Order **10**: 317–328, 1993.

Posets with Large Dimension and Relatively Few Critical Pairs

PETER C. FISHBURN
AT&T Bell Laboratories, 600 Mountain Avenue, Murray Hill, NJ 07974, U.S.A.

and

WILLIAM T. TROTTER
Bell Communications Research, 445 South Street 2L-367, Morristown, NJ 07962, U.S.A., and Department of Mathematics, Arizona State University, Tempe, AZ 85287, U.S.A.
E-mail: *wtt@bellcore.com*

Communicated by D. Kelly

(Received: 20 December 1992; accepted: 10 September 1993)

Abstract. The dimension of a poset (partially ordered set) $\mathbf{P} = (X, P)$ is the minimum number of linear extensions of P whose intersection is P. It is also the minimum number of extensions of P needed to reverse all critical pairs. Since any critical pair is reversed by some extension, the dimension t never exceeds the number of critical pairs m. This paper analyzes the relationship between t and m, when $3 \leqslant t \leqslant m \leqslant t + 2$, in terms of induced subposet containment. If $m \leqslant t + 1$ then the poset must contain $\mathbf{S}_t$, the standard example of a t-dimensional poset. The analysis for $m = t+2$ leads to dimension products and David Kelly's concept of a split. When $t = 3$ and $m = 5$, the poset must contain either $\mathbf{S}_3$, or the 6-point poset called a chevron, or the chevron's dual. When $t \geqslant 4$ and $m = t + 2$, the poset must contain $\mathbf{S}_t$, or the dimension product of the Kelly split of a chevron and $\mathbf{S}_{t-3}$, or the dual of this product.

Mathematics Subject Classifications (1991). 06A07, 05C35.

Key words. Partially ordered set, poset, dimension, critical pair.

1. Introduction

A *poset* (partially ordered set) $\mathbf{P} = (X, P)$ consists of a finite set X and a reflexive, antisymmetric and transitive binary relation P on X. A nonempty family $\mathcal{R}$ of linear extensions L of P is a *realizer* of P if $P = \bigcap_{\mathcal{R}} L$. When $\mathcal{R}$ is a realizer of P and $|\mathcal{R}| = t$, $\mathcal{R}$ is a *t-realizer* of P. Dushnik and Miller [1] defines dim($\mathbf{P}$), the *dimension* of $\mathbf{P}$, as the least positive integer t for which P has a t-realizer.

In this paper we study another definition of dim($\mathbf{P}$) that is based on incomparable pairs. Given a poset $\mathbf{P} = (X, P)$, $x \leqslant y$ in P means that $(x, y) \in P$, and $x \parallel y$ in P, the relation of incomparability for $x, y \in X$, means that neither (x, y) nor (y, x) is in P. We often omit 'in P' when this is clear from the context. The symmetric set of *ordered* incomparable pairs is

$$\mathrm{inc}(\mathbf{P}) = \{(x, y) \in X \times X : x \parallel y\}.$$

A member (x, y) of inc(**P**) is *critical* if

(1) for all $u \in X$, $u < x \Rightarrow u < y$,
(2) for all $v \in X$, $y < v \Rightarrow x < v$,

where $x < y$ when $x \leqslant y$ in P and $x \neq y$. The set of all critical ordered pairs in inc(**P**) is denoted by crit(**P**).

Critical pairs were introduced in Rabinovitch and Rival [8] which shows that a family $\mathcal{R}$ of linear extensions of P is a realizer of P if, and only if, for every $(x, y) \in \mathrm{crit}(\mathbf{P})$, there is an $L \in \mathcal{R}$ with $y < x$ in L. This is an easy consequence of the following observation.

PROPOSITION 1.1. *If* $\mathbf{P} = (X, P)$ *is a poset and* $(a, b) \in \mathrm{inc}(\mathbf{P})$, *then some* $(x, y) \in \mathrm{crit}(\mathbf{P})$ *has* $x \leqslant a$ *in* P *and* $b \leqslant y$ *in* P.

We say that a linear order L on X *reverses* a subset S of inc(**P**) if $y < x$ in L for every $(x, y) \in S$. Since every singleton subset of inc(**P**) is reversed by some linear extension of P, we note the following. Here and later a poset is *nonlinear* if it is not a linear order.

PROPOSITION 1.2. *If* $\mathbf{P} = (X, P)$ *is a nonlinear poset, then*

$$\dim(\mathbf{P}) \leqslant |\mathrm{crit}(\mathbf{P})|.$$

Our primary goal in this paper is to solve the following extremal problem.

CHARACTERIZATION PROBLEM 1.3. *For integers* t *and* m *with* $3 \leqslant t \leqslant m \leqslant t + 2$, *find the minimum set* $\mathcal{P}(t, m)$ *so that if* $\mathbf{P} = (X, P)$ *is a poset with* $\dim(\mathbf{P}) = t$ *and* $|\mathrm{crit}(\mathbf{P})| = m$, *then* $\mathbf{P}$ *contains a poset in* $\mathcal{P}(t, m)$ *as an induced subposet.*

The main analysis of the problem is in Section 4. Prior to that we develop background for the factors that contribute to the solution. We complete this introduction with remarks on cycles in inc(**P**) and related hypergraphs. Section 2 discusses the standard examples, and Section 3 introduces splits and dimension products. In Section 5, we give a brief discussion of the motivation for investigating our characterization problem. Throughout, the notation and terminology adhere to [9], which along with [10] offers general background on dimension theory and combinatorial problems for posets.

