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ON-LINE AND FIRST-FIT COLORING OF GRAPHS THAT DO NOT INDUCE P_5 *

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Abstract. For a graph H, let Forb(H) be the class of graphs that do not induce H, and let P_5 be the path on five vertices. In this article, we answer two questions of Gyárfás and Lehel. First, we show that there exists a function $f(\omega)$ such that for any graph $G \in \text{Forb}(P_5)$, the on-line coloring algorithm First-Fit uses at most $f(\omega(G))$ colors on G, where $\omega(G)$ is the clique size of G. Second, we show that there exists an on-line algorithm A that will color any graph $G \in \text{Forb}(P_5)$ with a number of colors exponential in $\omega(G)$. Finally, we extend some of our results to larger classes of graphs defined in terms of a list of forbidden subgraphs.

Key words. on-line algorithm, graph coloring, greedy algorithm

AMS subject classification. 05C35

Introduction. An on-line graph is a structure $G^< = (V, E, <)$, where $G = (V, E)$ is a graph and $<$ is a linear ordering of V. We allow V to be finite or countably infinite. We call $G^<$ an *on-line presentation* of the graph G. The on-line subgraph of $G^<$ induced by a subset $S \subset V$ is the on-line graph $G^<[S] = (S, E', <')$, where E' is the set of edges in E both of whose end points are in S, and $<'$ is $<$ restricted to S. We shall always assume that $V = \{x_1, x_2, \ldots\}$, where $x_i < x_j$ iff $i < j$. Let $V_i = \{x_j : j \leq i\}$ and $G_i^< = G^<[V_i]$. An algorithm for coloring the vertices of $G^<$ is said to be *on-line* if the color of a vertex v_i is determined solely by $G_i^<$. Intuitively, the algorithm colors the vertices of $G^<$ one at a time in some externally determined order $x_1, \ldots, x_n$, and at the time a color is irrevocably assigned to the vertex x_i, the algorithm can only see $G_i^<$. A simple but important example of an on-line algorithm is the algorithm First-Fit, denoted by FF, which colors the vertices of G with an initial sequence of the colors $\{1, 2, \ldots\}$ by assigning to the vertex x_i the least possible color not already assigned to any vertex of V_{i-1}, adjacent to x_i.

The *clique size* and *chromatic number* of a graph G are denoted by $\omega(G)$ and $\chi(G)$ respectively. Let A be an on-line graph-coloring algorithm. Then $\chi_A(G^<)$ denotes the number of colors A uses to color the on-line graph $G^<$ and $\chi_A(G)$ denotes the maximum of $\chi_A(G^<)$ over all on-line presentations $G^<$ of G. A class of graphs Γ is said to be *χ-bounded* if there exists a function f such that for all $G \in \Gamma, \chi(G) \leq f(\omega(G))$. The function f is called a *χ-binding function* for Γ. Easy examples of χ-bounded classes include the class of perfect graphs, the class of line graphs, and, more generally, the class of claw-free graphs. Similarly, for an on-line algorithm A, the class Γ is χ_A-bounded if there exists a function f such that for all $G \in \Gamma, \chi_A(G) \leq f(\omega(G))$. In this case we say that A is a *χ-binding algorithm* for Γ and f is an *on-line χ-binding function* for Γ. The class Γ is *on-line χ-bounded* if Γ is χ_A-bounded for some on-line algorithm A. In this article, we are interested in classes of graphs that are on-line χ-bounded. The class of perfect graphs is not on-line χ-bounded. In fact, the subclass of trees is not on-line χ-bounded (see, for example, Bean [1]). However, the class of claw-free

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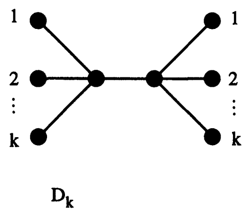

FIG. 1.

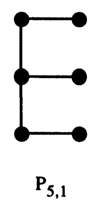

FIG. 2.

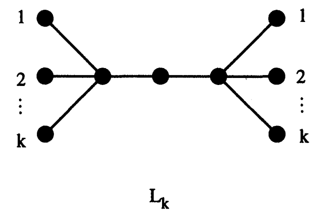

FIG. 3.

graphs is on-line χ-bounded. Here we shall be interested in other classes defined by forbidding certain induced subgraphs.

For a graph H, let Forb(H) be the class of graphs that do not induce H. Similarly, let Forb$(H_1, \dots, H_n)$ be the class of graphs that do not induce any of the graphs $H_1, \dots, H_n$. Gyárfás [3] and Sumner [15] independently conjectured that if T is a tree, then Forb(T) is χ-bounded. Gyárfás [4] has shown this to be the case when T is a path. Gyárfás, Szemerédi, and Tuza [8] verified the conjecture for triangle-free graphs in Forb(T) when T is a radius-two tree, and Kierstead and Penrice [11] extended this result by showing that Forb(T) is χ-bounded whenever T has radius two.

