Super-Greedy Linear Extensions of Ordered Sets[a]

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INTRODUCTION AND NOTATION

In this paper, we continue the investigation initiated in [5] into the problem of representing an ordered set as the intersection of a family of linear extensions where restrictions are placed on the methods by which the linear extensions can be constructed. Our primary purpose is to introduce a dimension theoretic parameter and to develop a number of inequalities relating this parameter to other more familiar parameters of ordered sets. To understand our results requires some familiarity with the basic combinatorial properties of ordered sets, especially those involving the concept of dimension. The experienced reader should proceed immediately to the next section. The remainder of this section is for the benefit of those who are relatively new to this subject.

Throughout this paper, we consider finite (partially) ordered sets. When P is an ordered set, and x and y are distinct points in P, we write $x \mid y$ in P when x and y are *comparable*, that is, either $x > y$ in P or $y > x$ in P. We say x *covers* y and write $x :> y$ (also $y <: x$) in P if $x > y$ in P and there is no point z so that $x > z$ and $z > y$ in P. When x and y are *incomparable*, we write $x \| y$ in P. A *chain* in an ordered set P is a subset C with $x \mid y$ in P for every distinct pair $x, y \in C$. An *antichain* is a subset A with $x \| y$ in P for every distinct pair $x, y \in A$. A chain (antichain) is *maximal* if no other chain (antichain) contains it as a proper subset. When $Q \subset P$, an element $x \in Q$ is called a *minimal* (*maximal*) *element* of Q when there is no point $y \in Q$ satisfying $y < x$ ($y > x$) in Q. For a subset $Q \subset P$, we let $\min(Q)$ ($\max(Q)$) denote the set of *minimal* (*maximal*) elements of Q.

The *width* of an ordered set P, denoted width (P), is the maximum number of points in an antichain of P. A well-known theorem of R. P. Dilworth asserts that if width $(P) = t$, then there exists a partition $P = C_1 \cup C_2 \cup \cdots \cup C_t$, where each C_i is a chain.

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Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set. To indicate that L is a linear order on X with $x_1 < x_2 < \cdots < x_n$ in L, we will write L: $[x_1, x_2, \ldots, x_n]$. Now let P be an ordered set and let X denote the underlying point set of P. A linear order L on X is called a *linear extension* of P if $x < y$ in L whenever $x < y$ in P for every $x, y \in X$. A family $\{L_1, L_2, \ldots, L_t\}$ of linear extensions of an ordered set P is called a *realizer* of P if $P = L_1 \cap L_2 \cap \cdots \cap L_t$, that is, $x < y$ in P if and only if $x < y$ in L_i for $i = 1, 2, \ldots, t$. The *dimension* of P, denoted $\dim(P)$, is the least positive integer t for which P has a realizer containing t linear extensions.

When L is a linear extension of P and A and B are subsets of P, we write A/B in L when $a > b$ in L for every $a \in A$, $b \in B$ for which $a \| b$ in P. If C is a chain in P, then there exist linear extensions L_1 and L_2 of P so that C/P in L_1 and P/C in L_2. If P is an ordered set with width $(P) = t$, $P = C_1 \cup C_2 \cup \cdots \cup C_t$ is a partition, and L_i is a linear extension of P with C_i/P in L_i for each $i = 1, 2, \ldots, t$, then the family $\{L_1, L_2, \ldots, L_t\}$ is a realizer of P. Thus dimension is well defined and, in fact, $\dim(P) \le$ width (P) for every ordered set P.

For an ordered set P, we let P^d denote the dual of P, that is, $x < y$ in P^d if and only if $x > y$ in P. A linear order L is a linear extension of P if and only if L^d is a linear extension of P^d. It is obvious that $\dim(P) = \dim(P^d)$. We refer the reader to the survey articles [4] and [10] for additional background material on the concept of dimension and other combinatorial parameters for ordered sets.

ALGORITHMIC CONSTRUCTIONS OF LINEAR EXTENSIONS

In this paper, we investigate the problem of representing an ordered set P as the intersection of a family $\{L_1, L_2, \ldots, L_t\}$ of linear extensions of P with the additional restriction that each L_i must be constructed according to a prescribed set of rules. We begin with the following elementary scheme, which we call algorithm LIN.