For poset $\mathbf{P} = (X, P)$ and $k \geqslant 2$, an *alternating cycle* (*of length* k) is a sequence $(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)$ of pairs of inc(**P**) such that $y_i \leqslant x_{i+1}$ for $i = 1, 2, \ldots, k$ $(x_{k+1} = x_1)$. The alternating cycle is *strict* if for all i and j, $y_i \leqslant x_j \iff j = i + 1$.

The *dual* S^d of $S \subseteq X \times X$ is defined by $S^d = \{(y,x)\colon (x,y) \in S\}$. The *dual* of poset $\mathbf{P} = (X,P)$ is $\mathbf{P}^d = (X,P^d)$. Clearly, $\dim(\mathbf{P}) = \dim(\mathbf{P}^d)$. The following is noted in [1].

LEMMA 1.4. *Suppose* $\mathbf{P} = (X,P)$ *is a poset and* $\varnothing \neq S \subseteq \mathrm{inc}(\mathbf{P})$. *Then* (1), (2) *and* (3) *are equivalent*:

(1) *some linear extension of* P *reverses* S;
(2) *no sequence of pairs from* S^d *is an alternating cycle*;
(3) *no sequence of pairs from* S^d *is a strict alternating cycle.*

The approach to dimension by reversing critical pairs suggests a reformation in terms of hypergraph coloring. Recall that the *chromatic number* $\chi(\mathcal{H})$ of a hypergraph $\mathcal{H}$ is the least positive integer t for which the vertex set can be partitioned into t subsets, called *color classes*, such that no edge of $\mathcal{H}$ is contained in one color class.

For any poset $\mathbf{P} = (X,P)$, we define $\mathcal{K}_{\mathbf{P}}$, the *hypergraph of critical pairs*, as follows. The vertex set of $\mathcal{K}_{\mathbf{P}}$ is $\mathrm{crit}(\mathbf{P})$, and $S \subseteq \mathrm{crit}(\mathbf{P})$ is an edge of $\mathcal{K}_{\mathbf{P}}$ if S^d contains an alternating cycle whose terms exhaust S^d. The *strict hypergraph of critical pairs* $\mathcal{K}^s_{\mathbf{P}}$ is defined similarly with $S \subseteq \mathrm{crit}(\mathbf{P})$ an edge if S^d contains a strict alternating cycle whose terms exhaust S^d.

For any hypergraph $\mathcal{H} = (\mathcal{X},\mathcal{E})$ let $\mathrm{graph}(\mathcal{H}) = (\mathcal{X}, \{E \in \mathcal{E}\colon |E| = 2\})$. Note that $\mathrm{graph}(\mathcal{K}_{\mathbf{P}}) = \mathrm{graph}(\mathcal{K}^s_{\mathbf{P}})$ for poset $\mathbf{P}$. A restatement of preceding observations gives the following result.

PROPOSITION 1.5. *If* $\mathbf{P} = (X,P)$ *is a nonlinear poset, then*

$$\dim(\mathbf{P}) = \chi(\mathcal{K}_{\mathbf{P}}) = \chi(\mathcal{K}^s_{\mathbf{P}}) \geqslant \chi\big(\mathrm{graph}(\mathcal{K}_{\mathbf{P}})\big).$$

Although $\dim(\mathbf{P}) > \chi(\mathrm{graph}(\mathcal{K}_{\mathbf{P}}))$ is possible, the following exception (which is implicit in the work of Ghoulà-Houri [2]) is noteworthy.

THEOREM 1.6. *A nonlinear poset* $\mathbf{P}$ *has* $\dim(\mathbf{P}) = 2$ *if and only if*

$$\chi\big(\mathrm{graph}(\mathcal{K}^s_{\mathbf{P}})\big) = 2.$$

The proof of Theorem 1.6 actually yields the following slightly stronger result that will be used later.

COROLLARY 1.7. *Let* $\mathbf{P} = (X,P)$ *be a poset, and let* S *be a nonempty subset of* $\mathrm{crit}(\mathbf{P})$. *Then there is a partition* $\{S_1,S_2\}$ *of* S *such that for each* S_i *some linear extension of* P *reverses* S_i *if, and only if, the subgraph of* $\mathrm{graph}(\mathcal{K}_{\mathbf{P}})$ *induced by* S *is bipartite.*

2. The Standard Examples

A poset $\mathbf{P} = (X, P)$ is *t-irreducible* for $t \geqslant 2$ if $\dim(\mathbf{P}) = t$ and the removal of any point in X leaves a poset having dimension less than t. We say that $\mathbf{P}$ is *irreducible* if it is t-irreducible for some $t \geqslant 2$. The only 2-irreducible poset is obviously a 2-element antichain. All 3-irreducible posets are identified in Kelly [4] and, independently, in Trotter and Moore [11].

For each $n \geqslant 2$ let $\mathbf{S}_n$ denote the bipartite poset on $2n$ points with n minimal elements $a_1, \ldots, a_n$, n maximal elements $b_1, \ldots, b_n$, and $a_i < b_j$ in $\mathbf{S}_n$ for all i and j if and only if $i \neq j$. Evidently, $\mathrm{crit}(\mathbf{S}_n) = \{(a_i, b_i)\colon\ i = 1, \ldots, n\}$, so $\dim(\mathbf{S}_n) \leqslant n$. It is also clear that no linear extension reverses more than one critical pair; in fact, $\mathbf{S}_n$'s hypergraph of critical pairs is an ordinary graph, namely the *complete graph* K_n. Therefore $\dim(\mathbf{S}_n) = n$. We call $\mathbf{S}_n$ the *standard example* of an n-dimensional poset. For $n \geqslant 3$, $\mathbf{S}_n$ is n-irreducible.

