Planar Graph Coloring with an Uncooperative Partner

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ABSTRACT

We show that the game chromatic number of a planar graph is at most 33. More generally, there exists a function f: $\mathbb{N} \to \mathbb{N}$ so that for each $n \in \mathbb{N}$, if a graph does not contain a homeomorph of K_n, then its game chromatic number is at most $f(n)$. In particular, the game chromatic number of a graph is bounded in terms of its genus. Our proof is motivated by the concept of p-arrangeability, which was first introduced by Guantao and Schelp in a Ramsey theoretic setting.

1. INTRODUCTION

Let $\mathbf{G} = (V, E)$ be a finite graph, and let X be a set whose elements will be referred to as *colors*. A function c: $V \to X$ is called a *proper coloring* (or just *coloring* for short) if $c(x) \neq c(y)$ whenever x and y are distinct nodes from V with $xy \in E$. If $|\{c(x): x \in V\}| = t$, the coloring c is also called a t-*coloring*. The *chromatic number* of $\mathbf{G}$, denoted $\chi(\mathbf{G})$, is the least positive integer t for which there exists a coloring c of $\mathbf{G}$ using a set X with $|X| = t$ as the set of colors.

In this paper, we will be concerned primarily with planar graphs. Because it is important to the spirit of the results that follow, we note that there is an elementary (and very fast) algorithm for coloring a planar graph with

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6 colors. By Euler's formula, a planar graph always has a node of degree at most 5. Given a graph $\mathbf{G} = (V, E)$ with n nodes, we can then label the nodes $x_1, x_2, \ldots, x_n$ so that for each $i = 2, 3, \ldots, n$, there are at most 5 neighbors of x_i in the set $\{x_j\colon 1 \le j < i\}$. The graph can then be 6-colored by applying First-Fit to the nodes in the order of their subscripts in this labeling, i.e., a node is colored with the least positive integer distinct from the colors given to those neighbors that precede it in the labeling.

We now consider a modified graph coloring problem posed as a two-person game, with one person (Alice) trying to color a graph and the other (Bob) trying to prevent this from happening. Let $\mathbf{G} = (V, E)$ be a graph, let t be a positive integer, and let X be a set of colors with $|X| = t$. Alice and Bob compete in a two-person game lasting at most $n = |V|$ *moves*. They alternate turns, with Alice having the first move. A move consists of selecting a previously uncolored node x and assigning it a color from X distinct from the colors assigned previously (by either player) to neighbors of x. If after n moves, the graph is colored, Alice is the winner. Bob wins if an impass is reached before all nodes in the graph are colored, i.e., for every uncolored node x and every color α from X, x is adjacent to a node having color α. The *game chromatic number* of a graph $\mathbf{G} = (V, E)$, denoted $\chi_g(\mathbf{G})$, is the least t for which Alice has a winning strategy. This parameter is well defined, since Alice always wins when $t = |V|$.

Example. Consider the planar graph shown in Figure 1. This graph has game chromatic number 6. To see that the game chromatic number is at least 6, here is a winning strategy for Bob if the set X of colors is $\{1, 2, 3, 4, 5\}$. Note that for each $j = 1, 2, \ldots, 6$, the two-element set $\{a_j, b_j\}$ is a dominating set, i.e., every other node in the graph is adjacent to at least one of these two nodes. Each time Alice colors a node from $\{a_j, b_j\}$, say

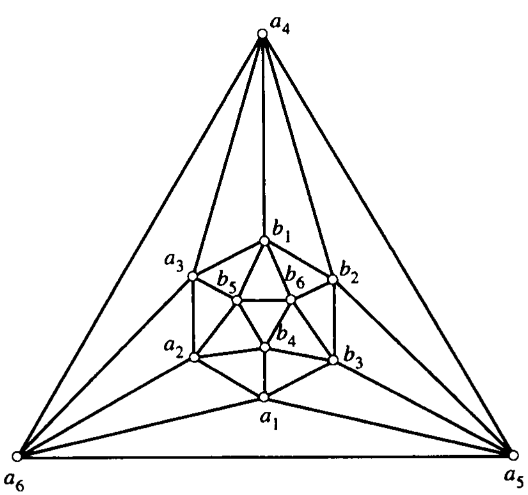

FIGURE 1

with color α, Bob responds by assigning color α to the other node in this set. It follows that α cannot be used by either player to color any other node in the graph. We leave it as an exercise to show that the game chromatic number is at most 6.

The *game chromatic* number of a family $\mathcal{F}$ of graphs, denoted $\chi(\mathcal{F})$, is then defined to be $\max\{\chi_g(G): G \in \mathcal{F}\}$, provided this value is finite; otherwise, we say that $\chi_g(\mathcal{F})$ is infinite.

The concept of game chromatic number was introduced by Bodlaender [1], who showed that the game chromatic number of the family of trees is at least 4 and at most 5. In [6], Faigle, Kern, Kierstead, and Trotter show that the game chromatic number of the family of trees is 4. In this paper, it is also shown that the family of bipartite graphs has infinite game chromatic number.

