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A Bound on the Dimension of Interval Orders

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We make use of the partially ordered set $(I(0, n), <)$ consisting of all closed intervals of real numbers with integer endpoints (including the degenerate intervals with the same right- and left-hand endpoints), ordered by $[a, b] < [c, d]$ if $b < c$, to show that there is no bound on the order dimension of interval orders. We then turn to the problem of computing the dimension of $I(0, n)$, showing that $I(0, 10)$ has dimension 3 but $I(0, 11)$ has dimension 4. We use these results as initial conditions in obtaining an upper bound on the dimension of $I(0, n)$ as a logarithmic function of n. It is our belief that this example is a "canonical" example for interval orders, so that the computation of its dimension should have significant impact on the problem of computing the dimension of interval orders in general.

1. INTRODUCTION

A finite interval order may be regarded as a collection of closed intervals of real numbers with the ordering $[a, b] < [c, d]$ if $b < c$. Fishburn [4] has shown how interval orders arise naturally in the theory of measurement. Dushnik and Miller [3] defined the *dimension* of a partial ordering P of a set X as the smallest number of linear orderings of X whose intersection is P, and Ore has noted that the dimension of P is also the smallest

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number of linearly ordered sets such that P is the restriction of the product ordering of these linearly ordered sets to some subset of the cartesian product of the sets [5]. Rabinovitch [6] has shown that the dimension of an interval order is bounded by 1 plus the base 2 logarithm of its height and has given a complete description of the dimension theory of semi-orders (interval orders in which all the intervals have equal length), showing among other things that the dimension of a semiorder is at most 3.

2. The Growth of Dimension

We use Ramsey's theorem to show that the dimension of $I(0, n)$ can be made as large as desired by choosing a sufficiently large n. Suppose that $I(0, n)$ may be realized as an intersection of k linear orders, $L_1, L_2, \ldots, L_k$. Partition the three-element sets of $\{0, 1, \ldots, n\}$ into k (or perhaps fewer) classes by the rule:

> The set $\{r, s, t\}$ (with $r < s < t$) is placed in class i if L_i is the first linear order in which the interval $[s, t]$ is less than the interval $[r, s]$.

Note that each three-element subset of $\{0, 1, 2, \ldots, n\}$ is placed in a class because if $r < s < t$, then $[r, s]$ and $[s, t]$ are incomparable, so in some one of the L_i's $[s, t]$ will be less than $[r, s]$.

We make use of the following form of Ramsey's theorem.

"For each triple of integers j, k, m there exists a number n_0 so that if $n > n_0$ and the m element subsets of an n element set are partitioned into k (or fewer) parts, there exists a j element subset of the n element set, all of whose m element subsets lie in one of the parts."

For our application we choose $m = 3$, $j = 4$, and k as above. Then for $n > n_0$, the guaranteed integer, there is a four-element set $\{r, s, t, u\}$ $(r < s < t < u)$ all of whose three-element subsets lie in the same class. Thus in one linear extension L_i of $I(0, n)$, $[r, s]$ is greater than $[s, t]$ which is greater than $[t, u]$, and this is impossible since $s < t$ implies that $[r, s] < [t, u]$ in $I(0, n)$, and thus in L_i. In other words, for $n > n_0$, more than k linear orders will be needed to realize $I(0, n)$. This gives our first theorem.

Theorem 1. *For each k there is an $n(k)$ such that for $n > n(k)$, the dimension of $I(0, n)$ is greater than k.*

A *semiorder* may be thought of as an interval order all of whose intervals have the same length. Semiorders are the orders which arise naturally in studying the concept of "just noticeable difference" [1]. Thus semiorders are natural generalizations of linear orderings. In fact Rabinovitch [6] has shown that a semiorder has dimension at most 3. The argument of Theorem 1 may be modified to show that there is no k such that $I(0, n)$ is an intersection of or is imbedded in a product of k semiorders for all n.

3. An Example

The use of Ramsey's theorem in the last section might make it appear that the computation of the dimension of an interval order is of the same level of difficulty as the computation of Ramsey numbers. Though we cannot yet compute the dimension of $I(0, n)$ for all n we feel the problem is not hopeless. In this section we shall prove that $I(0, 11)$ has dimension 4 and exhibit three linear orders whose intersection is $I(0, 10)$. First we note that $I(0, 3)$ has dimension 3, for it contains one of Trotter's seven-element partially ordered sets of dimension 3, the one shown in Fig. 1 [7].