The standard examples, introduced in [1], have played a key role in many combinatorial problems in dimension theory. An example is Hiraguchi's inequality [3] which says that $\dim(\mathbf{P}) \leqslant |X|/2$ if poset $\mathbf{P} = (X, P)$ has $|X| \geqslant 4$. And if $\dim(\mathbf{P}) \geqslant n \geqslant 4$ and $|X| \leqslant 2n + 1$ then $\mathbf{P}$ contains $\mathbf{S}_n$ as a subposet ([6], [8, p. 94]). The standard examples also belong to more general classes of posets studied in [6], [9].

There are exactly three 3-irreducible posets on six points. They are $\mathbf{S}_3$, the chevron $\mathbf{C}$ of Figure 1, and $\mathbf{C}^d$. Note that $\mathcal{K}^s_{\mathbf{C}}$ is an ordinary graph cycle on five vertices.

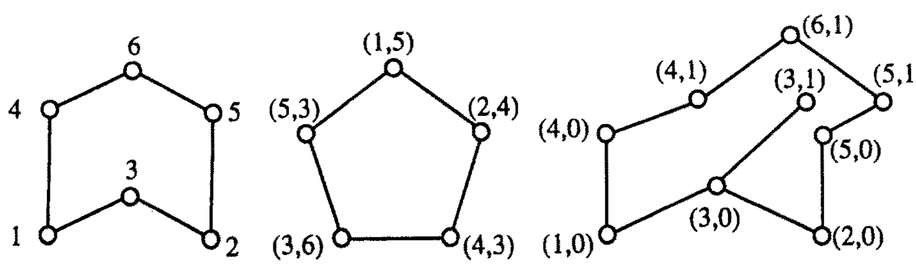

diagram of $\mathbf{C}$ graph$(\mathcal{K}^s_{\mathbf{C}}) = (\mathcal{K}^s_{\mathbf{C}})$ diagram of split of $\mathbf{C}$

Fig. 1.

3. Splits and Dimension Products

In this section we introduce splits and dimension products as defined in Kelly [5] (see [9], Chapter 2, for a more extensive discussion). Let $\mathbf{P} = (X, P)$ be a nonlinear poset. Let

$$A(\mathbf{P}) = \{x \in X\colon\ (x, y) \in \mathrm{crit}(\mathbf{P}) \text{ for some } y \in X\},$$

$$B(\mathbf{P}) = \{y \in X\colon\ (x, y) \in \mathrm{crit}(\mathbf{P}) \text{ for some } x \in X\}.$$

We define the *split* of $\mathbf{P}$ as the poset $\mathbf{Q} = (Y, Q)$ with

$$Y = \big(A(\mathbf{P}) \times \{0\}\big) \cup \big(B(\mathbf{P}) \times \{1\}\big)$$

and with order Q defined in three parts as follows:

(1) for all $a, a' \in A(\mathbf{P})$, $(a, 0) \leqslant (a', 0)$ in Q if $a \leqslant a'$ in P;
(2) for all $b, b' \in B(\mathbf{P})$, $(b, 1) \leqslant (b', 1)$ in Q if $b \leqslant b'$ in P;
(3) for all $(a, b) \in A(\mathbf{P}) \times B(\mathbf{P})$, $(a, 0) < (b, 1)$ in Q if $a \leqslant b$ in P.

See Figure 1 for the split of $\mathbf{C}$.

The *dimension product* $\mathbf{P}_1 \otimes \mathbf{P}_2$ of nonlinear posets $\mathbf{P}_1 = (X_1, P_1)$ and $\mathbf{P}_2 = (X_2, P_2)$ is the poset $\mathbf{Q} = (Y, Q)$ which consists of the disjoint union of the splits of $\mathbf{P}_1$ and $\mathbf{P}_2$ plus the following additional comparabilities on $Y = [(A(\mathbf{P}_1) \times \{0\}) \cup (B(\mathbf{P}_1) \times \{1\})] \cup [(A(\mathbf{P}_2) \times \{0\}) \cup (B(\mathbf{P}_2) \times \{1\})]$:

(4) for all $(a, b) \in A(\mathbf{P}_1) \times B(\mathbf{P}_2)$, $(a, 0) < (b, 1)$ in Q;
(5) for all $(a, b) \in A(\mathbf{P}_2) \times B(\mathbf{P}_1)$, $(a, 0) < (b, 1)$ in Q.

Kelly [5] proves the following result for dimension products.