With these remarks as background, we can now state the principal result of this paper.

1.1 Theorem. The game chromatic number of the family of planar graphs is at most 33. ∎

Furthermore, we will produce a very fast procedure for implementing the winning strategy. As an added bonus, we obtain the following more general result.

1.2 Theorem. There exists a function f: $\mathbb{N} \rightarrow \mathbb{N}$ so that for each $n \in \mathbb{N}$, if a graph does not contain a homeomorph of K_n, then its game chromatic number is at most $f(n)$. ∎

It follows from Theorem 1.2 that there exists a function g: $\mathbb{N} \rightarrow \mathbb{N}$ so that $\mathbf{G}$ is a graph of genus n; then the game chromatic number of $\mathbf{G}$ is at most $g(n)$.

2. ARRANGEABILITY AND RAMSEY THEORY

Let $\mathbf{G} = (V,E)$ be a graph and let L be a linear order on the node set V. For each node $x \in V$, we define the *back degree of x relative to L* as $|\{y \in V: xy \in E \text{ and } x > y \text{ in } L\}|$. The *back degree* of L is then the maximum back degree of the nodes relative to L. The graph $\mathbf{G} = (V,E)$ is said to be k-*degenerate* if there is a linear order L on V that has back degree at most k. If $\mathbf{G}$ is k-degenerate, then $\chi(\mathbf{G}) \leq k + 1$, since First-Fit will use at most $k + 1$ colors when the nodes are processed in the linear order that witnesses that the graph is k-degenerate.

Again, let L be a linear order on the node set V of a graph $\mathbf{G} = (V,E)$, and let $x \in V$. We define the *arrangeability of* x *relative to* L as $|\{y \in V: y \leq x$ in L and there is some $z \in V$ with $yz \in E$, $xz \in E$ and $x < z$ in $L\}|$. In Figure 2, we illustrate a linear order L on the nodes of a graph. In this

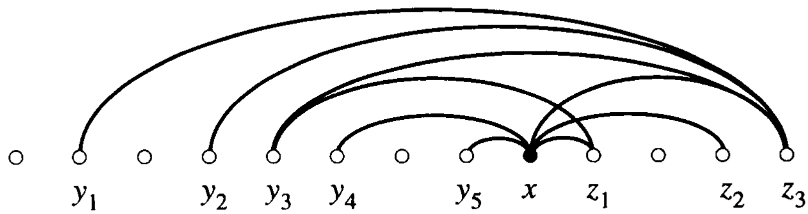

FIGURE 2

example, the back degree of the node x is 2. The set $S = \{x, y_1, y_2, y_3\}$ shows that the arrangeability of x relative to L is 4.

The *arrangeability* of L is then the maximum value of the arrangeability of the nodes relative to L. Following G. Chen and R. Schelp [3], we say that the graph $\mathbf{G}$ is p-*arrangeable* if there is a linear order L on the nodes having arrangeability at most p.

2.1 Proposition. A p-*arrangeable graph* $\mathbf{G} = (V, E)$ is p-*degenerate.*

Proof. A linear order L on V that has arrangeability at most p also has back degree at most p. ∎

We now present a brief discussion of the Ramsey theoretic problems that led Chen and Schelp [3] to introduce and investigate the concept of p-arrangeability. Let $\mathbf{G} = (V, E)$ be a graph. Define the *Ramsey number* of $\mathbf{G}$, denoted $r(\mathbf{G})$, as the least positive integer t so that if the edges of a complete graph K_t on t nodes are colored with two colors, then there is always a monochromatic copy of $\mathbf{G}$. If $|V| = n$, then the Ramsey number $r(\mathbf{G})$ satisfies

$$2n - 1 \leq r(\mathbf{G}) \leq \binom{2n-2}{n-1}. \tag{2.1}$$

The lower bound in this inequality is trivial, and the upper bound is just the well-known bound of Erdös and Szekeres [5] for the Ramsey number $r(\mathbf{K}_n)$. On the one hand, the exponential form of this upper bound is correct in the sense that $r(\mathbf{K}_n) \geq 2^{n/2}$. On the other hand, there are some interesting cases where the lower bound is closer to the truth. Examples include cycles and trees.

Recall that the *arboricity* of a graph $\mathbf{G} = (V, E)$ is the least t so that the edge set E can be partitioned into t forests. The following beautiful conjecture was made 17 years ago by S. Burr and P. Erdös [2].

2.2 Conjecture. For each positive integer a, there exists a positive constant c so that if $\mathbf{G}$ is an n-node graph having arboricity at most a, then the Ramsey number of $\mathbf{G}$ satisfies $r(\mathbf{G}) \leq cn$. ∎

Progress in resolving this conjecture has been slow. However, in 1983, V. Chvatál, V. Rödl, E. Szemerédi, and W.T. Trotter proved [4] a linear bound on the Ramsey numbers of graphs of bounded maximum degree.