> Set $M(0) = \min(P)$ and choose $x_1 \in M(0)$. Suppose $x_1, x_2, \ldots, x_i$ have been chosen for some i with $1 \le i < |P|$. Let $M(i) = \min(P - \{x_1, x_2, \ldots, x_i\})$. Choose $x_{i+1} \in M(i)$.

It is easy to see that every linear extension of P is obtained from algorithm LIN by a suitable sequence of choices of the points $x_{i+1} \in M(i)$ for $i = 0, 1, 2, \ldots, |P| - 1$. In [5] we studied a restricted class of linear extensions obtained by adding a tie-breaking condition for the selection of x_{i+1}. We call the following algorithm GREEDY.

> Set $G(0) = \min(P)$ and choose $x_1 \in G(0)$. Suppose $x_1, x_2, \ldots, x_i$ have been chosen for some i with $1 \le i < |P|$. Let $M(i) = \min(P - \{x_1, x_2, \ldots, x_i\})$. Then set $H(i) = \{x \in M_i\colon\ x_i <: x \text{ in } P\}$. If $H(i) \ne \varnothing$, set $G(i) = H(i)$. Else, set $G(i) = M(i)$. Choose $x_{i+1} \in G(i)$.

Linear extensions obtained from this algorithm are called *greedy* linear extensions. The original motivation for studying this class of linear extensions stemmed from their connection with the jump number problem about which we will have more to say in the section after the next one.

GREEDY DIMENSION

In [2], Bouchitte *et al.* observed that every ordered set has a realizer $\{L_1, L_2, \ldots, L_t\}$, where each L_i is a greedy linear extension of P. Such a family is called a *greedy realizer* of P. The existence of greedy realizers follows immediately from the following basic result [2], [5].

LEMMA 1: If C is a chain in an ordered set P, then there exists a greedy linear extension L so that C/P in L.

Examples are presented in [2] and [5] to show that there may not be a greedy linear extension of P with P/C in L. However, Lemma 1 and Dilworth's theorem are enough to show that every ordered set is the intersection of a family of greedy linear extensions. Accordingly, we may then define the *greedy dimension* of P, denoted $\dim_G(P)$, as the least integer t for which P has a greedy realizer consisting of t greedy linear extensions. Evidently, greedy dimension is well defined. In fact, $\dim(P) \leq \dim_G(P) \leq \text{width}(P)$ for every ordered set P.

In [5], the authors derived the inequalities given in the next two theorems. These theorems bound the greedy dimension of an ordered set in terms of the cardinality and width of the complement of a maximal antichain.

THEOREM 1: Let A be an antichain in an ordered set P. If $|P - A| \geq 2$, then $\dim_G(P) \leq |P - A|$.

THEOREM 2: Let A be an antichain in an ordered set P with width $(P - A) = n \geq 1$. Then the following inequalities hold.

(a) $\dim_G(P) \leq n^2 + n$ when $n \geq 2$, and $\dim_G(P) \leq 3$ when $n = 1$.
(b) If $A = \min(P)$, $\dim_G(P) \leq 2n - 1$ when $n \geq 2$, and $\dim_G(P) \leq 2$ when $n = 1$.
(c) If $A = \max(P)$, $\dim_G(P) \leq n + 1$.

Theorems 1 and 2 have analogues in ordinary dimension, and we refer the reader to [8] and [9] for details.

SUPER-GREEDY LINEAR EXTENSIONS

In this paper, we study a more restrictive tie-breaking scheme. We call the following algorithm SUPER-GREEDY.

Set SG (0) = $\min(P)$ and choose $x_1 \in$ SG (0). Suppose $x_1, x_2, \ldots, x_i$ have been chosen for some i with $1 \leq i < |P|$. Let $M(i) = \min(P - \{x_1, x_2, \ldots, x_i\})$. Then let $J(i) = \{j\colon 1 \leq j \leq i$ and there exists $x \in M(i)$ so that $x_j <: x$ in $P\}$. If $J(i) \neq \varnothing$, let k be the largest integer in $J(i)$ and set SG $(i) = \{x \in M(i)\colon x_k <: x$ in $P\}$. Else, set SG $(i) = M(i)$. Choose $x_{i+1} \in$ SG (i).