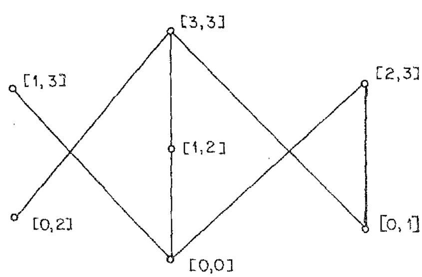

Figure 1

Thus $I(0, n)$ has dimension at least 3 for $n \geqslant 3$. To prove that the dimension of $I(0, 11)$ is 4, we consider a partition of it into three parts, $I_1 = I(6, 11)$, $I_2 = I(0, 5)$ and $I_3 = I(0, 11) - (I_1 \cup I_2)$. Each element of I_3 is incomparable with something in I_1 and something in I_2, for the intervals in I_3 have left endpoints 5 or less and right endpoints 6 or more. An element of I_3 cannot simultaneously be below an element of I_2 and above an element of I_1 in a linear extension of $I(0, 11)$, for each element of I_2 is below each element of I_1. Thus if $I(0, 11)$ is an intersection of three linear orders, then each element of I_3 must either be below everything in I_2 with which it is incomparable in one of these orders, or must be above everything in I_1 with which it is incomparable in one of these orders.

Suppose now that in a linear extension of $I(0, 11)$, the interval $[a, b]$ of I_3 is ordered below everything it is incomparable with in I_2. Then every interval of I_2 whose right-hand endpoint is greater than or equal to a is above $[a, b]$ in this linear order, and every interval whose right-hand endpoint is less than a is below $[a, b]$ in this linear order.

When each interval in a set X with right-hand endpoint e or greater is above each interval with right-hand endpoint less than e in a linear ordering of X, we say the ordering *splits X at right-hand endpoint e.* Similarly if each interval with left-hand endpoint e or less is listed below each interval with left-hand endpoint greater than e by a linear ordering of X, we say the linear ordering *splits X at left-hand endpoint e.* In this terminology, if $[a, b]$ is below everything in I_2 with which it is incomparable in a linear order L, then L splits I_2 at right-hand endpoint a. In what follows a linear ordering of X is regarded as a list of the elements of X. We say x is listed above y if x is greater than or equal to y in the linear ordering.

Our computation of the dimension of $I(0, 11)$ begins with the following lemma.

LEMMA 1. *If the linear order L and two other linear orders may be intersected to yield $I(0, 4)$, then L does not split $I(0, 4)$ at all of the right-hand endpoints* 1, 2, 3 *and* 4.

Proof. Suppose, contrary to the lemma, that L does split $I(0, 4)$ at right-hand endpoints 1, 2, 3, and 4. Then in two linear orderings L_1 and L_2, we must put the first element of each of the following pairs of intervals over the second.

[0, 2],	[1, 3],
[1, 3],	[3, 4],
[0, 1],	[1, 3],
[0, 0],	[0, 1],
[0, 1],	[1, 2],
[1, 2],	[2, 3],
[2, 3],	[3, 4],
[0, 2],	[2, 3],

Since for any two successive rows above we cannot put the first interval of each row over the second in the same linear order, [0, 2] must be over [1, 3] in one of L_1 or L_2, and [0, 2] must be over [2, 3] in the other. Thus [0, 0] cannot be over [0, 2] in either of L_1 or L_2—and also not in L. This proves the lemma.

LEMMA 2. *If L_1, L_2, and L_3 are three linear orderings whose intersection is $I(0, 4)$, then it is impossible to have L_1 split $I(0, 4)$ at right-hand endpoint* 4 *and L_2 split $I(0, 4)$ at right-hand endpoints* 1, 2, *and* 3.

Proof. Suppose to the contrary that L_1 and L_2 split $I(0, 4)$ as described above. This determines many of the comparisons made by L_3. In particular [0, 1] must be listed above [1, 4] in L_3 and [1, 2] must be listed above [2, 4] in L_3. However, since [0, 1] cannot be listed above [2, 4] and L_2 splits $I(0, 4)$ at right-hand endpoint 2, [0, 1] must be listed above [1, 2] in L_1. However, then there is no list in which [0, 0] may be listed above [0, 1], a contradiction.