2.3 Theorem. For each positive integer d, there is a positive constant c so that if $\mathbf{G}$ is an n-node graph and the maximum degree of $\mathbf{G}$ is at most d, then the Ramsey number of $\mathbf{G}$ satisfies $r(\mathbf{G}) \leq cn$. ▮

In [3], G. Chen and R. Schelp develop an interesting strengthening of Theorem 2.3.

2.4 Theorem. For each positive integer p, there is a positive constant c so that if $\mathbf{G}$ is an n-node graph and $\mathbf{G}$ is p-arrangeable, then the Ramsey number of $\mathbf{G}$ satisfies $r(\mathbf{G}) \leq cn$. ▮

In order to demonstrate that their theorem applied to important examples not covered under Theorem 2.3, Chen and Schelp [3] then proved the following result, which is of primary importance in this paper.

2.5 Theorem. Every planar graph is 761-arrangeable. ▮

Before closing this section, we make three remarks concerning Theorem 2.5 and the concept of arboricity. First, it is not immediately clear to us why planar graphs are p-arrangeable for *any* value of p, regardless of how large p is taken to be. Second, the proof of Theorem 2.3 depends heavily on Szemerédi's regularity lemma [8] that he first used to resolve the Erdös/Turán conjecture: Any subset of the positive integers having positive upper density contains arbitrarily long arithmetic progressions. The regularity lemma has become a much used tool in combinatorics (see [9] for a short proof of the lemma), but it involves constants that are just barely finite. For this reason, Chen and Schelp were not motivated to find the least value of p for which every planar graph is p-arrangeable. Third, the family of bipartite graphs used in [6] to show that the family of bipartite graphs has infinite game chromatic number is also a family of graphs of arboricity 2. So bounded arboricity is not enough to bound the game chromatic number.

3. ARRANGEABILITY AND GRAPH COLORING

Let $\mathbf{G} = (V, E)$ be a graph and let L be a linear order on V. For each node $x \in V$, we say that a subset $S \subseteq V$ is *admissible for* x if (1) $y \leq x$ in L, for every $y \in S$, and (2) there is an injection f that maps the subset $S' = \{y \in S\colon xy \notin E\}$ to V so that $yf(y) \in E$, $xf(y) \in E$ and $x < f(y)$ in L, for every $y \in S'$. The *admissibility of* x *relative to* L is then defined as the maximum size of a subset S that is admissible for x, and the *admissibility* of L is the maximum value of the admissibility of the nodes relative to L.

A graph $\mathbf{G} = (V, E)$ is m-*admissible* if there is a linear order L on V that has admissibility at most m. For the example given in Figure 2, note that the set $S = \{x, y_2, y_3, y_4, y_5\}$ is admissible for x relative to L.

The following results are immediate.

3.1 Proposition. An m-admissible graph is m-degenerate. ∎

3.2 Proposition. A p-arrangeable graph is $2p$-admissible. ∎

3.3 Proposition. An m-admissible graph is $m^2 - m + 1$-arrangeable. ∎

Throughout the remainder of the paper, we let $[q]$ denote the set $\{1, 2, \ldots, q\}$. The next theorem explains why the concepts of arrangeability and admissibility are important in the adversarial graph coloring environment.

3.4 Theorem. Let $\mathbf{G} = (V, E)$ be an m-admissible graph, and let $\chi(\mathbf{G}) = r$. Then the game chromatic number of $\mathbf{G}$ is at most $rm + 1$.

Proof. We take the set X of colors as $\{*\} \cup \{(\alpha, j): \alpha \in [r], j \in [m]\}$. As the game is played, we will denote the color assigned to a node x be denoted by $g(x)$. When $g(x) = (\alpha, j)$ for some $j \in [r]$, we let $g_1(x) = \alpha$.

We now describe a winning strategy for Alice. This strategy is given in terms of a decision process for selecting a node to color and for then coloring it with one of the colors from X. The fact that the strategy results in a win for Alice requires us to prove that this decision process always results in a legitimate assignment. Alice's strategy is based on a fixed r-coloring c of $\mathbf{G}$ and a linear order L of the node set that has admissibility at most m.

In the remainder of the argument, we will describe a number of different subsets of the set of nodes. In order to assist the reader in keeping track of these sets, they will be defined as acronyms of capital letters.

At some intermediate point in the game, we let C denote the set of *colored* nodes, and we let U denote the set of *uncolored* nodes. For each node x, let $P(x)$ denote the set of nodes that *precede* x in L, let $F(x)$ denote the set of nodes that *follow* x in L, and let $N(x)$ denote the set of *neighbors* of x. Then let $CN(x) = C \cap N(x)$, let $PN(x) = P(x) \cap N(x)$, $PU(x) = P(x) \cap U$, $PUN(x) = P(x) \cap U \cap N(x)$, etc.

For $y \in C$, let $T(y) = \{x \in UN(y)$: if $g(y) \neq *$, then $g_1(y) = c(x)\}$, and for $x \in U$, let $D(x) = \{y \in CN(x)$: if $g(y) \neq *$, then $g_1(y) = c(x)\}$.