THEOREM 3.1. *Let $\mathbf{P}_1$ and $\mathbf{P}_2$ be nonlinear posets. Then*

$$\dim(\mathbf{P}_1 \otimes \mathbf{P}_2) = \dim(\mathbf{P}_1) + \dim(\mathbf{P}_2).$$

We find it useful to extend the definition of $\mathbf{S}_n$ for $n \geqslant 2$ by defining $\mathbf{S}_1 = (\{a_1, b_1\}, \{(a_1, a_1), (b_1, b_1)\})$, the 2-point antichain. When $n \geqslant 2$, note that the split of $\mathbf{S}_n$ is isomorphic to $\mathbf{S}_n$. This is not true for $\mathbf{S}_1$. However, when $\mathbf{P}$ is a nonlinear poset, we define $S_1 \otimes \mathbf{P}$ so that it is consistent with the definition of $\mathbf{S}_n \otimes \mathbf{P}$ for $n \geqslant 2$. Accordingly, for all $n \geqslant 1$, we take $\mathbf{S}_n \otimes \mathbf{P}$ as the union of disjoint copies of $\mathbf{S}_n$ and the split of $\mathbf{P}$ with the additional comparabilities

(4′) $a_i < (b, 1)$ in $\mathbf{S}_n \otimes \mathbf{P}$ for all $i \in \{1, \ldots, n\}$ and all $b \in B(\mathbf{P})$,
(5′) $(a, 0) < b_i$ in $\mathbf{S}_n \otimes \mathbf{P}$ for all $i \in \{1, \ldots, n\}$ and all $a \in A(\mathbf{P})$.

With $\mathbf{C}$ the chevron of Figure 1, note that for all $t \geqslant 4$, the dimension product $\mathbf{S}_{t-3} \otimes \mathbf{C}$ is a t-dimensional poset (see Theorem 3.1) with exactly $t+2$ critical pairs.

4. Characterization Theorems

This section establishes a complete characterization of posets for which the number of critical pairs exceeds the dimension by at most two. We assume throughout that

$$\dim(\mathbf{P}) = t \geqslant 3,$$

$$\big|\mathrm{crit}(\mathbf{P})\big| = m \leqslant t + 2.$$

The m critical pairs are labelled so that

$$\operatorname{crit}(\mathbf{P}) = \{(x_i, y_i)\colon\ i = 1, \ldots, m\},$$

and we often identify (x_i, y_i) simply by i. Thus $\{1, 2, \ldots, m\}$ identifies the vertex set of hypergraphs $\mathcal{K}_{\mathbf{P}}$ and $\mathcal{K}_{\mathbf{P}}^s$. For brevity, we denote graph($\mathcal{K}_{\mathbf{P}}$) by $\mathbf{G}$ and speak of i in the vertex set of $\mathbf{G}$ as the critical pair (i.e. (x_i, y_i)) thus identified. We say also that poset $\mathbf{P} = (X, P)$ *contains* $\mathbf{Q} = (Y, Q)$ when $\mathbf{P}$ contains an induced subposet isomorphic to $\mathbf{Q}$.

We begin the characterization process with the following elementary result.

LEMMA 4.1. *If* $\mathbf{G}$ *contains* K_t *then* $\mathbf{P}$ *contains* $\mathbf{S}_t$.

Proof. Suppose the subgraph of $\mathbf{G}$ induced by $\{1, 2, \ldots, t\}$ is a complete graph K_t. For all distinct $i, j \in \{1, 2, \ldots, t\}$ we know that $x_i \leqslant y_j$ and $x_j \leqslant y_i$ in P. The conclusion of the lemma follows if $x_i \neq y_j$ for all distinct $i, j \in \{1, 2, \ldots, t\}$.

To the contrary, suppose for definiteness that $x_1 = y_2$. Since $t \geqslant 3$, we conclude that $x_3 \leqslant y_2 = x_1 \leqslant y_3$. Then $x_3 \leqslant y_3$, thus contradicting $(x_3, y_3) \in \operatorname{crit}(\mathbf{P})$. □

Note that the conclusion of Lemma 4.1 fails if $\mathbf{G}$ contains K_2.

THEOREM 4.2. *If* $m \leqslant t + 1$, *then* $\mathbf{P}$ *contains* $\mathbf{S}_t$.

Proof. If $\mathbf{G}$ contains a K_t, the desired result follows from Lemma 4.1. Assume henceforth without loss of generality that $\{1, 2\}$ is not an edge of $\mathbf{G}$. Then a single linear extension reverses critical pairs 1 and 2. Hence $\dim(\mathbf{P}) \leqslant m - 1$, so $m = t + 1$. Moreover, $\mathbf{G}$ is complete on the $t - 1$ vertices in $\{3, 4, \ldots, t + 1\}$, else $\dim(\mathbf{P}) \leqslant t - 1$.

For $i \in \{1, 2\}$ let

$$N_i = \{j\colon\ 3 \leqslant j \leqslant t + 1 \text{ and } \{i, j\} \text{ is } \textit{not} \text{ an edge of } \mathbf{G}\}.$$

If either N_i is empty, Lemma 4.1 shows that $\mathbf{P}$ contains $\mathbf{S}_t$. Assume henceforth that $N_1 \neq \varnothing \neq N_2$.

Suppose $j \in N_1$, $k \in N_2$, and $j \neq k$. Then some one linear extension of $\mathbf{P}$ reverses 1 and j, and another reverses 2 and k. To avoid the resulting contradiction that $\dim(\mathbf{P}) < t$, assume henceforth without loss of generality that $N_1 = N_2 = \{3\}$.

Then $\{1, 2, 3\}$ is an edge of $\mathcal{K}_{\mathbf{P}}$, since otherwise a single linear extension reverses all three critical pairs in violation of $\dim(\mathbf{P}) = t$. Suppose for definiteness that the alternating cycle for $\{1, 2, 3\}$ is (y_1, x_1), (y_2, x_2), (y_3, x_3). Then $x_1 \leqslant y_2$, $x_2 \leqslant y_3$ and $x_3 \leqslant y_1$, and it follows that $\{x_1, x_3, x_4, \ldots, x_{t+1}\} \cup \{y_1, y_2, y_4, \ldots, y_{t+1}\}$ yields a copy of $\mathbf{S}_t$. □

Throughout the rest of this section, $[n] = \{1, 2, \ldots, n\}$.