Linear extensions obtained from this algorithm are called *super-greedy* linear extensions. We illustrate this definition with the ordered set shown in FIGURE 1, which is also used as an example in [5]. For this ordered set, it is easy to see that

$$L_1\colon [a_1, a_2, \ldots, a_{n-1}, x, b_1, c_1, b_2, c_2, \ldots, b_{n-1}, c_{n-1}, y]$$

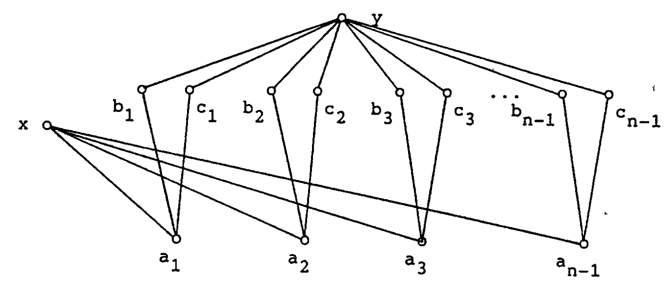

FIGURE 1.

is a linear extension but not a greedy linear extension of P;

$$L_2\colon [a_1, b_1, a_2, b_2, \ldots, a_{n-1}, x, b_{n-1}, c_1, c_2, \ldots, c_{n-1}, y]$$

is a greedy linear extension but not a super-greedy linear extension; and

$$L_3\colon [a_1, b_1, c_1, a_2, b_2, c_2, \ldots, a_{n-1}, x, b_{n-1}, c_{n-1}, y]$$

is a super-greedy linear extension.

The construction of a super-greedy linear extension by the SUPER-GREEDY algorithm involves backtracking through the list of elements previously chosen to find the first one covered by a minimal element among the remaining points. For this reason, it would have been just as natural to call a super-greedy linear extension a *backtracking* linear extension. We have chosen the super-greedy terminology to emphasize that the concept evolved from the notion of greedy linear extensions.

Greedy linear extensions were originally introduced as a method of attacking the following scheduling problem known as the *jump number problem* (see [7]). Suppose an ordered set P represents a set of calculations to be performed on a single processor where $x < y$ in P means the output of calculation x is required for calculation y. A *feasible* schedule for performing the calculations in P is then just a linear extension of P. Whenever $x <: y$ in both P and L, some savings in the loading cost for calculation y can be achieved by leaving the output of calculation x in the small internal storage of the processor. The jump number problem asks for a schedule L that maximizes the savings by maximizing the number of pairs of calculations (x, y) so that $x <: y$ in both L and P. The optimal schedule can always be realized by some greedy linear extension, although Pulleyblank [6] has shown that the jump number problem is NP-complete.

Now suppose we modify the problem by providing the processor with a large stack to which data can be pushed and from which data can be popped at very low cost. It is assumed that the transferal of data between the processor and other external storage locations is much less efficient. The *stack jump number problem* is to devise a schedule so that for as many calculations as possible, some input necessary for the calculation can be popped from the top of the stack rather than loaded from an external storage location. We claim that any super-greedy linear extension provides an optimal solution to the stack jump number problem; in fact, using any super-greedy linear extension, we can arrange to pop input off the top of the stack for every calculation.

