every $f \in H_1 \times H_2 \times \cdots \times H_n$.* □

For each $n \geqslant 2$, we apply the preceding theorem with

$$p = \binom{n^2 + n - 1}{n} + 2 \quad \text{and} \quad r = n^{n^2+n-1}$$

and let $m = m_0$ be the integer provided by the Product Ramsey Theorem. Then set $R = R(n, m)$; note that R actually depends only on n. To complete the proof, we show that $\dim_g(R) \geqslant n^2 + n$. To accomplish this we suppose to the contrary that $\dim_g(R) < n^2 + n$. It follows that there exists a greedy realizer $\Sigma = \{L_1, L_2, \dots, L_t\}$ of R where $t = n^2 + n - 1$.

Let $f \in \mathbf{m}^n$; in what follows, we refer to the n-element antichain $\{d(f, j) : 1 \leqslant j \leqslant n\}$ as the *f-level* of D. Similarly, we refer to the n-element antichain $\{u(f(j), j) : 1 \leqslant j \leqslant n\}$ as the *f-level* of U. The elements of the f-level of D always occur at the same height in R, but this is not the case with the points in the f-level of U.

For each $f \in \mathbf{m}^n$, we consider the points in the f-level of U, and we consider the orderings imposed on them by the linear orders in Σ. In particular, we record which element of the f-level of U is the lowest in L. This is accomplished by a sequence $(k_1, k_2, \dots, k_t)$ where the least element in the restriction of L_i to the f-level of U is $u(f(k_i), k_i)$. With this convention, we have defined a function Ψ mapping $\mathbf{m}^n$ to $\mathbf{n}^t$. We conclude that there is some sequence $\alpha = (k_1, k_2, \dots, k_t)$ and n subsets $H_1, H_2, \dots, H_n$ of $\mathbf{m}$ with

$$|H_j| = p = \binom{t}{n} + 2 \quad \text{for each } j \in \mathbf{n},$$

so that $\Psi(f) = \alpha$ for every $f \in H_1 \times H_2 \times \cdots \times H_n$.

For each $j \in \mathbf{n}$, we let $V_j = \{u(i, j) : i \in H_j\}$ and relabel the points of V_j as $\{v(i, j) : i \in q\}$, where $v(i, j) < v(i', j)$ in R iff $i < i'$. We also let $V = V_1 \cup V_2 \cup \cdots \cup V_n$. The basic purpose behind this Ramsey construction is stated in the next claim which follows immediately from the fact that each $f \in H_1 \times H_2 \times \cdots \times H_n$ is mapped by Ψ to the same $\alpha \in \mathbf{n}^t$.

CLAIM 0. *For each $L \in \Sigma$ there exists $k \in \mathbf{n}$ such that V/V_k in L.* □

For each $i \in \mathbf{p-1}$, let $f_i \in \mathbf{m}^n$ be such that $f_i(j) = k$ where $u(k, j) = v(i + 1, j)$. Thus, for all j, $k \in \mathbf{n}$, $a(f_i, j) < v(l, k)$ in R iff $i < l$. Since Σ is a realizer of R, Σ must meet the following requirements for each i, $i' \in \mathbf{p-1}$ with $i < i'$ and j, j', $k \in \mathbf{n}$.

Requirement $S(i, j, k)$: There exists $L \in \Sigma$ such that $v(i, k) < a(f_i, j)$ in L.

Requirement $T(i, i', j, j')$: There exists $L \in \Sigma$ so that $a(f_{i'}, j') < a(f_i, j)$ in L.

To complete the proof, we show that since $|\Sigma| = n^2 + n - 1$, Σ cannot satisfy all of these requirements. We begin with a series of claims which restrict the number of requirements that can be met by a single linear extension $L \in \Sigma$.

CLAIM 1. *For each* $i \in \mathbf{p} - \mathbf{1}$ *and each* $j \in \mathbf{n}$, *there is at most one* $k \in \mathbf{n}$ *such that* L *satisfies requirement* $S(i, j, k)$.

Proof. Suppose that L satisfies $S(i, j, k)$. Then $v(i, k) < a(f_i, j) < v(p, l)$ in L, for any $l \in \mathbf{n}$. Thus by Claim 0, V/V_k in L. Now suppose that $l \in \mathbf{n}$ and $l \neq k$. Then $a(f_i, j) < v(p, k) < v(i, l)$ in L, so L does not satisfy $S(i, j, l)$. □

CLAIM 2. *Suppose that* $i, i' \in \mathbf{p} - \mathbf{1}$, $i < i'$, *and* $j, j', k \in \mathbf{n}$. *If* L *satisfies* $S(i, j, k)$ *or* $T(i, i', j, j')$, *then* $d(f_i, j)$ *is the* L*-largest element of the* f_i*-level of* D.