THEOREM 4.3. *If $m \leqslant 5$, then* **P** *contains* $\mathbf{S}_3$, **C**, *or* $\mathbf{C}^d$.

THEOREM 4.4. *If $t \geqslant 4$, then* **P** *contains* $\mathbf{S}_t$, $\mathbf{S}_{t-3} \otimes \mathbf{C}$, *or* $\mathbf{S}_{t-3} \otimes \mathbf{C}^d$.

The proofs consume the rest of the section.

Proof of Theorem 4.3. We argue by contradiction. Assume that $(t, m) = (3, 5)$, since otherwise the conclusion of Theorem 4.3 follows from Theorem 4.2. We know also that the chromatic number of **G** is 3. If **G** contains a triangle then **P** contains $\mathbf{S}_3$. Assume that **P** doesn't contain $\mathbf{S}_3$. Then **G** is a cycle on five vertices: see Corollary 1.7. Relabelling if necessary, assume for each $i \leqslant 5$ that $x_i \leqslant y_{i+1}$ and $x_i \leqslant y_{i-1}$ $(y_6 = y_1, y_0 = y_5)$. We proceed by cases through a series of claims. Indices are interpreted cyclically in all cases.

CLAIM 1. *For all $i \leqslant 5$, either $x_i \parallel y_{i+2}$ or $x_i < y_{i+2}$, and either $x_i \parallel y_{i-2}$ or $x_i < y_{i-2}$.*

Proof. If $y_{i+2} \leqslant x_i$, then $x_{i+1} \leqslant y_{i+2} \leqslant x_i \leqslant y_{i+1}$, so $x_{i+1} \leqslant y_{i+1}$, a contradiction to $(x_{i+1}, y_{i+1}) \in \mathrm{crit}(\mathbf{P})$. Hence either $x_i \parallel y_{i+2}$ or $x_i < y_{i+2}$. A symmetric proof applies to x_i and y_{i-2}. □

CLAIM 2. *For all $i \leqslant 5$, either $x_i \parallel y_{i+2}$ or $x_{i+2} \parallel y_i$.*

Proof. Else, by Claim 1, $x_i < y_{i+2}$ and $x_{i+2} < y_i$, so $\{x_i, x_{i+1}, x_{i+2}, y_i, y_{i+1}, y_{i+2}\}$ is a copy of $\mathbf{S}_3$. □

CLAIM 3. *For all $i \leqslant 5$, $x_i \parallel x_{i+1}$ and $y_i \parallel y_{i+1}$.*

Proof. Otherwise our labelling scheme for **G** gives $x_i \leqslant y_i$. □

CLAIM 4. *For all $i \leqslant 5$,*

$$(x_i \parallel y_{i+2}) \Longrightarrow (x_{i+2} \leqslant x_i \text{ or } y_{i+2} \leqslant y_i),$$

and

$$(x_{i+2} \parallel y_i) \Longrightarrow (x_i \leqslant x_{i+2} \text{ or } y_i \leqslant y_{i+2}).$$

Proof. Suppose $x_i \parallel y_{i+2}$. By Proposition 1.1, let (x_j, y_j) in crit(**P**) have $x_j \leqslant x_i$ and $y_{i+2} \leqslant y_j$. If $j = i+1$ then $x_{i+1} \leqslant x_i \leqslant y_{i+1}$, so $x_{i+1} \leqslant y_{i+1}$. This contradiction shows that $j \neq i+1$. If $j = i-1$, a similar contradiction obtains, so $j \neq i-1$. If $j = i-2$, then $x_{i+3} \leqslant y_{i+2} \leqslant y_{i-2} = y_{i+3}$, another contradiction. So $j \neq i-2$. If $j = i$ then $y_{i+2} \leqslant y_i$; if $j = i+2$ then $x_{i+2} \leqslant x_i$. The argument for x_{i+2} and y_i is similar. □

Now for each $i \in [5]$ let

$$U(x_i) = \{ j \in [5]\colon\ x_i \leqslant y_j \},$$

$$D(y_i) = \{ j \in [5]\colon\ x_j \leqslant y_i \}.$$

CLAIM 5. *For all* $i, j \in [5]$,

$$U(x_i) \subseteq U(x_j) \Longrightarrow (x_j, x_i) \notin \mathrm{inc}(\mathbf{P})$$

and

$$D(y_i) \supseteq D(y_j) \Longrightarrow (y_j, y_i) \notin \mathrm{inc}(\mathbf{P}).$$

Proof. Suppose otherwise for the first implication that

$$U(x_i) \subseteq U(x_j) \quad \text{and} \quad (x_j, x_i) \in \mathrm{inc}(\mathbf{P}).$$