We argue as follows. Suppose L: $[x_1, x_2, \ldots, x_n]$ is an arbitrary super-greedy linear extension of P and that $Y = \min(P)$. Denote the restriction of L to Y by $[y_1, y_2, \ldots, y_m]$. For each $j = 1, 2, \ldots, m$, we push the input for calculation y_j onto the stack so that the input for y_1 is on the top of the stack, the input for y_2 is in the second highest position, etc. Note that $x_1 = y_1$ so that in performing the first calculation, the input data for x_1 is popped from the stack. Now suppose that for some $i \geq 0$ we have just popped some necessary data from the stack. In the case that these data are the output of some calculation x_k and there exists an integer $\alpha > i$ so that α is the largest integer in $J(\alpha)$, we push a copy of these data back on top of the stack. All other data necessary for calculation x_{i+1} are retrieved and the calculation is made. The output from this calculation is sent to external storage. If there is an integer $\beta > i$ so that $i + 1$ is the largest integer in $J(\beta)$, then we also push a copy of the output data from calculation x_{i+1} onto the top of the stack.

It is easy to see that this scheme has the property that for each $i \geq 0$, we can always retrieve one item of the input data necessary for calculation x_{i+1} from the top of the stack, so that choosing L as a super-greedy linear extension of P was an optimal schedule for processing the calculations. However, we should emphasize that the savings gained from implementing this scheme comes from the efficient transferral of data and not from reducing the storage space required.

There is another important observation to be made concerning super-greedy linear extensions. The family of all super-greedy linear extensions of an ordered set P is exactly the same as the family of all linear extensions of P that arise from a depth-first search of P. Add an artificial least element to P and consider the ordered set as a digraph in which there is a directed edge from x to y whenever $x <: y$ in P. Starting at the artificial element, a depth-first search of P is performed and the elements of P are recorded in the order they are last visited in this search. The resulting linear order is a super-greedy linear extension of P in reverse order. We refer the reader to [1] for extensive background material on the concept of depth-first search and to the article [3] by Bouchitte *et al.* for a discussion of depth-first search in ordered sets. In their article, the term super-greedy is replaced by *depth-first greedy*.

In the remainder of this paper, we let SG (P) denote the set of all super-greedy linear extensions of an ordered set P. We will follow the convention of abbreviating super-greedy to SG.

SUPER-GREEDY DIMENSION

At the Oberwolfach meeting on ordered sets, the authors presented the results given in Theorems 1 and 2 for greedy dimension. In his lecture, Kierstead gave a short proof of Lemma 1. Subsequently, O. Pretzel proposed the definition of a super-greedy linear extension (which he good humoredly suggested should be called an imperial linear extension) and remarked that Kierstead's proof of Lemma 1 was still valid for super-greedy linear extensions. For the sake of completeness, we include this proof below. The authors gratefully acknowledge Pretzel's insightful remarks and suggestions concerning super-greedy linear extensions.

LEMMA 2: Let C be a chain in an ordered set P. Then there exists a super-greedy linear extension L of P so that C/P in L.

Proof: Choose a maximal chain C_1 containing C. Then construct a SG linear extension L of P by adding the following tie-breaking constraint to the SG algorithm.

T: Avoid elements of C_1 if possible.

By this we mean that when selecting x_{i+1} from SG (i), we always take a point from SG $(i) - C_1$ if this set is nonempty. We only take x_{i+1} from C_1 when SG $(i) \subset C_1$. We claim that the addition of this simple tie-breaking rule to the SG algorithm always yields an SG linear extension L with C_1/P in L. Since $C \subset C_1$, we would then conclude that C/P in L.

Suppose our claim that C_1/P in L is false. Choose the least nonnegative integer i so that $x_{i+1} \in C_1$ and there exists $y \in P$ such that $x_{i+1} \| y$ in P but $x_{i+1} < y$ in L. Choose $z \in M(i)$ so that $z \leq y$ in P. Since $x_{i+1} \in C_1$, we know that z does not belong to SG (i). It follows that $J(i)$ is nonempty. Let k be the largest integer in $J(i)$. Then $x_k <: x_{i+1}$ in P, but $x_k \| z$ in P. Since $k \leq i$, we know that $x_k \notin C_1$. Since $x_k < x_{i+1}$ in P and C_1 is a maximal chain, x_{i+1} is not the least element of C_1. Let x_m be the element of C_1 that is immediately under x_{i+1}. Since $x_k \neq x_m < x_{i+1}$ in P, the choice of k as the largest element in $J(i)$ requires $m < k$. However, since C_1 is maximal, we must have $x_k \| x_m$ in P. But this contradicts the choice of i. This completes the proof. □

Just as was the case with greedy dimension, the preceding lemma justifies the definition of the super-greedy dimension of an ordered set P, denoted $\dim_{SG}(P)$, as the least t for which P has a *super-greedy realizer*, which consists of t super-greedy linear extensions of P. It is clear that super-greedy dimension is well defined and that $\dim(P) \leq \dim_G(P) \leq \dim_{SG}(P)$ for every ordered set P.