Gyárfás and Lehel [7], [6] opened up an exciting and unexpected new area for study when they proved that Forb(P_5) is on-line χ-bounded, where P_n is a path on n vertices. They also showed that Forb(P_6) is not on-line χ-bounded. These results led to many interesting questions. The Gyárfás–Lehel algorithm was quite complicated and gave a superexponential on-line χ-binding function. They asked whether Forb(P_5) had an exponential on-line χ-binding function and whether the simple algorithm First-Fit was an χ-binding algorithm for Forb(P_5). In this article we prove the following theorems.

THEOREM 1.1. *There exists an on-line algorithm A and an exponential function $f(\omega) = (4^\omega - 1)/3$ such that $\chi_A(G) \le f(\omega(G))$, for any graph $G \in$* Forb(P_5).

THEOREM 2.1. Forb(P_5) *is χ_{FF}-bounded.*

The smallest function known to be an (off-line) χ-binding function for Forb(P_5) is 2^n. Our on-line χ-binding function for Forb(P_5) is within a power of two of this function. In light of the Gyárfás–Sumner conjecture, one is led to ask for which trees T, Forb(T) is on-line χ-bounded. Since Forb(P_6) is not on-line χ-bounded, neither is Forb(T) if T has radius greater than two. The authors [13] have recently proved that Forb(T) is on-line χ-bounded if T has radius at most two.

Theorem 2.1, together with some observations of Gyárfás and Lehel [5], allow us to characterize those trees T for which Forb(T) is χ_{FF}-bounded.

THEOREM 2.2. *Let T be a tree.* Forb(T) *is χ_{FF}-bounded if and only if T does not induce $K_2 + 2K_1$.*

For other trees T, we shall try to determine reasons why Forb(T) is not χ_{FF}-bounded. In [11] Kierstead and Penrice showed that for any tree T and integer t, Forb$(T, K_{t,t})$ is χ_{FF}-bounded. This fact is used in [13] to prove that Forb(T) is on-line χ-bounded for any radius-two tree T. However, the fact that a graph contains $K_{t,t}$ does not explain why it might have a large First-Fit chromatic number. Let B_t be the graph obtained from $K_{t,t}$ by removing a perfect matching M. If $B_t^<$ is the on-line presentation of B_t, where adjacent vertices of M appear consecutively, then it is easy to see (and is explained in more detail in the proof of Theorem 2.2) that $\chi_{\text{FF}}(B_t^<)$ is t. Thus if a graph induces B_t, then we have a certificate that its First-Fit chromatic number is at least t. We shall prove the following two theorems which are quite satisfying from this point of view. The trees D_k and $P_{5,1}$ are shown in Figs. 1

and 2 (also see the definition at the end of this section).

THEOREM 2.3. *For every positive integer* k, $\mathrm{Forb}(P_{5,1}, B_t)$ *is* χ_{FF}*-bounded.*

THEOREM 2.4. *For every positive integer* k, $\mathrm{Forb}(D_k, B_t)$ *is* χ_{FF}*-bounded.*

It should be noted that B_3 is just the six cycle C_6.

Finally, we mention some related results on First-Fit. Woodall [16] showed that the class of interval graphs is χ_{FF}-bounded. Gyárfás and Lehel [5] gave an improved bound for this problem and introduced the notion of a wall, which we shall also use. Recently Kierstead [10] showed that the binding function is linear. Interval graphs are cocomparability graphs of interval orders. The class of comparability graphs contains the class of trees and so is not on-line χ-bounded. However, Kierstead [12] has proved that the class of comparability graphs of interval orders is χ_{FF}-bounded. A consequence of a theorem of Chvátal [2] is that First-Fit uses exactly $\omega(G)$ colors to color $G \in \mathrm{Forb}(P_4)$, where T is the path on four vertices.

Theorem 1.1 is proved in §1. Theorems 2.1–2.4 are proved in §2. In §3, we discuss some problems for further research. In the remainder of this section we review our terminology and notation.

Let $G = (V, E)$ be a graph. Adjacency between two vertices x and y is denoted by $x \sim y$ and nonadjacency is denoted by $x \not\sim y$. The neighborhood of a vertex x is denoted by $N(x) = \{y \in V : y \sim x\}$. The clique number, the independence number, the chromatic number, and the number of vertices of G are denoted by $\omega(G), \alpha(G), \chi(G)$, and $\nu(G)$, respectively. The following special notation is used to denote certain graphs.

P_k : path on k vertices.

S_k : star with k leaves, i.e., a star on $k+1$ vertices.