Since $U(x_i) \not\subseteq U(x_{i+1})$ and $U(x_i) \not\subseteq U(x_{i-1})$, we may assume that $j = i+2$. Choose $(x_k, y_k) \in \mathrm{crit}(\mathbf{P})$ so that $x_k \leqslant x_j = x_{i+2}$ and $x_i \leqslant y_k$. Inequality $x_k \leqslant x_{i+2}$ implies that $k \neq i+1$ and $k \neq i+3 = i-2$; $x_i \leqslant y_k$ implies $k \neq i$. If $k = i+2$ then $U(x_i) \subseteq U(x_{i+2})$ implies $x_{i+2} \leqslant y_{i+2}$, which is false. Finally, if $k = i-1$ then $U(x_i) \subseteq U(x_{i+2})$ and hence $x_{i+2} \leqslant y_{i-1}$. However, $x_k = x_{i-1} \leqslant x_{i+2}$ then gives $x_{i-1} \leqslant y_{i-1}$, a contradiction. We conclude that $U(x_i) \subseteq U(x_j) \Rightarrow (x_j, x_i) \notin \mathrm{inc}(\mathbf{P})$. The proof of the D implication is similar. $\square$

CLAIM 6. *For some* $i \in [5]$, *either*

(1) $x_i \leqslant x_{i+2}$ *and* $x_i \leqslant x_{i-2}$,

or

(2) $y_{i+2} \leqslant y_i$ *and* $y_{i-2} \leqslant y_i$.

Proof. Suppose to the contrary that (1) and (2) fail for every i. By Claim 1 and symmetry, we may assume that $x_1 \parallel y_3$. By Claim 4, $x_3 \leqslant x_1$ or $y_3 \leqslant y_1$. Using duality, we may assume that $x_3 \leqslant x_1$. Then $x_3 \leqslant y_5$, so $y_3 \parallel x_5$ by Claim 2. Therefore $x_3 \leqslant x_5$ or $y_3 \leqslant y_5$. However, $x_3 \leqslant x_5$ satisfies (1), so we have $x_3 \not\leqslant x_5$ and $y_3 \leqslant y_5$. It follows that $x_2 \leqslant y_5$ and $x_5 \parallel y_2$. This in turn requires $y_2 \not\leqslant y_5$ and $x_2 \leqslant x_5$. Then $x_2 \leqslant y_4$ and $y_2 \parallel x_4$, so $x_2 \not\leqslant x_4$ and $y_2 \leqslant y_4$. Thus $x_1 \leqslant y_4$ and $x_4 \parallel y_1$. Claim 4 applied to $x_4 \parallel y_1$ gives $x_1 \leqslant x_4$ or $y_1 \leqslant y_4$. However, $x_1 \leqslant x_4$ and $x_3 \leqslant x_1$ imply $x_3 \leqslant x_4$, contrary to Claim 3, and $y_1 \leqslant y_4$ and $y_2 \leqslant y_4$ violate our supposition that (2) fails. $\square$

In view of Claim 6, assume henceforth without loss of generality that $y_3 \leqslant y_1$ and $y_4 \leqslant y_1$. We show that $\mathbf{P}$ contains $\mathbf{C}$ or $\mathbf{C}^d$. In anticipation of the proof of Theorem 4.4, we prove that either

(I) $\mathbf{P}$ contains a copy of $\mathbf{C}$ with $A(\mathbf{C}) \subseteq \{x_1, \ldots, x_5\}$ and $B(\mathbf{C}) \subseteq \{y_1, \ldots, y_5\}$,

or

(II) $\mathbf{P}$ contains a copy of $\mathbf{C}^d$ with $A(\mathbf{C}) \subseteq \{x_1, \ldots, x_5\}$ and $B(\mathbf{C}) \subseteq \{y_1, \ldots, y_5\}$.

Since $x_1 \parallel y_1$ by our initial labelling of $\mathrm{crit}(\mathbf{P})$ in this section, we require $x_1 \not\leqslant y_3$, and $x_1 \not\leqslant y_4$, so $U(x_1) = \{2, 5\}$. We divide further arguments into two cases.

Case 1. HYPOTHESIS: either $y_2 \parallel x_4$ or $x_3 \parallel y_5$.

Assume without loss of generality that $y_2 \parallel x_4$. By Claim 4, $x_2 \leqslant x_4$ or $y_2 \leqslant y_4$. Since $y_2 \leqslant y_4$ along with $x_1 \leqslant y_2$ and $y_4 \leqslant y_1$ yields $x_1 \leqslant y_1$, which is false, we have $y_2 \nleqslant y_4$ and $x_2 \leqslant x_4$. Then $x_2 \leqslant y_5$ and $y_2 \parallel x_5$. Therefore $D(y_2) = \{1, 3\}$.

If $x_3 \parallel y_5$ then $\{x_1, x_3, x_4, y_1, y_2, y_5\}$ yields $\mathbf{S}_3$, so we have $x_3 \leqslant y_5$ and $y_3 \parallel x_5$. At this point we know that $D(y_5) = \{1, 2, 3, 4\}$ and $D(y_3) = \{2, 4\}$, so Claim 5 gives $(y_5, y_3) \notin \mathrm{inc}(\mathbf{P})$. It follows that $y_3 \leqslant y_5$. We also have $U(x_1) = \{2, 5\}$, $U(x_3) = \{2, 4, 5\}$, $U(x_4) = \{1, 3, 5\}$ and $U(x_5) = \{1, 4\}$. Recalling that $x_2 \leqslant x_4$, Case 1 now splits into two subcases.

Subcase 1a. HYPOTHESIS: $x_2 < x_4$.