Proof. Since L is greedy, if $d(f_i, j)$ is not the L-largest element of the f_i-level of D, then $a(f_i, j) :> d(f_i, j)$ in L. Thus $a(f_i, j) < d(f_i, m)$ in L, where $d(f_i, m)$ is the L-largest element of the f_i-level of D. Since $d(f_i, m) < a(f_{i'}, j')$ in L and $d(f_i, m) < v(i, k)$ in L, L satisfies neither $S(i, j, k)$ nor $T(i, i', j, j')$. □

CLAIM 3. *Suppose that* $i, i' \in \mathbf{p} - \mathbf{1}$ *with* $i < i'$ *and* $j, j' \in \mathbf{n}$. *If* L *satisfies* $T(i, i', j, j')$, *then* L *does not satisfy* $S(i', j', k)$ *for any* $k \in \mathbf{n}$.

Proof. Since L satisfies $T(i, i', j, j')$ we have $a(f_{i'}, j') < a(f_i, j) < v(i', k)$ in L for all $k \in \mathbf{n}$.

We are now in a position to conclude the proof of Part IV. For each $i \in \mathbf{p} - \mathbf{1}$ and $j \in \mathbf{n}$ let $\Sigma_{i,j} = \{L \in \Sigma : d(f_i, j)$ is the L-largest element of the f_i-level of $D\}$. By Claim 1, for each $i \in \mathbf{p} - \mathbf{1}$ and $j \in \mathbf{n}$, it takes n distinct linear extensions to satisfy $S(i, j, k)$ for all $k \in \mathbf{n}$. Thus, by Claim 2, $|\Sigma_{i,j}| \geq n$. Clearly $\Sigma_{ij} \cap \Sigma_{i,j'} = \emptyset$ if $j \neq j'$. Since $t = n^2 + n - 1$, for each $i \in \mathbf{p} - \mathbf{1}$ there exists $j(i)$ such that $|\Sigma_{i, j(i)}| = n$. Let $\Sigma_{i, j(i)} = \Sigma_i$. Note that if Σ meets requirements $S(i, j(i), k)$ for some $k \in \mathbf{n}$, then some $L \in \Sigma_i$ satisfies $S(i, j(i), k)$. Since $p = \binom{t}{n} + 2$, there exist $i, i' \in \mathbf{p} - \mathbf{1}$ with $i < i'$ such that $\Sigma_i = \Sigma_{i'}$. By Claim 2, if $L \in \Sigma$ satisfies $T(i, i', j(i), j(i'))$, then $L \in \Sigma_i = \Sigma_{i'}$. By Claim 3, L does not satisfy $S(i', j(i'), k)$ for any $k \in \mathbf{n}$. By Claim 1 the remaining $n - 1$ linear extensions in $\Sigma_{i'}$ cannot satisfy all of the requirements $S(i', j(i'), k)$ for $k \in \mathbf{n}$. This contradicts the assumption that Σ is a realizer of R and completes the proof. □

PART V. *There exists an ordered set* R *and an antichain* $A \subset R$ *such that* $R - A$ *is a chain and* $\dim_g(R) = 3$.

Proof. The ordered set shown in Figure 5 is the required example. It is a well known example of a three-dimensional ordered set. □

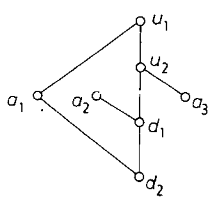

Fig. 5.

References

1. V. Bouchitté, M. Habib, and R. Jégou (1985) On the greedy dimension of a partial order, *Order* **1**, 219–224.
2. R. Dilworth (1950) A decomposition theorem for partially ordered sets, *Ann. Math.* **51**, 161–166.
3. M. El-Zahar and I. Rival (1985) Greedy linear extensions to minimize jumps, *Discrete Appl. Math.*, to appear.
4. R. Graham, B. Rothschild, and J. Spencer (1984) *Ramsey Theory*, Wiley-Interscience.
5. T. Hiraguchi (1951) On the dimension of partially ordered sets, *Sci. Rep. Kanazawa Univ.* **1**, 77–94.
6. D. Kelly and W. Trotter (1982) Dimension theory for ordered sets, in *Ordered Sets* (ed. I. Rival), D. Reidel, Dordrecht, pp. 171–211.
7. R. Kimble (1973) Extremal problems in dimension theory for partially ordered sets, PhD Thesis, MIT.
8. W. Pulleyblank (198x) On minimizing setups in precedence constrained scheduling, *Discrete Appl. Math.*, to appear.
9. I. Rabinovitch (1978) An upper bound on the dimension of interval orders, *J. Comb. Theory* **A25**, 68–71.
10. W. Trotter (1975) Inequalities in dimension theory for posets, *Proc. Amer. Math. Soc.* **47**, 311–316.
11. W. Trotter (1974) Irreducible posets with arbitrarily large height exist, *J. Comb. Theory* **A17**, 337–344.
12. W. Trotter, J. Moore, and D. Sumner (1976) The dimension of a comparability graph, *Proc. Amer. Math. Soc.* **60**, 35–38.