D_k : tree obtained by adding $k-1$ leaves to the second and third vertices of P_4.

L_k : tree obtained by adding $k-1$ leaves to the second and fourth vertices of P_5 (see Fig. 3).

$P_{n,k}$: tree obtained by adding k leaves to the third vertex of P_n.

$P^2_{n,k}$: broom obtained by adding $k-1$ leaves to the second vertex of P_n.

LS_k : tree on $2k+1$ vertices consisting of k independent edges and a vertex which is adjacent to exactly one vertex of each of these edges.

K_s : clique on s vertices.

$K_{t,t}$: complete bipartite graph with t vertices in each part.

B_t : bipartite graph obtained by deleting a perfect matching from $K_{t,t}$.

For a set S, let $[S]^2 = \{A \subset S : |S| = 2\}$. We may write $(\alpha > \beta)$ for the two element set $\{\alpha, \beta\}$ to denote that $\alpha > \beta$. For a coloring $p: [S]^2 \to \{1, \ldots, n\}$, we say that $H \subset S$ is *homogeneous* if p restricted to $[H]^2$ is constant. Let $R_i(t)$ be the Ramsey function such that for every coloring $p: [S]^2 \to \{1, \ldots, i\}$, with $|S| \geq R_i(t)$, there exists a homogeneous subset $H \subset S$ such that $|H| = t$. Similarly, let $R(t_1, \ldots, t_i)$ be the Ramsey function such that for every coloring $p: [S]^2 \to \{1, \ldots, i\}$, with $|S| \geq R(t_1, \ldots, t_i)$, there exists a homogeneous set H with $|H| = t_j$ such that $p(\alpha > \beta) = j$, for all $\alpha, \beta \in H$ with $\alpha \neq \beta$. Note that if G is a graph such that $\nu(G) \geq R(\omega, \alpha)$, then either $\omega(G) \geq \omega$ or $\alpha(G) \geq \alpha$.

1. An exponential on-line algorithm for P_5-free graphs. In this section we prove Theorem 1. Before starting the proof, we establish two useful properties of $\mathrm{Forb}(P_5)$.

LEMMA 1.2. *Suppose* $G \in \mathrm{Forb}(P_5)$. *If* M_1 *and* M_2 *are distinct maximal cliques in the same connected component of* G, *then there exist vertices* $x \in M_1$ *and* $y \in M_2$ *such that* x *is adjacent to* y.

Proof. Suppose not. Then M_1 and M_2 are disjoint. Since M_1 and M_2 are in the same connected component, there exists a path from M_1 to M_2, i.e., a path $P = (x_1, \dots, x_m)$ such that $x_1 \in M_1, x_m \in M_2$. Let P be a shortest such path. Then P is an induced path and, by our initial assumption, $m \geq 3$. Since M_1 is a maximal clique, there exists $x_0 \in M_1$ not adjacent to x_2. Since P is a shortest path, y is not adjacent to x_i for $i \geq 3$. Similarly, there exists $x_{m+1} \in M_2$ such that x_{m+1} is not adjacent to x_i for $i \leq m-1$. Thus $x_0 + P + x_{m+1}$ is an induced path of length at least five, which is a contradiction. □

COROLLARY 1.3. *If x is a cut vertex of a connected graph G in* Forb(P_5) *and M_1 and M_2 are maximal cliques in distinct components of $G - x$, then either $M_1 \cup \{x\}$ or $M_2 \cup \{x\}$ is a clique.*

Proof of Theorem 1.1. We first present an on-line algorithm A and then prove that it properly colors any on-line presentation $G^<$ of a graph $G \in \text{Forb}(P_5)$ with at most $f(\omega(G))$ colors. An important point is that the algorithm must be independent of the clique size of G. For this reason, the algorithm is defined recursively. Let $A(x, G^<)$ denote the color A assigns to x, when x is considered as a vertex of the on-line graph $G^<$. When a new vertex x_i is presented, the algorithm first assigns x_i to one of the sets $S_{j,k}$, where $1 \leq j \leq 4$ and k is the clique size of the connected component of x_i in $G_i^<$. This will be done so that for $j < 4$, each $S_{j,k}$ has clique size less than k and each $S_{4,k}$ is an independent set. Then A assigns x_i a color derived from the color $A(x_i, G^<[S_{j,k}])$ obtained by a recursive call of itself. Vertices in distinct $S_{j,k}$ receive colors from disjoint sets of colors. Note that the exponential function $f(w) = (4^w - 1)/3$ is defined recursively by $f(1) = 1$ and $f(k) = 4f(k-1) + 1$.