On her first turn, Alice colors the L-least node v with the color $(c(v), 1)$. At each succeeding turn, she selects the node she will color as follows:

Selection Rule. Let v denote the node colored by Bob on his last turn. If $PT(v) \neq \emptyset$, let x denote the L-least node in $PT(v)$. If $PT(v) = \emptyset$, let x be the L-least node in U.

Next we describe a scheme for coloring the selected node. Let x be the node that will be colored, and let $c(x) = \alpha$.

Coloring Rule. If $CN(x)$ contains a subset $\{z_1, \ldots, z_m\}$ and these nodes have already been assigned distinct colors from $\{(\alpha, 1), \ldots, (\alpha, m)\}$, assign color $*$ to x. Otherwise, assign color (α, j) to x, where j is the least positive integer for which x is not adjacent to a node assigned color (α, j).

We will now show that these rules yield a legitimate color for the node Alice has selected. In fact, we show something stronger. We show that the same rules could—on any turn—be used by either player, under the assumption that they have always been followed by Alice. This will show that neither player can ever be trapped without a legal move. First, observe that by following the rules given above, a player is avoiding (almost all) conflict with color assignments made previously by Alice. This is accomplished by the expedient of using a two-coordinate color where the first coordinate is taken from a proper r-coloring of **G**. A player using this coloring scheme must avoid conflicts with nodes colored by Bob and nodes that one of the two players has colored $*$.

Claim. At any stage in the game, if the next player uses the Selection Rule to determine a node x to be colored and Alice has consistently followed both the Selection Rule and the Coloring Rule at each preceding turn, then

$$|PUN(x) \cup D(x)| \leq m\,.$$

Proof. We proceed by induction on the number of turns. The base step is trivial since there are no colored nodes at the start of the game. Now consider the inductive step. Let x be the uncolored node that has been selected, and let $c(x) = \alpha$. Since Alice has always used the Selection Rule and the Coloring Rule, we know that all the nodes in $D(x)$ except possibly those assigned color $*$ have been colored by Bob. We now show that any node in $FD(x)$ has been colored by Bob.

Suppose to the contrary that Alice colored some node $y \in FD(x)$ with $*$. Since Alice has consistently played by the rules, we know that at the moment she colors y, it is adjacent to at least m nodes in $D(y)$. Note that $x \in PUN(y)$. Thus $|PUN(y) \cup D(y)| > m$, which contradicts the inductive hypothesis.

For each $z \in FD(x)$, let y_z denote the element that Alice chooses to color immediately after Bob colors z. Then let $Y = \{y_z \colon z \in FD(x)\}$. Note that if $y \in Y$, then Alice has colored y, unless $y = x$. In particular, $Y \cap PUN(x) = \varnothing$. We now show that $Y \cap PD(x) = \varnothing$.

Suppose to the contrary that $y \in Y \cap PD(x)$. Choose $z \in FD(x)$ so that $y \neq y_z$. Since $y \in D(x)$ and Alice colored y, we know that Alice assigned it color $*$. If Bob assigned color $*$ to z, then our inductive hypothesis would have been violated when Bob colored z, so we may assume that $g(z) \neq *$. Since $z \in D(x)$, we conclude that $g(z) = (\alpha, j)$, for some $j \in [r]$. Now the fact that Alice chose to color y rather than x implies that $c(y) = \alpha$. This is a contradiction, since x and y are adjacent.

Thus $|PUN(x) \cup D(x)| = |PUN(x) \cup PD(x) \cup Y| \leq m$, since $|FD \cdot (x)| = |Y|$ and $PUN(x) \cup PD(x) \cup Y$ is an admissible set for x. ∎

To complete the proof of our theorem, we need only remark that if a player uses the Selection Rule to choose x, then the Coloring Rule can always be used to provide x with a legal color. ∎

4. PLANAR GRAPHS ARE 8-ADMISSIBLE

In this section, we present the following theorem, which also yields an improvement of the bound in Theorem 2.5.

4.1 Theorem. Let $\mathbf{G} = (V, E)$ be a planar graph. Then there is a linear order L of node set V that has back degree at most 5, admissibility at most 8 and arrangeability at most 10.

Proof. Fix a plane drawing of $\mathbf{G}$ that has no edge crossings. We will define the linear order L as a labeling $x_1, x_2, \ldots, x_n$ of the nodes in V. The definition proceeds in reverse order and begins with the choice of x_n as a node of degree at most five in $\mathbf{G}$. Note that the admissibility of x_n is at most 5 and the arrangeability of x_n is 0.

At step i, we assume that we have chosen nodes $x_{i+1}, x_{i+2}, \ldots, x_n$, and that each of these nodes have back degree at most 5, admissibility at most 8 and arrangeability at most 10. Next, we describe how the node x_i is chosen. The ordering on the nodes $x_1, x_2, \ldots, x_6$ is arbitrary so we may assume that $i > 7$.