In order for the reader to gain some experience in working with super-greedy dimension, we prove an elementary result providing conditions under which the super-greedy dimension of an ordered set is the same as its dimension. The conditions are identical to those known to hold for greedy dimension [2]. Some preliminary remarks are necessary. Recall that a subset D of an ordered set Q is called a *down set* when $x \in D$ and $y \leq x$ in Q imply $y \in D$ for all $x, y \in Q$. *Up sets* are defined analogously. The ordered set P consisting of all down sets of an ordered set Q ordered by set inclusion is a distributive lattice with $\dim(P) = \text{width}(Q)$. Furthermore, every distributive lattice arises from this process. For an element $x \in P$, we let $D(x)$ denote the down set $\{y \in P: y < x \text{ in } P\}$ and $D[x] = D(x) \cup \{x\}$.

THEOREM 3: Let P be an ordered set. Then

$$\dim(P) = \dim_G(P) = \dim_{SG}(P)$$

if either of the following conditions is satisfied:

(a) $\dim(P) \leq 2$.
(b) P is a distributive lattice.

Proof: Suppose first that $\dim(P) = 2$. Choose two linear extensions L_1 and L_2 so that $P = L_1 \cap L_2$. We show that L_1 and L_2 are SG. Suppose that one of them, say L_1, is not SG. Let L_1: $[x_1, x_2, \ldots, x_n]$. Choose a nonnegative integer i for which the selection of x_{i+1} violates the SG algorithm. Clearly, we must have $i > 0$. Let $M(i)$, $J(i)$, and SG (i) be defined by the SG algorithm. Then x_{i+1} belongs to $M(i)$ but not to SG (i). This implies that $J(i)$ is nonempty. Let k be the least integer in $J(i)$. Choose $z \in$ SG (i); then $x_k <: z$ in P. It follows that x_{i+1} is incomparable with both of x_k and z, but is between them in L_1. This would require that $z < x_{i+1} < x_k$ in L_2, which is impossible.

Now let P be a distributive lattice. Choose an ordered set Q so that P is isomorphic to the set of all down sets of Q ordered by set inclusion. We observe that if L: $[x_1, x_2, \ldots, x_n]$ is a super-greedy linear extension of P, then $x_1 = \emptyset$. Furthermore, for each integer i with $2 \leq i < n$, it is easy to see that $J(i) \neq \emptyset$, so that when a down set $x_{i+1} = D$ is chosen from SG (i), there is a down set $x_j = E \in J(i)$ with $E <: D$ in P. This implies that $E \subset D$ and $|D - E| = 1$.

Now let $t = \text{width}(Q) = \dim(P)$, and let $Q = C_1 \cup C_2 \cup \cdots \cup C_t$, where each C_i is a chain. For each $j = 1, 2, \ldots, t$, form an SG linear extension L_j of P by adding the following tie-breaking rule to the SG algorithm.

T_j: For each i with $2 \leq i < n$, choose x_{i+1} as a down set D from SG (i) so that if k is the largest integer in $J(i)$, $x_k = E$, and $D - E = \{x\}$, then $|\{c \in C_j : x \leq c$ in $P\}|$ is as small as possible.

We claim that the tie-breaking rule T_j yields a SG linear extension L_j so that $D > F$ in L_j whenever D and F are down sets of Q and $|D \cap C_j| > |E \cap F_j|$. Suppose the claim is false. Choose the least nonnegative integer i for which $x_{i+1} = D$, and there exists a down set F so that $|D \cap C_j| > |F \cap C_j|$, and $D < F$ in L_j. Without loss of generality, $F \in M(i)$. Furthermore, $D \neq \emptyset$, so that $i > 0$. Let c be the largest element of $D \cap C_j$. Then $c \notin F$. The minimality of i requires that $D = D[c]$.