At each stage of the algorithm, each connected component C of $G_i^<$ will have a special maximum clique K, called the *active clique*. Once a clique becomes active, it remains active until a larger clique is formed in its component. The choice of which $S_{j,k}$ in which to put x_i is completely determined by $\omega(C)$ and the adjacencies between x_i and elements of K, where C denotes the connected component of x_i in $G_i^<$ and K denotes the active clique of C.

Suppose A has colored G_{i-1}. We specify how A assigns a color to the next vertex x_i.

ALGORITHM $A(x_i, G^<)$.

Find C, the connected component of x_i in $G_i^<$, and set $k = \omega(C)$.

Case 1: $k > \omega(C - \{x_i\})$. Put x_i in $S_{4,k}$. [Claim 1: $S_{4,k}$ is independent.] Set $A(x_i, G^<) = f(k)$. Let K be a k-clique in C. Deactivate any active cliques of $C - \{x_i\}$ and designate K as the active clique of C.

Case 2: $k = \omega(C - \{x_i\})$. Let K be the active clique of C. [Claim 2: K is unique.]

Case 2a: For some $v \in K$, both $x_i \sim v$ and $u \sim v$, for all $u \in C \cap S_{1,k}$. Put x_i in $S_{1,k}$. [Claim 3: $\omega(S_{1,k}) < k$.] Set $A(x_i, G^<) = f(k-1) + A(x_i, G^<[S_{1,k}])$.

Case 2b: Not Case 2a and for some $v \in K$, both $x_i \sim v$ and $u \sim v$, for all $u \in C \cap S_{2,k}$. Put x_i in $S_{2,k}$. [Claim 4: $\omega(S_{2,k}) < k$.] Set $A(x_i, G^<) = 2f(k-1) + A(x_i, G^<[S_{2,k}])$.

Case 2c: Not Case 2a or Case 2b. Put x_i in $S_{3,k}$. [Claim 5: $\omega(S_{3,k}) < k$.] Set $A(x_i, G^<) = 3f(k-1) + A(x_i, G^<[S_{3,k}])$.

Next we show that the algorithm produces a proper $f(\omega(G))$-coloring of $G^<$, assuming the five claims above. Then we shall verify the claims. We argue by induction on $\omega(G)$. If $\omega(G) = 1$, then every point is isolated. Thus each x_i is assigned to the independent set $S_{4,1}$ and colored $f(1) = 1$.

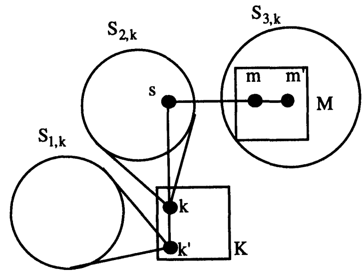

FIG. 4

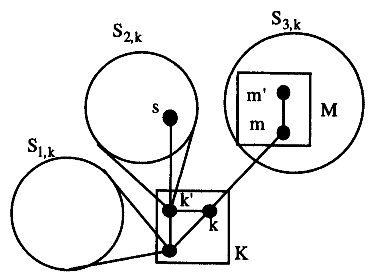

FIG. 5

Now suppose $\omega(G) = k > 1$. By Claims 3–5, for each $j \leq 3$ and $m \leq k, \omega(S_{j,m}) \leq k-1$. By the induction hypothesis, $S_{j,m}$ is properly colored with colors from the set $\{jf(m-1)+1, \ldots, (j+1)f(m-1)\}$, since $A(x_i, G^{<}[S_{j,m}]) \leq f(m-1)$ for each $x_i \in S_{j,k}$. By Claim 1, for each $m \leq k$, the set $S_{4,m}$ is independent and colored with $f(m) = 4f(m-1)+1$. Since these sets of colors are easily seen to be pairwise disjoint, we are done.

Claim 3 is easy because all vertices in the same connected component of $S_{1,k}$ have a common neighbor in their active k-clique. Claim 4 is similar. We shall prove Claims 1 and 2 by induction on i. When $i = 1$, both claims are trivial. Suppose $i > 1$. If Claim 1 fails, then by induction x_i is adjacent to some $x_j \in S_{4,k}$, where $j < i$. Clearly the component C' of x_j in $G_j^{<}$ is contained in C since $x_j \in S_{4,k}, \omega(C') = k$, which contradicts $\omega(C - \{x_i\}) < k$. By the induction hypothesis, the only way that Claim 2 could fail is if x_i is a cut vertex of C and two distinct components of $C - \{x_i\}$ contain k-cliques. Then by Corollary 1.3, x_i is in a $(k+1)$-clique of C, contradicting $\omega(C) = k$.