We call the nodes in the set $C = \{x_{i+1}, x_{i+2}, \ldots, x_n\}$ the *chosen* nodes, and we let $U = V - C$ denote the *unchosen* nodes. Of course, the node x_i will be chosen from U. We refer to the edges in E as *real* edges. Let $\mathbf{G}' = (V, E')$ denote the planar graph obtained by removing all real edges with both end points in C. For each $z \in C$, let U_z denote the unchosen neighbors of z, and let $d_z = |U_z|$. Then let $C' = \{z \in C\colon d_z \geq 2\}$.

For each $z \in C'$, we let $A_z = \{e_z(x,y)\colon x, y \in U_z, x \neq y\}$. Then set $A = \cup\{A_z\colon z \in C'\}$. We refer to the elements in A as *artificial* edges, and we consider $e_z(x,y)$ as an edge having x and y as end points. We intend to distinguish between the edges $e_z(x,y)$ and $e_{z'}(x,y)$ whenever $z \neq z'$. Of course, we also distinguish between real and artificial edges.

For each $z \in C'$, we label the nodes in U_z as $u_1(z), u_2(z), \ldots, u_d(z)$, where $d = d_z$ and the labeling proceeds in clockwise order around z (see Figure 3). The choice of the starting node $u_1(z)$ is arbitrary.

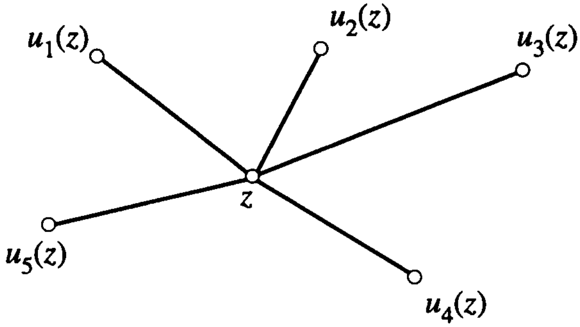

FIGURE 3

Then let $SA(z) = \{e_z(u_j(z), u_{j+1}(z))\colon j \in [d]\}$. Of course, we intend this definition to be interpreted cyclically so that $u_1(z) = u_{d+1}(z)$. Then let $SA = \cup\{SA(z)\colon z \in C'\}$. Edges in SA are called *strong artificial* edges. As remarked previously, we also intend that $SA(z) \cap SA(z') = \emptyset$ when $z \neq z'$. Now let $\mathbf{G}'' = (V, E' \cup SA)$, and note that $\mathbf{G}''$ is a planar multigraph. In fact, $\mathbf{G}''$ can be drawn so that if $z \in C'$ and $d_z \geq 4$, elements of $\mathbf{G}''$ in the interior of the star-shaped region R_z bounded by the strong artificial edges with end points in U_z are the node z and real edges of the form $xu_j(z)$ (see Figure 4).

Now let $\mathbf{U}_0$ denote the subgraph of $\mathbf{G}''$ induced by the unchosen nodes. By successively removing strong artificial edges that belong to two-sided faces, we obtain a planar multigraph $\mathbf{U}_1$ such that the following holds:

(1) $\mathbf{U}_1$ contains no two-sided faces.
(2) If $z \in C$ and $d_z \geq 4$, then $\mathbf{U}_1$ contains a face F_z whose boundary cycle consists of the nodes from U_z. Furthermore the edges of F_z belong to $E \cup SA$.
(3) If $z, z' \in C$, $d_z \geq 4$ and $d_{z'} \geq 4$, then the face F_z contains z in its interior while z' is in its exterior.

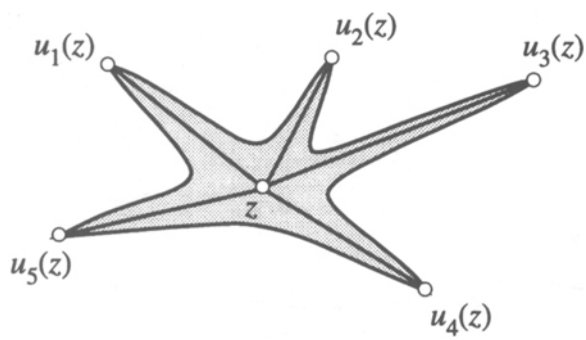

FIGURE 4

For each $z \in C'$ with $d_z \geq 4$ and for each $j \in [d_z]$, let $WA_j(z) = \{e_z(u_j(z), v(z))\colon v \in U_z$ and $v \notin \{u_{j+1}, u_{j-1}\}\}$. We let $WA(z) = \cup\{WA_j(z)\colon j \in [d_z]\}$ and $WA = \cup\{WA(z)\colon z \in C, d_z \geq 4\}$. The edges in WA are called *weak artificial* edges.