Let k be the largest integer in $J(i)$ and let x_k be the down set E. Then it is clear that $E = D(c)$. Let α be the largest integer so that $k \leq \alpha < i + 1$ and $x_k \leq x_\alpha$ in P. Let $G = x_\alpha$. Then G is not a subset of F. Choose $f \in \min(F - G)$ and let $H = G \cup \{f\}$. The choice of α requires $D < H$ in L_j. However, the SG algorithm will always prefer H to D since it is obvious that f is not less than c in P. □

We comment that there is a third condition that is known to guarantee that $\dim(P) = \dim_G(P)$. The diagram of an ordered set P is said to be N-free if P does not contain points a, b, c, d so that $a <: b$, $a <: c$, $d <: c$, and d is not covered by b in P. If the diagram of P is N-free, then $\dim(P) = \dim_G(P)$ [2]. In fact, Zaguia [11] has extended this result to W-free ordered sets. However, as observed by Bouchitte *et al.* [3], the ordered set in Figure 2 is N-free, but it is easy to see that $\dim(P) = \dim_G(P) = 3$, while $\dim_{SG}(P) = 4$.

A full characterization of those ordered sets P satisfying $\dim(P) = \dim_{SG}(P)$ appears to be hopelessly difficult. We conjecture that the problem of deciding whether an ordered set satisfies $\dim(P) = \dim_{SG}(P)$ is NP-complete. Nevertheless, it would be of interest to determine nontrivial conditions that force $\dim(P) = \dim_{SG}(P)$. It would also be of interest to produce inequalities that bound $\dim_{SG}(P)$ in terms of various combinatorial properties of P. In the remainder of this paper, we discuss some results of this general flavor.

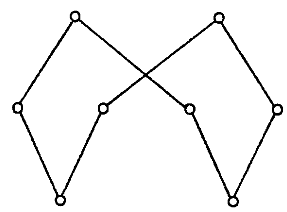

FIGURE 2.

INEQUALITIES FOR SUPER-GREEDY DIMENSION

Let P be an ordered set and let $\{Q_x : x \in P\}$ be a family of ordered sets indexed by the points in P. The *ordinal sum* of $\{Q_x : x \in P\}$, denoted $\sum \{Q_x : x \in P\}$, is the ordered set R whose points are the pairs (x, y) with $x \in P$ and $y \in Q_x$. The ordering in R is given by $(x, y) < (u, v)$ in R if and only if either $x < u$ in P or $x = u$ and $y < v$ in Q_x. The first part of the following result was given in [5].

THEOREM 4: Let P be an ordered set, and let $\{Q_x : x \in P\}$ be a family of ordered sets indexed by the points of P. Also let $t_1 = \max\{\dim_G(Q_x) : x \in P\}$ and $t_2 = \max\{\dim_{SG}(Q_x) : x \in P\}$. Then the following inequalities hold:

(a) $\max\{t_1, \dim(P)\} \leq \dim_G(\Sigma\ \{Q_x : x \in P\}) \leq \max\{t_1, \dim_G(P)\}$.
(b) $\max\{t_2, \dim(P)\} \leq \dim_{SG}(\Sigma\ \{Q_x : x \in P\}) \leq \max\{t_2, \dim_{SG}(P)\}$.

Proof of (b): The upper bound follows from the observation that if L is an SG linear extension of P and L_x is an SG linear extension of Q_x for each $x \in P$, then $\Sigma\ \{L_x : x \in L\}$ is an SG linear extension of $\Sigma\ \{Q_x : x \in P\}$. The lower bound follows from two observations. First, for each $x \in P$, the restriction of an SG linear extension of $\Sigma\ \{Q_x x \in P\}$ to Q_x is an SG linear extension of Q_x. Second, the ordered set $\Sigma\ \{Q_x : x \in P\}$ contains an isomorphic copy of P so $\dim(P) \leq \dim(\Sigma\ \{Q_x : x \in P\}) \leq \dim_{SG}(\Sigma\ \{Q_x : x \in P\})$. □