It remains to prove Claim 5. Suppose it is false. Let M be a k-clique in $S_{3,k}$. Let $v_{i_1}, \ldots, v_{i_k}$ be the vertices of M with $i_1 < i_2 < \cdots i_k$. If K is the active clique of v_{i_1} in $G_{i_1}^{<}$, it is easy to show by induction that K is the active clique of the component of v_{i_j} in $G_{i_j}^{<}$ whenever $1 \leq j \leq k$, since K could only be deactivated by the addition of a vertex which raised the clique size of the component. But in that case, v_{i_k} would have been assigned to $S_{j,k'}$ for some $j \leq 4, k' > k$. We consider two cases.

Case 1. There exist $s \in S_{2,k}$ and $m \in M$ such that $s \sim m$. Since $\omega(C) = k$, there exists $m' \in S_{2,k}$ such that $m' \not\sim s$. (See Fig. 4.) Since m and m' are not in $S_{2,k}$, there exists $k \in K$ such that $k \sim s, k \not\sim m$, and $k \not\sim m'$. Also since m, m', and s are not in $S_{1,k}$, there exists $k' \in K$ such that $k' \not\sim m, k' \not\sim m'$, and $k' \not\sim s$. But this is a contradiction, since (k', k, s, m, m') is an induced P_5.

Case 2. For all $s \in S_{2,k}$ and $m \in M, s \not\sim m$. (See Fig. 5.) By Lemma 1.2 there exist $m \in M$ and $k \in K$ such that $m \sim k$. Also there exists $m' \in M$ such that $k \not\sim m'$.

Since $m \notin S_{2,k}$, there exists $s \in S_{2,k}$ such that $s \not\sim k$. Since neither m nor m' is in $S_{2,k}$, there exists $k' \in K$ such that $s \sim k', m \not\sim k'$, and $m' \not\sim k'$. By the hypothesis of this case, $s \not\sim m$ and $s \not\sim m'$. Thus (s, k', k, m, m') is an induced P_5, which is a contradiction. □

2. First-Fit. We begin this section with the proof of Theorem 2.2 assuming Theorem 2.1. We then state and prove a series of lemmas which lead eventually to the proofs of Theorems 2.3 and 2.4. Along the way we pause to prove Theorem 2.1.

Proof of Theorem 2.2. The reader may check that if T does not induce $K_2 + 2K_1$, then T is either a star or a path on 5 or fewer vertices. Thus it suffices to show that $\mathrm{Forb}(K_2 + 2K_1)$ is not χ_{FF}-bounded, whereas $\mathrm{Forb}(S_k)$ and $\mathrm{Forb}(P_5)$ are.

Gyárfás and Lehel [5] noted that $\mathrm{Forb}(K_2 + 2K_1)$ is not χ_{FF}-bounded as follows. Recall that B_t is the graph formed by deleting a perfect matching from a complete bipartite graph with t vertices in each part. Let $\{a_1, \ldots, a_t\}$ and $\{b_1, \ldots, b_t\}$ be the independent sets of the bipartition of B_t, and assume that the pairs $\{a_i, b_i\}$ are independent. It is easy to check that, for all positive integers t, B_t does not induce $K_2 + 2K_1$ and that First-Fit will use t colors on B_t if the vertices are presented in the order $a_1, b_1, a_2, b_2, \ldots, a_t, b_t$.

Now note that $\mathrm{Forb}(S_t)$ is χ_{FF}-bounded. If $G \in \mathrm{Forb}(S_t)$ and $\omega(G) = k$, First-Fit will use no more than $R(k,t)$ colors on G. If a vertex x receives color $R(k,t)+1$, it must have $R(k,t)$ neighbors. Because $\omega(G) = k, x$ must in fact have t independent neighbors, which, together with x, form an induced S_t.

$\mathrm{Forb}(P_5)$ is χ_{FF}-bounded by Theorem 2.1, and thus we are done. □

For the arguments to follow, it is useful to be able to analyze the performance of First-Fit in terms of static substructures rather than on-line presentations of graphs. To this end, we introduce the notion of a *wall,* which is due originally to Gyárfás and Lehel [5]. A *colored graph* is a pair $W = (G(W), f(W))$, where $G(W) = (V(W), E(W))$ is a graph and $f(W)$ is a proper coloring of $G(W)$. Let $C(W)$ be the range of $f(W)$. For $I \subset C(W)$, let $[I]_W$ denote the set of vertices in G which are colored with some color in I. If $I = \{\alpha\}$, we may write $[\alpha]_W$ for $[\{\alpha\}]_W$. The colored graph W is called a *wall* if, for all $\alpha > \beta$ in $C(W)$ and for every vertex $x \in [\alpha]_W$, there exists a vertex $y \in [\beta]_W$ such that x is adjacent to y. The color classes of the wall, $[\alpha]_W$, are called *levels,* and we say that $[\alpha]_W$ is a *higher* level than $[\beta]_W$, if $\alpha > \beta$. The *height* $h(W)$ of a wall W is $|C(W)|$, or, equivalently, the number of levels. A wall W is said to *support* a vertex x if x is adjacent to some vertex at every level of W. We say that a wall W is *in* a graph G if $G(W)$ is an induced subgraph of G. We say that a colored graph W' is an induced subwall of a wall W if W' is a wall in $G(W)$ and $f(W')$ is the restriction of $f(W)$ to W'. Note that if W is a wall, then $[I]_W$ induces a subwall of W for any $I \subset C(W)$. We call $[I]_W$ a level induced subwall. The following easy observation allows us to discuss walls rather than on-line graphs when considering First-Fit.