Let $E(\mathbf{U}_1)$ denote the edge set of $\mathbf{U}_1$. For each $j \in \{1, 2, \ldots, 5\}$, let $\mathbf{H}_j = (U, E_j)$ be the multigraph with

$$E_j = E(\mathbf{U}_1) \cup \cup\{A_j(z)\colon z \in C, d_z = 5\} \cup \cup\{A_1(z)\colon z \in C, d_z = 4\}.$$

For each $j \in \{6, 7, \ldots, 10\}$, let $\mathbf{H}_j = (U, E_j)$ be the multigraph with

$$E_j = E(\mathbf{U}_1) \cup \cup\{A_{j-5}\colon z \in C, d_z = 5\} \cup \cup\{A_2(z)\colon z \in C, d_z = 4\}.$$

For each $j \in [10]$, the multigraph $\mathbf{H}_j$ is planar. Furthermore, $\mathbf{H}_j$ can be drawn without crossings by inserting each weak artificial edge of the form $e_x(x, y)$ present in $\mathbf{H}_j$ in the star-shaped region R_z, which is always contained inside the face F_j in the drawing of $\mathbf{U}_1$. In Figure 5, we illustrate how this process works for the node z of Figures 3 and 4 in the drawing of $\mathbf{H}_5$. In this instance, the face F_z has been formed by deleting (at least) three strong artificial edges with both end points in U_z.

It is important to note that for such a drawing of $\mathbf{H}_j$, there are no two-sided faces. Now for each $j \in [10]$ and for each $x \in U$, let $\deg_j(x)$ denote the *degree* of x in $\mathbf{H}_j$. Also, let $\mathcal{H} = \{\mathbf{H}_1, \mathbf{H}_2, \ldots, \mathbf{H}_{10}\}$. That each $\mathbf{H}_j \in \mathcal{H}$ is a planar multigraph with no two-sided faces implies that

$$2|E_j| = \sum_{x \in U} \deg_j(x) < 6|U|.$$

It follows that there is a node $x = x_i \in U$ for which

$$\text{total degree}(x) = \sum_{j=1}^{10} \deg_j(x) < 60.$$

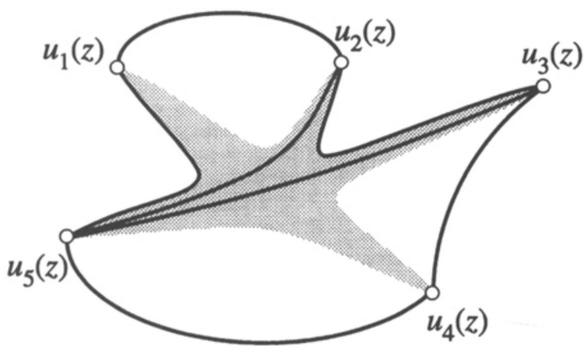

FIGURE 5

We now show that x has back degree at most 5, admissibility at most 8, and arrangeability at most 10. The first statement is trivial since $xy \in E_j$ for every $j \in [10]$ whenever $y \in U$ and $xy \in E$. Let $T = T(x) = \{y \in U\colon xy \in E\} \cup \{y \in U\colon$ there is some $z \in C'$ with $e_z(x,y) \in SA(z)\}$ and let $Z = Z(x) = \{z \in C\colon d_z \geq 4, x \in U_z\}$. Our arguments for the last two statements are simplified by the following observation concerning the relative sizes of these two sets.

Claim. $|T| \geq |Z|$.

Proof. For each $z \in Z$, the edge xz lies (except for the end point x) entirely in the interior of the face F_z whose boundary edges belong to $E \cup SA$. Each such F_z contains 2 boundary edges incident with x, and these edges are either real or strong artificial edges. Since an edge is a boundary edge of at most two faces, the claim follows. ∎

Now let S be an admissible set for x. We show that $|S| \leq 8$; in fact, we show that $|S - \{x\}| \leq 7$. Note that any node in $S - \{x\}$ is adjacent to x by a real edge or an artificial edge. Let $S_1 = \{y \in S - \{x\}\colon xy \in E\} \cup \{y \in S - \{x\}\colon$ there is some $z \in C$ with $d_z \geq 4$ and $e_z(x,y) \in SA(z)\}$. Note that $S_1 \subseteq T$. Let $S_2 = \{y \in S - S_1\colon y \neq x\}$. For each $y \in S_2$, there is a unique $z = z_y \in Z$ with $yz \in E$. It follows that the weak artificial edge $e_z(x,y)$ appears in at least 4 of the 10 planar multigraphs. This implies that the total degree of x is at least $10|T| + 4(|S_2|$, so that $10|T| + 4|S_2| < 60$. Using the claim, we know that $|T| \geq |Z| \geq |S_2|$. These two inequalities imply that $|T| + |S_2| \leq 7$, so that $|S| \leq 8$.

We now show that the arrangeability of x is at most 10. Let S be an arrangeable set for x. As before, let $S_1\{y \in S - \{x\}\colon xy \in E\} \cup \{y \in S - \{x\}\colon$ there is some $z \in C$ with $d_z \geq 4$ and $e_z(x,y) \in SA(z)\}$. Also, let $S_2 = \{y \in S - S_1\colon y \neq x\}$.