Since (x_4, y_4) is a critical pair, $x_2 < y_4$. Therefore $U(x_2) = \{1, 3, 4, 5\}$, and we know $U(x_i)$ and $D(y_i)$ for every $i \in [5]$. Furthermore, it follows from Claim 5 that the subposets of $\mathbf{P}$ determined by the y_i and by the x_i are as shown in Figure 2.

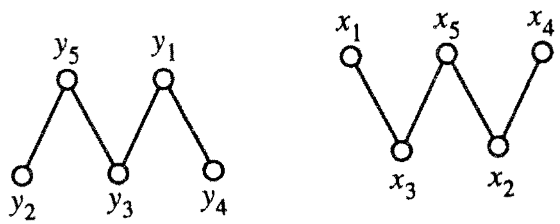

Fig. 2.

However, we do not know whether $\leqslant$ in each of $x_1 \leqslant y_2$, $x_5 \leqslant y_4$ and $x_4 \leqslant y_3$ is $=$ or $<$. Figure 3 gives a pseudo-diagram of $\mathbf{P}$ that encloses these three $\leqslant$ in ovals. It follows easily that both (I) and (II) hold.

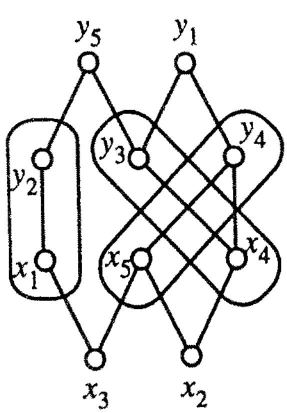

Fig. 3.

Subcase 1b. HYPOTHESIS: $x_2 = x_4$.

In this subcase, we know that the subposet of **P** determined by the y_i is the same as shown in Figure 2. Figure 4 shows the new subposet on the x_i.

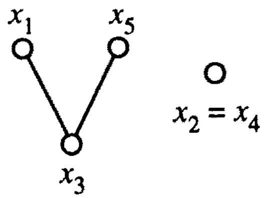

Fig. 4.

We do not know whether $\leqslant$ in each of $x_2 \leqslant y_3$, $x_1 \leqslant y_2$ and $x_5 \leqslant y_4$ is $=$ or $<$. Figure 5 represents this in a manner similar to Figure 3. Regardless of whether any or all of the oval pairs are distinct, we observe that (II) holds for **P**.

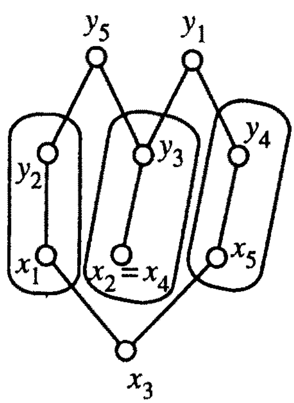

Fig. 5.

Case 2. HYPOTHESIS: $x_4 < y_2$ and $x_3 < y_5$.

By Claim 2, $x_2 \parallel y_4$ and $x_5 \parallel y_3$. Moreover, either $x_2 \parallel y_5$ or $y_2 \parallel x_5$. By symmetry, we may assume that $x_2 \parallel y_5$. Then $U(x_1) = \{2, 5\}$, $U(x_2) = \{1, 3\}$, $U(x_3) = \{1, 2, 4, 5\}$, and $U(x_4) = \{1, 2, 3, 5\}$. If $x_5 < y_2$, relabelling yields the situation of Subcase 1a, where both (I) and (II) hold. If $x_5 \parallel y_2$, then $y_2 = y_5$ and we get the dual of Subcase 1b where (I) holds for **P**.

Proof of Theorem 4.4. We again argue by contradiction under the assumption that **P** does not contain $\mathbf{S}_t$. We know by Theorem 4.2 that $m = t + 2$. Moreover, for each n with $4 \leqslant n \leqslant t + 2$, no set of n critical pairs can be reversed by only $n - 3$ linear extensions. Also, by Lemma 4.1, **G** does not contain K_t. We may therefore assume that $\{1,3\}$ is not an edge of **G**. Since the subgraph of **G** induced by $\{2,4,5,\ldots,t+2\}$ is not complete, we may assume also that $\{2,4\}$ is not an edge of **G**. For each $i \in \{1,2,3,4\}$ let

$$N_i = \{j\colon\ 5 \leqslant j \leqslant t+2 \text{ and } \{i,j\} \text{ is } \textit{not} \text{ an edge of } \mathbf{G}\}.$$

We continue with a series of claims.

CLAIM 1. $N_1 \cap N_3 = \varnothing = N_2 \cap N_4$.

Proof. Suppose to the contrary of $N_1 \cap N_3 \neq \varnothing$ that $j \in N_1 \cap N_3$. Consider the subgraph of **G** induced by $\{1,2,3,4,j\}$. It contains no triangle and is not induced by a cycle on five vertices. It follows that two linear extensions reverse the five critical pairs in $\{1,2,3,4,j\}$, a contradiction. Thus $N_1 \cap N_3 = \varnothing$. Similarly, $N_2 \cap N_4 = \varnothing$. □

CLAIM 2. *Either* $N_1 = \varnothing$ *or* $N_3 = \varnothing$. *Either* $N_2 = \varnothing$ *or* $N_4 = \varnothing$.