LEMMA 2.5. *Let G be a graph. Then $\chi_{\mathrm{FF}}(G) = \max h(W)$, where the maximum is taken over all walls in G.*

Proof. Suppose $G^<$ is an on-line presentation of G with $\chi_{\mathrm{FF}}(G^<) = t$. Then (G, f) is a wall of height t, where f is the coloring produced by First-Fit when applied to $G^<$. Alternatively, suppose that W is a wall in G with $h(W) = t$. Then $\chi_{\mathrm{FF}}(G^<) \geq t$ if $G^<$ is any on-line presentation in which the vertices of the lowest level of W precede the vertices of the second-lowest level, which precede those of the third level, and so on through W, and all vertices of W precede all vertices of $G - W$. □

Let W be a wall in G, which supports a vertex x. Define a coloring $p = p_{W,x}$ on the two element subsets of $C(W)$ by $p(\alpha > \beta) = c$, where

$$\begin{aligned} c = 1 \text{ iff } &(a)\ \exists y \in ([\alpha]_W \cap N(x))\ \forall z \in ([\beta]_W \cap N(x))(y \not\sim z) \text{ and} \\ &(b)\ \exists y' \in ([\alpha]_W - N(x))\ \forall z' \in ([\beta]_W - N(x))(y' \not\sim z'); \\ c = 2 \text{ iff } &\text{not (a); and} \\ c = 3 \text{ iff } &\text{both (a) and not (b).} \end{aligned}$$

If (a) holds for y, we call y a *left witness point* for the pair (α, β). Note that since W is a wall, in this case y must be adjacent to some vertex $z' \in [\beta]_W - N(x)$. If (b) holds for y' we call y' a *right witness point* for the pair (α, β). Note that y' is adjacent to some vertex $z \in [\beta]_W \cap N(x)$. W is said to have the *cross property* if for some $x, p_{W,x}(\alpha > \beta) = 1$, for every pair of colors $\alpha, \beta \in C(W)$. The following observation is crucial to our arguments. If $p(\alpha > \beta) = 2$ for every pair $\alpha, \beta \in C(W)$, then $W \cap N(x)$ is a wall, and if $p(\alpha > \beta) = 3$ for every pair $\alpha, \beta \in C(W)$, then $W - N(x)$ is a wall.

The *path number* $\pi_G(W) = \pi(W)$ of a wall W in G is the length of the longest induced path $P = (x_1, \ldots, x_n)$ in G such that $x_1 \in [\alpha]_W$, where α is the largest color in $C(W)$, and no vertex of $(x_2, \ldots, x_n)$ is adjacent to any vertex of $W - \{x_1\}$.

LEMMA 2.6. *There exists a function $g(h)$ such that for any graph $G = (V, E)$, if W is a wall in G such that $h(W) \geq g(h)$, then there exists an induced subwall W' of W such that $h(W') \geq h$ and*

(i) *W' is a level induced subwall and has the cross property*; *or*

(ii) *$\omega(W') < \omega(W)$; or*

(iii) *both $\omega(W') = \omega(W)$ and $\pi(W') > \pi(W)$.*

Proof. First note that for any induced subwall W' of $W, \omega(W') \leq \omega(W)$. Let $g = g(h) = R_3(2h) + 1$. Suppose W is a wall in G with $h(W) \geq g$. Let $P = (x = x_1, \ldots, x_\pi)$ be a path that witnesses the value of $\pi(W)$. Let I be the set of the $g - 1$ smallest colors of $C(W)$ and $W_0 = [I]_W$. Let $p = p_{W_0,x}$ be the coloring of 2-subsets of I defined above. By Ramsey's theorem there exists a homogeneous $2h$-subset $H \subset I$. Let $p(\alpha > \beta) = c$, for any $\alpha, \beta \in H$ with $\alpha \neq \beta$.

Case 1. $c = 1$. Then $H' = [H]_W$ is a level induced subwall of W with the cross property and $h(W') \geq h$.

Case 2. $c = 2$. Then $W' = [H]_{W \cap N(x)}$ is a wall with $h(W') \geq h$. Since $V(W') \subset N(x)$ and $x \in V(W), \omega(W') < \omega(W)$.