LEMMA 3. *If L_1, L_2 and L_3 are three linear orderings whose intersection is $I(0, 4)$ then it is impossible for L_1 to split $I(0, 4)$ at right-hand endpoints 2 and 4 and for L_2 to split $I(0, 4)$ at right-hand endpoints 1 and 3.*

Proof. If three lists as described above exist then in L_3 [0, 0] must be listed above [0, 2] and [0, 2] must be listed above [2, 4], and this is clearly impossible.

LEMMA 4. *If L_1, L_2 and L_3 are three linear orderings whose intersection is $I(0, 5)$ then there is at least one right-hand endpoint at which none of L_1, L_2 and L_3 split $I(0, 5)$.*

Proof. If L_1, L_2, and L_3 were each to split $I(0, 5)$ in at least one right-hand endpoint, then [0, 0] would not be listed over [0, 4] in any of the lists. Now four or more splits other than the trivial split at zero cannot be made in one list by Lemma 1. Thus if among L_1, L_2, and L_3 there are five distinct splits (other than the trivial split at zero) at right-hand endpoints, they must be distributed in two lists, say three in list 1 and 2 in list 2. Now unless the three splits in list 1 are at endpoints 5, 4, and 3 and the two in list 2 are at endpoints 2 and 1, we may remove enough elements from $I(0, 5)$ to get an isomorphic copy of $I(0, 4)$ and three lists that violate either Lemma 2 or Lemma 3. Now assume list 1 splits $I(0, 5)$ at right-hand endpoints 3, 4 and 5 and list 2 splits $I(0, 5)$ at right-hand endpoints 1 and 2. Then [0, 0] cannot be listed over [0, 3] in either list 1 or 2 so it must be listed over [0, 3] in list 3. Then [0, 3] cannot be listed over [1, 4] in either list 1 or list 3, so it must be listed over [1, 4] in list 2. However [1, 1] cannot be listed over [1, 4] in either list 2 or list 1, thus [1, 1] must be listed over [1, 4] in list 3. Thus there is no list in which [1, 4] may be listed above [4, 5]. This proves the Lemma.

THEOREM 2. *The dimension of $I(0, 11)$ is* 4.

Proof. Recall that $I_1 = I(6, 11)$, $I_2 = I(0, 5)$ and $I_3 = I(0, 11) - (I_1 \cup I_2)$. In particular, I_3 consists of all intervals whose left-hand endpoints are less than or equal to 5 and whose right-hand endpoints are greater than or equal to 6. Suppose that there are three linear orders L_1, L_2 and L_3 whose intersection is $I(0, 11)$. We say an element of I_3 "goes down" in L_i if in L_i this element is below some element of I_2. An element of I_3 "goes up" in L_i if in L_i this element is above some element I_1. If $[a, b]$ goes down in exactly one L_i, then in that L_i, it must get below everything in I_2 with which it is incomparable. Thus this L_i splits I_2 at right-hand endpoint a. Similarly if $[a, b]$ goes up in exactly one L_j, then L_j splits I_1 at left-hand endpoint b.

Now suppose a set of intervals of I_3 contains an interval with left-hand endpoint a for $a = 1, 2, 3, 4, 5$. Then by Lemma 4 at least one element of this set must go down in two different linear orderings L_i. Similarly if a set of intervals of I_3 contains an interval with right-hand endpoint b for $b = 6, 7, 8, 9, 10$ then at least one of the intervals in this set must go up in two different linear orderings L_i. We construct a set S as follows. For each $b = 6, 7, 8, 9, 10$ some element of the set

$$\{[1, b], \quad [2, b], \quad [3, b], \quad [4, b], \quad [5, b]\}$$

must go down in two different L_i's. Let S consist of one such interval for each b. Since S contains an $[a, b]$ for all $b = 6, 7, 8, 9, 10$, some element of S must go up in two different L_i's. This is not possible for there are only 3 L_i's by hypothesis. Thus the dimension of $I(0, 11)$ must be at least 4. For many reasons, one of which appears in Section 4, the dimension can be no more than 4, so the theorem is proved.