For each $z \in Z$, there are at most 2 nodes in $U_z \cap S_2$. It follows that $|S_2| \leq 2|Z|$. By the claim, $|S_2| \leq 2|T|$. Since the total degree of x is at least $10|T| + 4|Z|$, it follows that $|T \cup S_2| \leq 9$. Since $S \subseteq \{x\} \cup T \cup S_2$, we conclude that $|S| \leq 10$. This completes the proof of our theorem. ∎

4.2 Corollary (Theorem 1.1). The game chromatic number of planar graph is at most 33. ∎

4.3 Corollary. If **G** is an outerplanar graph, then there exists a linear order L on the node set that has back degree at most 2, arrangeability at most 3, and admissibility at most 3.

Proof. It is straightforward to modify the proof of Theorem 4.1 to construct the desired linear order. In fact, in this case, we do not even have to consider a family of outerplanar multigraphs. By maintaining the property that the back degree of L is at most 2, all artificial edges are strong. ∎

4.4 Corollary. The game chromatic number of an outerplanar graph is at most 10. ∎

5. LOWER BOUNDS

In this section, we produce lower bounds on the game chromatic number, admissibility, and arrangeability of planar graphs. The result for admissibility is tight, and the gap for arrangeability is modest, but we leave a relatively large gap with our lower bound on game chromatic number.

In Section 1, we gave an example of a planar graph with game chromatic number 6. We can do just a bit better.

5.1 Theorem. The game chromatic number of the class of planar graphs is at least 7.

Proof. Consider the planar graph $\mathbf{G}_1$ shown in Figure 6. This diagram is intended to suggest that for each pair selected from $\{a, b, c\}$, there are 7 common neighbors of degree 2.

Then form a planar graph $\mathbf{G}_2$ by taking two copies of $\mathbf{G}$ and identifying the nodes labeled w_1. Then form $\mathbf{G}_3$ by taking two disjoint copies of $\mathbf{G}_2$.

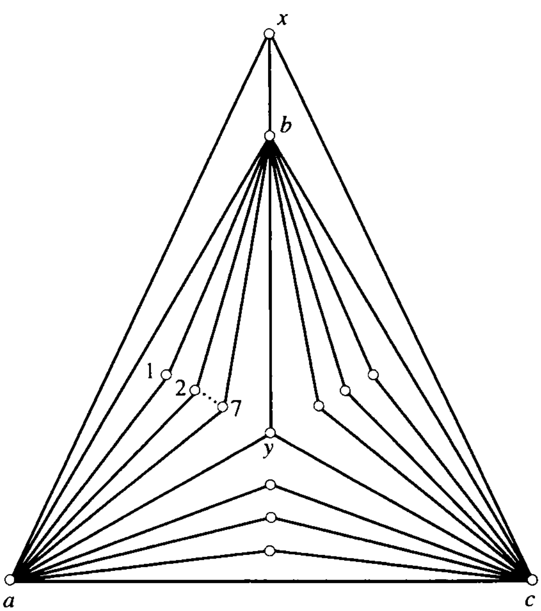

FIGURE 6

We show that the game chromatic number of $\mathbf{G}_3$ is at least 7 by providing a winning strategy for Bob when the colors come from [6].

On her first move, Alice colors a node from one of the copies of $\mathbf{G}_2$. Bob will color only in the other copy of $\mathbf{G}_2$, and on his first turn assigns color 1 to the node x. Alice makes her second move, and Bob then selects one of the copies of $\mathbf{G}_1$ containing the node he colored on his first turn, but containing no nodes colored by Alice. In this copy, Bob assigns color 2 to y. Note that the nodes in $T = \{a, b, c\}$ form a triangle, and each node in T is adjacent to both x and y. Let $S = \{3, 4, 5, 6\}$. As the game progresses, we will delete nodes from T and colors to S. Also, it will always be the case that the nodes in T must be assigned colors from S.

Alice now makes her third move. Bob's strategy unfolds as follows.

(1) Bob will never color a node from T. Until Alice colors a node from T, Bob colors nodes of degree 2 adjacent to both a and b with distinct colors from S. Clearly, Alice must eventually color a node from T.
(2) When Alice first assigns color α to a node u from T, Bob assigns a color $\beta \in S - \{\alpha\}$ to one of the nodes of degree 2 adjacent to the other two nodes in T. Let $S = S - \{\alpha, \beta\}$, and $T = T - \{u\}$. Note that $|T| = |S| = 2$, and that the two nodes in T must be assigned colors from S.
(3) Now Bob colors nodes of degree 2 that are adjacent to both nodes of T with distinct colors from S until Alice colors a node from T.
(4) When Alice assigns color $\gamma \in S$ to a node $v \in T$, let $S = S - \{\gamma\} = \{\delta\}$ and $T = T - \{v\} = \{w\}$. Bob then assigns color δ to one of the degree 2 nodes adjacent to w.
(5) Bob wins because w is adjacent to 6 nodes assigned distinct colors from [6]. ∎

In the argument to follow, we let $\mathbf{G}^d$ denote the *planar dual* of the graph G.

5.2 Theorem. There is a planar graph with admissibility 8.

Proof. Consider the planar graph $\mathbf{G}$ shown in Figure 7.