Case 3. $c = 3$. Then $[H]_{W-N(x)}$ is a wall of height at least $2h - 1$. Let $\alpha > \beta$ be the two largest colors in H and let y be a left witness point for the pair (α, β). Let $J = \{\gamma \in H : y \sim z', \text{ for some } z' \in [\gamma]_W - N(x)\}$. If $|J| \geq h - 1$, let $W' = \{y\} \cup [J]_{W-N(x)}$. Then $y + P$ witnesses that $\pi(W') > \pi(W)$. See Fig. 6. Otherwise let $W' = [(H - J) \cup \{\beta\}]_{W \cap (V - N(x))}$. Then $z' + y + P$, where $y \sim z'$ and $z' \in [\beta] - N(x)$, witnesses that $\pi(W') > \pi(W)$. In either case, $h(W') \geq h$ and $\omega(W') \leq \omega(W)$. See Fig. 7. □

We now use Lemma 2.6 to give an inductive proof of Theorem 2.1.

Proof of Theorem 2.1. Consider a graph $G \in \text{Forb}(P_5)$. First note (1) if W is a wall in G with the cross property, then $h(W) = 1$: Otherwise, let y and y' be left and right witness points for the pair (α, β), where $\{\alpha, \beta\} \subset C(W)$. Thus by our remark above, there exist $z' \in [\beta]_W - N(x)$ and $z \in [\beta]_W \cap N(x)$ such that $y \sim z'$ and $y' \sim z$. Since y and y' are witness points, $y \not\sim z$ and $y' \not\sim z'$. Thus $\{z', y, x, z, y'\}$ induces P_5, which is a contradiction. See Fig. 8.

Next note (2) if W is a wall in G with $\pi(W) \geq 3$, then W contains an induced subwall W_0 such that $h(W_0) = h(W) - 1$ and $\omega(W_0) < \omega(W)$. Suppose $P = (x_1, x_2, x_3)$

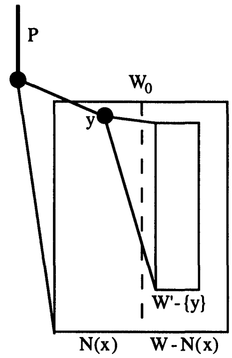

FIG. 6.

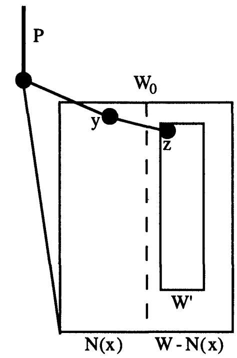

FIG. 7.

is a path that witnesses that $\pi(W) \geq 3$. If $W_0 = W \cap N(x)$ is an induced wall, we are clearly done; otherwise there exist $\alpha > \beta \in C(W_0)$ and $y_1 \in [\alpha]_{W_0}$ such that y_1 is not supported in $[\beta]_{W_0}$. Thus y_1 is supported by some $y_2 \in [\beta]_W - N(x)$. But then $x_1 \sim y_1, y_1 \sim y_2, x_1 \not\sim y_2$, and $\{x_3, x_2, x_1, y_2, y_3\}$ induces P_5, which is a contradiction.

Let g be the function defined in Lemma 2.6. We claim that the function f, defined recursively by $f(1) = 1$ and $f(\omega + 1) = g \circ g(1 + f(\omega))$, is a χ_{FF}-binding function for Forb(P_5). We show by induction on ω that if $G \in$ Forb(P_5) and $\omega(G) \leq \omega$, then $\chi_{\mathrm{FF}}(G) \leq f(\omega)$. The base step is trivial. For the inductive step, suppose $\omega(G) \leq \omega$ and $\chi_{\mathrm{FF}}(G) > f(\omega)$. By Lemma 2.5, G contains a wall W of height $\chi_{\mathrm{FF}}(G)$. Thus by Lemma 2.6, G has a wall W_1 of height $g(1+f(\omega))$ such that either (i) W_1 has the cross property, (ii) $\omega(W_1) < \omega$, or (iii) $\pi(W_1) = \omega$ and $\pi(W_1) \geq 2$. But (i) is impossible by (1) above and (ii) is impossible by the induction hypothesis and Lemma 2.5. Thus (iii) holds. Applying Lemma 2.6 to W_1, and using the same reasoning, we obtain a wall W_2 such that $h(W_2) \geq 1 + f(\omega)$ and $\pi(W_2) \geq 3$. Thus by (2) above, W_2 contains an induced subwall W_3 such that $\omega(W_3) < \omega$ and $h(W_3) \geq f(\omega) > f(\omega(W_3))$, which, using Lemma 2.5, contradicts the induction hypothesis. □