[0, 4]	[0, 4]	[4, 4]
[1, 4]	[4, 4]	[3, 3]
[2, 4]	[3, 3]	[1, 4]
[3, 4]	[0, 3]	[2, 3]
[4, 4]	[3, 4]	[1, 3]
	[1, 2]	[3, 4]
[0, 3]	[0, 2]	[2, 2]
[1, 3]	[2, 4]	[2, 4]
[2, 3]	[2, 2]	[0, 1]
[3, 3]	[2, 3]	[1, 1]
	[0, 1]	[1, 2]
[0, 2]	[1, 1]	[0, 0]
		→ ←
[1, 2]	[1, 3]	[0, 2]
[2, 2]	[1, 4]	[0, 3]
		[0, 4]
[1, 1]	[0, 0]	
[0, 0]		
→ ←		
[0, 1]		

FIG. 2. Three special linear orders that realize $I(0, 4)$.

In fact the computations made in the proofs of the lemmas suggested to us the three linear orderings described as lists in Fig. 3 whose intersection is $I(0, 10)$. The elements of $I(0, 4)$ are listed in the order in which they are shown in Fig. 2; the elements of $I(6, 10)$ are listed according to a similar linear order of $I(6, 10)$. (To get the orderings of $I(6, 10)$, reverse the roles of left- and right-hand endpoints in the orderings at $I(0, 4)$, and interchange ordering 1 with ordering 3.)

Linear order of	[10, 10]	[l, 10]'s
$I(6, 10)$ with no splits		[9, 10]
	[l, 9]'s with [1, 9] on top	
[5, j]'s and [1, j]'s	Rest of $I(6, 10)$	[10, 10]
other than [1, 9]		[9, 9]
	All things incomp. with	[l, 8]'s
[k, 4]'s	[5, 5] except [1, j]'s and	[8, i]'s
[4, j]'s	[l, 9]'s	[l, 7]'s
[k, 3]'s		[7, i]'s
[3, j]'s	All of $I[0, 4]$ but [0, 0]	[l, 6]'s
[k, 1]'s		[6, i]'s
[2, j]'s	All [1, j]'s but [1, 9]	
[1, 1]		[l, 5]'s and [l, 9]'s except
[1, 9]	[0, 0]	[1, 9]
[0, 0]		
[0, 1]		All of $I(0, 4)$ but [0, 0],
[0, j]'s		[0, 4], [0, 3] and [0, 2]
		[1, 9]
		[0, 0]
		[0, 2]
		[0, 3]
		[0, 4]

FIG. 3. Three special linear orders that realize $I(0, 10)$ (convention $j \geqslant 5$; $k \leqslant 4$, $i \geqslant 6$, $l \leqslant 5$).

4. An Upper Bound on Dimension

Suppose now that, for each k, $n(k)$ denotes the largest n such that the dimension of $I(0, n)$ is k. For example $n(1) = 0$, since $I(0, 1)$ has an antichain with two elements [0, 0] and [0, 1]. Also, $n(2) = 2$, for $I(0, 2)$ has six elements and is not one of the examples in [2], and $I(0, 3)$ contains the three-dimensional interval order first exhibited in [7] and reproduced Fig. 1. In Section 3 we saw that $n(3) = 10$. We shall use a recursion on $n(k)$ to obtain a bound on the dimension of $I(n)$. For this purpose we need a lemma from [6].

LEMMA (*Rabinovitch*). *If A and B are disjoint subsets of an interval order I then there is a linear extension of I in which each element of B is above each element of A with which it is incomparable relative to I.*

We use the ambiguous notation $A \leftarrow B$ for an order on $A \cup B$ of the type described in the lemma; this notation is ambiguous because there may be several orders of the type the lemma describes. Since any of these orders will do for our purpose we will make no attempt to be specific. Conceptually it is convenient to think of this notation in two different ways; the second way amounts to a change in notation. The notation $A \rightarrow B$ denotes a linear extension of the restriction of the interval order to $A \cup B$ in which each element of A is below each element of B with which it is incomparable.

A second notion from [6] is the following.

> (**) If x and y are each above and below exactly the same elements of $X - \{x, y\}$ in a poset (X, P), then the restriction of P to $X - \{x\}$ and the restriction of P to $X - \{y\}$ have the same dimension. (This is also the dimension of P unless P linearly orders $X - \{x\}$ or $X - \{y\}$.)