Proof. Suppose both N_1 and N_3 are nonempty. By Claim 1 we have $j \in N_1$, $k \in N_3$ and $j \neq k$. It follows that three linear extensions (one each for $\{1,j\}$, $\{3,k\}$, $\{2,4\}$) reverse the six critical pairs in $\{1,2,3,4,j,k\}$, a contradiction. The argument for N_2 and N_4 is symmetric. □

We assume henceforth without loss of generality that $N_1 = N_4 = \varnothing$. Then $\{1,4\}$ is not an edge of G, since otherwise **P** contains $\mathbf{S}_t$.

CLAIM 3. *There is a unique* $j \in \{5,6,\ldots,t+2\}$ *for which* $N_2 = N_3 = \{j\}$.

Proof. Suppose $N_2 = \varnothing$. If $\{1,2\}$ is an edge of **G**, we get $\mathbf{S}_t$. Hence $\{1,2\}$ is not an edge of **G**. Take $j \in \{5,6,\ldots,t+2\}$ and consider the five critical pairs in $\{1,2,3,4,j\}$. Since we need three linear extensions to reverse these five pairs, $\{3,4,j\}$ induces a triangle in **G**, contrary to an hypothesis since it implies that **P** contains $\mathbf{S}_t$. We conclude that $N_2 \neq \varnothing$. Similarly, $N_3 \neq \varnothing$. If $j \in N_2$, $k \in N_3$ and $j \neq k$, we get a set of six critical pairs which are reversed by three linear extensions, a contradiction. Hence $N_2 = N_3 = \{j\}$ for some j. □

Assume without loss of generality that $N_2 = N_3 = \{5\}$. It follows that $\{1,2,3,4,5\}$ induces a 5-cycle in **G** and that $\{i,i+1\}$ (cyclically) is an edge of **G** for $i = 1,2,3,4,5$.

CLAIM 4. *For all* $i \in \{1,\ldots,5\}$ *and* $j \in \{6,\ldots,t+2\}$, $\{i,j\}$ *is an edge of* **G**.

Proof. Otherwise three linear extensions reverse the six critical pairs in $\{1,2,3,4,5,j\}$. □

CLAIM 5. **P** *contains* $\mathbf{S}_{t-3} \otimes \mathbf{C}$ *or* $\mathbf{S}_{t-3} \otimes \mathbf{C}^d$.

Proof. It is routine to see that all six claims in the proof of Theorem 4.3 are valid in the present setting. The proof of Claim 5 and hence of Theorem 4.4 is completed by the following observations. First, for all $i, j \in [5]$, $x_i \not\leqslant y_j$. This follows from the fact that $x_i \leqslant y_6$ and $x_6 \leqslant y_j$ for all $i, j \in [5]$. Second, if the hypothesis of Case 1 near the end of the proof of Theorem 4.3 holds, then $\mathbf{P}$ contains $\mathbf{S}_{t-3} \otimes \mathbf{C}^d$, and if the hypothesis of Case 2 holds then $\mathbf{P}$ contains $\mathbf{S}_{t-3} \otimes \mathbf{C}$. □

5. Motivation: A Conjecture Concerning Incomparable Pairs

The original motivation for investigating the characterization problem discussed in this paper is the following conjecture made by the first author.

CONJECTURE 5.1. *For each $n \geqslant 4$, any poset $\mathbf{P}$ with dimension at least n contains at least n^2 incomparable pairs. Furthermore, if* $\dim(\mathbf{P}) = n$ *and $\mathbf{P}$ contains exactly n^2 incomparable pairs, then $\mathbf{P}$ contains the standard example $\mathbf{S}_n$ as a subposet.*

In [7], Jun Qin verifies this conjecture when $n = 4$. He also shows that any 5-dimensional poset has at least 24 incomparable pairs. The argument for this partial result is quite complicated and seems to suggest that a more complete understanding of the relationship between dimension and the number of incomparable pairs (and for that matter, the number of critical pairs) remains to be discovered.

References

1. B. Dushnik and E. W. Miller (1941) Partially ordered sets, *Amer. J. Math.* **63**, 600–610.
2. A. Ghoulà-Houri (1962) Caractérisation des graphes non orientes dont on peut orienter les arêtes de manier a obtenir le graphe d'une relation d'ordre, *C. R. Acad. Sci. Paris* **254**, 1370–1371.
3. T. Hiraguchi (1955) On the dimension of orders, *Sci. Rep. Kanazawa Univ.* **4**, 1–20.
4. D. Kelly (1977) The 3-irreducible partially ordered sets, *Can. J. Math.* **29**, 367–383.
5. D. Kelly (1981) On the dimension of partially ordered sets, *Discrete Math.* **35**, 135–156.
6. R. J. Kimble (1973) *Extremal Problems in Dimension Theory for Partially Ordered Sets*, Ph.D. Thesis, Massachusetts Institute of Technology.
7. Jun Qin (1991) *Some Problems Involving the Dimension Theory for Ordered Sets and the First-Fit Coloring Algorithm*, Ph.D. Dissertation, Arizona State University.
8. I. Rabinovitch and I. Rival (1979) The rank of a distributive lattice, *Discrete Math.* **25**, 275–279.
9. W. T. Trotter (1992) *Combinatorics and Partially Ordered Sets: Dimension Theory*, Johns Hopkins University Press, Baltimore.
10. W. T. Trotter (to appear) Partially ordered sets, in R. Graham, M. Grötschel and L. Lovász (eds), *Handbook of Combinatorics*.
11. W. T. Trotter and J. I. Moore (1976) Characterization problems for graphs, partially ordered sets, lattices, and families of sets, *Discrete Math.* **16**, 361–381.