The ordered set P_n in Figure 1 illustrates many of the curious properties of greedy and super-greedy dimension. Let Q_n denote the ordered set obtained from P_n by removing the points $\{c_1, c_2, \ldots, c_{n-1}\}$. Observe that P_n is an ordinal sum $\Sigma\ \{E_x : x \in Q_n\}$ where each R_x is either a one- or two-element antichain. It is straightforward to verify that the following statements are valid for all $n \geq 3$:

1. $\dim(P_n) = \dim(Q_n) = \dim_G(Q_n) = \dim_{SG}(P_n^d) = 3$.
2. $\dim_G(P_n) = \dim_{SG}(P_n) = \dim_{SG}(Q_n) = n$.

3. $\dim_{SG}(Q_n - x) = 2$.
4. There is no greedy linear extension L of P_n, so that $P_n/\{x\}$ in L.

It follows from these remarks that the behavior of ordinal sums with respect to greedy dimension is fairly well understood in the sense that both the upper and lower bounds in the inequalities of part (a) of Theorem 4 are tight. For super-greedy dimension, the situation is not clear, and we make the following conjecture.

CONJECTURE: For every ordered set P and every family $\{Q_x: x \in P\}$, $\dim_{SG}(\Sigma\ \{Q_x: x \in P\}) = \max\{\dim_{SG}(P), \max\{\dim_{SG}(Q_x): x \in P\}\}$.

A chain C in an ordered set P is called an *initial chain* if $D[x]$ is a chain for every $x \in C$. An initial chain is *maximal* if it is not a proper subset of another initial chain. If C is a maximal initial chain of P and L is a greedy linear extension of $P - C$, then it is easy to see that the linear extension M of P whose restriction to $P - C$ is L with $(P - C)/C$ in M is a greedy linear extension of P. When combined with Lemma 1, we have proved the following result [5].

LEMMA 3: If C is a maximal initial chain in an ordered set P, then $\dim_G(P) \leq 1 + \dim_G(P - C)$.

As observed in [3], the example in FIGURE 2 shows that Lemma 3 does not hold for super-greedy dimension. We believe a stronger conclusion can be made.

CONJECTURE: For every positive integer t, there exists an ordered set P so that $\dim_{SG}(P) \geq t + \dim_{SG}(P - C)$ for every chain C.

Despite the implications of the preceding conjecture, super-greedy linear extensions enjoy one particularly nice property with respect to subsets. By way of contrast, note that this property does not hold for greedy linear extensions.

LEMMA 4: If U is an up set of an ordered set and L is a super-greedy linear extension of P, then the restriction of L to $P - U$ is a super-greedy linear extension of $P - U$.

By far, the most complex research on super-greedy dimension we have been involved in to date has been concentrated on finding the appropriate generalization of Theorem 2. We announce our results in the next three theorems. The proofs, which are lengthy, will appear elsewhere.

THEOREM 5: Let A be a maximal antichain in an ordered set P so that $|P - A| \geq 2$. Let $D = D(A)$, $U = U(A)$, $m = |D|$, and $n = |U|$. Then each of the following inequalities is valid.

(a) If $mn \geq 1$, then $\dim_{SG}(P) \leq mn + 1$.
(b) If $A = \min(P)$, then $\dim(P) \leq n$.
(c) If $A = \max(P)$, then $\dim(P) \leq m$.

THEOREM 6: For every positive integer m, there exists an ordered set P that contains a maximal antichain A so that width $(P - A) = 1$ and $\dim_{SG}(P)n$.

THEOREM 7: Let A be a maximal antichain in an ordered set P with width $(P - A) = n \geq 1$. Then each of the following inequalities is valid.

(a) If $A = \min(P)$, then $\dim_{SG}(P) \leq 2n$.
(b) If $A = \max(P)$, then $\dim_{SG}(P) \leq n + 1$.

We have also been able to construct examples to show that the inequalities in these theorems are best possible.

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