The lemma below gives us a recursive lower bound on $n(k)$ which in turn will give us an upper bound on the dimension of $I(0, n)$. In the proof of the lemma if P and Q are linear orderings of disjoint sets, the notation PQ means the linear ordering of the union of the sets which places each element in the domain of P before each element in the domain of Q and agrees with P and Q on their domains.

LEMMA 5. $n(k) \geqslant 2n(k-1) + n(k-2)$.

Proof. We must show that the dimension of $I = I(0, 2n(k-1) + n(k-2))$ is at most k. We split I into three parts:

$$I_1 = I(n(k-1) + n(k-2), 2n(k-1) + n(k-2)),$$
$$I_2 = I(0, n(k-1)),$$
$$I_3 = I - (I_1 \cup I_2).$$

The ordering of I_1, I_2, and I_3 is that inherited from I. The dimension of I_1 and I_2 is $k-1$. For each element x in I_3 there is a y in $I(n(k-1), n(k-1) + n(k-2))$ which is above and below exactly the same things in I_3 that x is. Thus by the remark (**) above, I_3 has the same dimension as $I(0, n(k-2))$; that is I_3 has dimension $k-2$.

Let $L(1, 1), L(1, 2), \ldots, L(1, k-1)$ be a set of linear orders whose intersection is I_1, let $L(2, 1), L(2, 2), \ldots, L(2, k-1)$ be a set of linear orders whose intersection is I_2 and let $L(3, 1), L(3, 2),..., L(3, k-2)$ be a set of linear orders whose intersection is I_3. We claim that the k linear orderings given by

$$\begin{gathered} L(1, 1)\, L(3, 1)\, L(2, 1) \\ \vdots \\ L(1, k-2)\, L(3, k-2)\, L(2, k-2) \\ L(1, k-1)\, I_3 \to I_2 \\ I_1 \leftarrow I_3 L(2, k-1) \end{gathered}$$

intersect to give I. Since they are clearly extensions of I we need only prove that if two elements x and y of I are incomparable then x is over y in some list and y is over x in some list. It is not possible for one of x and y to be in I_1 and the other in I_2, and if both are in I_1 or in I_2 or in I_3, then x is over y in one of the linear orderings. Thus we may assume one of x or y is in I_3 and the other is in I_1 or I_2. Suppose x is in I_1 and y is in I_3. Then x is over y in the first linear ordering, and y is over x in the last one. The other possibilities are covered similarly, and thus the dimension of I is at most k.

Now we apply the lemma to get the following theorem.

THEOREM 3. $\dim I(0, n) < \log_a(n) + \frac{1}{2}$ *where* $a = 1 + 2^{1/2}$.

Suppose $m(k) = 2m(k-1) + m(k-2)$ and $n(2) = m(2)$ and $n(3) = m(3)$. Then by the lemma, $n(k) \geqslant m(k)$ for $k \geqslant 2$. Substituting $m(k) = a^k$ gives

$$a^2 - 2a - 1 = 0,$$

or

$$a = 1 \pm 2^{1/2}.$$

Thus

$$m(k) = C_1(1 + 2^{1/2})^k + C_2(1 - 2^{1/2})^k.$$

Since $n(2) = 2$ and $n(3) = 10$, we have

$$C_1 = 4(2)^{1/2} - 5, \qquad C_2 = -4(2)^{1/2} - 5.$$

Thus

$$n(k) \geqslant (4(2)^{1/2} - 5)(1 + 2^{1/2})^k - (4(2)^{1/2} + 5)(1 - 2^{1/2})^k.$$

By rearranging terms and taking logarithms it is possible to show that for $k \geqslant 4$, if $a = 1 + 2^{1/2}$

$$\log_a(n(k)) > k - \tfrac{1}{2}.$$

(In fact, by using the computer it is possible to show that for $k = 4$

$$n(k) - a^{k-1/2} < 0.004,$$

so that this bound is very sharp indeed.) Thus if $n(k+1) > n \geqslant n(k)$,

$$\dim(I(0, n)) = k < \log_a(n(k)) + \tfrac{1}{2} \leqslant \log_a(n) + \tfrac{1}{2}.$$

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