Observe that $\mathbf{G}$ is constructed as follows. Begin with the 8 node cube. Add the 6 nodes and 12 edges of the planar dual of the cube. Perform a $Y - \Delta$ transformation at each node of degree 3. Insert a 4-gon at each place where edges of the cube and its dual cross. Now let $\mathbf{H} = \mathbf{G}^d$ denote its planar dual. For each node x in $\mathbf{H}$, let $N(x)$ denote the set of neighbors of x. It is straightforward to verify the following properties of $\mathbf{H}$.

(1) There are 24 nodes of degree 8.
(2) Each node of degree 8 is adjacent to four other nodes of degree 8, three nodes of degree 4 and one node of degree 3.
(3) If x is a node of degree 8 and z_1, z_2, and z_3 are its three neighbors of degree 4, then there are three nodes x_1, x_2, and x_3 so that for each $i \in [3]$:

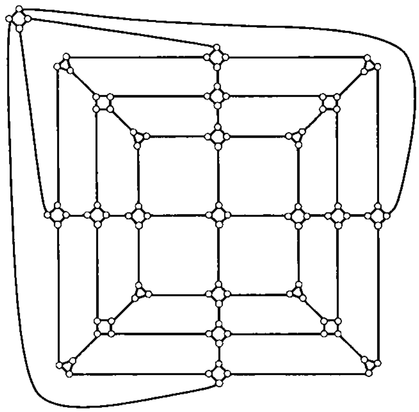

FIGURE 7

(a) The degree of x_i is 8;
(b) x and x_i are not adjacent; and
(c) x_i and z_i are adjacent.

Now let L be any linear order on the nodes of **H**, and let x be the last node of degree 8 to occur in L. We show that the admissibility of x in L is at least 8. Let y_1, y_2, y_3, and y_4 be the four neighbors of x that have degree 8. Note that $y_i < x$ in L for each $i \in [4]$. Now let $\{x_1, x_2, x_3, z_1, z_2, z_3\}$ be the set of 6 nodes satisfying the third property listed above. For each $j \in [3]$, let $u_j = z_j$ if $z_j < x$ in L; otherwise, let $u_j = x_j$. Finally, let w be the unique node of degree 3 adjacent to x. Set $v = w$ if $w < x$ in L; otherwise, set $v = x$. It follows that the set $S = \{y_1, y_2, y_3, y_4, u_1, u_2, u_3, v\}$ is admissible for x. ∎

5.3 Corollary. There is a planar graph with arrangeability at least 8.

Proof. Let **G** be any planar graph with admissibility 8. Form a planar graph **H** from **G** as follows. For each edge $e = xy$ in **G** add 7 new nodes of degree 2 each adjacent to both x and y. We claim that the arrangeability of **H** is at least 8. To show this, let L be any linear order of the nodes of **H**. Then let M be the restriction of L to the nodes of **G**. Choose a node x and set S of 8 nodes from **G** so that S is admissible for x relative to M. If S is arrangeable for x relative to L, then the arrangeability of L is at least 8. Now suppose that S is not arrangeable for x relative to L. Then there is a node $y \in S$ so that xy is an edge in **G**, but none of the 8 nodes of degree 2 adjacent to both x and y added in the formation of **H** follows x in L. This implies that the back degree of x in L is at least 8, so that the arrangeability of L is at least 8. ∎

By this time, the reader may enjoy the task of constructing an example to provide a lower bound on the game chromatic number of outerplanar graphs.

5.4 Exercise. There is an outerplanar graph with game chromatic number at least 5. ∎

6. CONCLUDING REMARKS AND OPEN PROBLEMS

Two obvious open problems that remain are to tighten the bounds we have produced for the game chromatic number of planar and outerplanar graphs. We have recently shown that the game chromatic number of the class of planar graphs is at least 8, and by the results presented in this paper, it is at most 33. For the class of outerplanar graphs, our bounds are 6 and 8.

A third open problem is to determine the least p for which every planar graph is p-arrangeable. We suspect that the upper bound of 10 provided in Theorem 4.1 is tight. For outerplanar graphs, the upper bound of 3 on the arrangeability and admissibility provided by Corollary 4.3 is tight—for both parameters.

Let C_n be the class of graphs that do not contain the subdivision of the complete graph K_n on n vertices as a subgraph. Then it is well known that there is some constant $c = c_n$ so that the average degree of any graph in C_n is at most c. It is then easy to modify the proof of Theorem 4.1 to obtain a bound on the admissibility and arrangeability of graphs in C_n. In particular, the game chromatic number of any proper minor closed class of graphs is bounded. Also, there is a bound on the game chromatic number of a graph in terms of its genus. We have no feel for what the best bounds for these functions might be.

We do not have a good lower bound for the inequality in Theorem 3.4, and we do not know if the bound in Theorem 3.5 is tight.

More generally, it seems to us to make good sense to investigate general classes of optimization problems that exhibit the key features of the uncooperative (adversarial) graph coloring problem we have studied in this paper.

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