Let W be a wall, which has the cross property with respect to x. Then for every $\alpha > \beta$ in $C(W)$, there exists a right witness point y_β in $[\alpha]_W \cap N(x)$ for the pair (α, β). However, for different values of β, the right witness points y_β may be distinct. We say that $y \in [\alpha]_W \cap N(x)$ is a left *-witness point for α if y is a left witness point for every pair (α, β), with $\beta \in C(W)$ and $\alpha > \beta$. Similarly, $y' \in [\alpha]_W - N(x)$ is a right *-witness point for α if y' is a right witness point for every pair (α, β), with $\beta \in C(W)$ and $\alpha > \beta$. We say that W has *-witnesses for α if there exist left and right *-witnesses for α. We say that W has the *strong cross property* if for every color $\alpha \in C(W), W$ has *-witnesses for α. In order to establish the existence of a relatively high wall with the strong cross property, we need the following lemma.

LEMMA 2.7. *There exists a function $j(h)$ such that, if W is a wall in a graph G, W has height $j = j(h)$, and W supports a vertex x, then there exists an induced subwall W' of W such that W' supports $x, h(W') \geq h$, and $(*)$ for every vertex y in W' and for all $\alpha > \beta$ in $C(W'), y$ is a left or right witness for (α, β) iff y is a left or right $*$-witness for α.*

Proof. Let $j(h) = 2^{2^h}$. We construct W' one level at a time starting at the top. At each new level, we must add points to support all the points from higher levels already added to W'. In order to ensure that regardless of how we later add points at lower levels, these new points will satisfy $(*)$, we remove certain lower levels from consideration. This idea is formalized as follows.

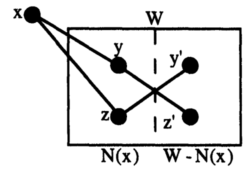

FIG. 8.

Stage 0. Let $I_0 = C(W), I'_0 = \varnothing$, and $v_0 = x$.
Stage $s+1$. Suppose we have constructed $V_s = \{v_0, \dots, v_n\}, I_n$, and I'_s such that:
(1) $n < 2^s, |I_n| \geq j2^{-n}$, and $|I'_s| = s$;
(2) $\forall \alpha \in I_n \ \forall \beta \in I'_s(\alpha < \beta)$;
(3) $[I']_{W \cap V_s}$ is a wall which supports x and satisfies (*);
(4) $\forall y \in (N(x) \cap V_s) \ \forall \alpha, \beta \in I'_s$,

$$[\exists z \in ([\alpha]_W \cap N(x))(y \sim z)] \Leftrightarrow [\exists z \in ([\beta]_W \cap N(x)(y \sim z)];$$

and

(5) $\forall y \in (V_s - N(x)) \ \forall \alpha, \beta \in I'_s$,

$$[\exists z \in ([\alpha]_W - N(x))(y \sim z)] \Leftrightarrow [\exists z \in ([\beta]_W - N(x))(y - z)].$$

Let α be the largest color in I_n. Set $I'_{s+1} = I'_s \cup \{\alpha\}$. For each $v_i \in V_s$, choose $v_{n+i} \in [\alpha]_W$ such that $v_i \sim v_{n+i}$. Set $V_{s+1} = \{v_0, \dots, v_{2n}\}$. Define I_i, for $i = n+1, \dots, 2n$ by induction on i as follows. Suppose I_i has been defined. Let $J = \{\gamma \in I : \exists z \in [\gamma]_W[v_{n+i+1} \sim z \text{ and } (v_{n+i+1} \sim x \Leftrightarrow z \sim x)]\}$. If $|J| \geq |I_i|/2$, set $I_{i+1} = J$; otherwise set $I_{i+1} = I_i - (J \cup \{\alpha\})$. It is easy to check that conditions (1)–(5) are maintained. This completes the proof. $\square$

Lemma 2.7 allows us to strengthen Lemma 2.6 as follows.

LEMMA 2.8. *There exists a function $g^*(h)$ such that for any graph $G = (V, E)$, if W is a wall in G and $h(W) \geq g^*(h)$, then there exists an induced subwall W' of W with $h(W') \geq h$ and*
(i) *W' has the strong cross property; or*
(ii) *$\omega(W') < \omega(W)$; or*
(iii) *both $\omega(W') \leq \omega(W)$ and $\pi(W') > \pi(W)$.*

Proof. Let $g^*(h) = j \circ g(h)$. Suppose $h(W) \geq g^*(h)$. Then by Lemma 2.7 there exists an induced subwall $W_1 \subset W$ such that $h(W_1) \geq g(h)$ and W_1 satisfies (*). By Lemma 2.6, there exists an induced subwall $W' \subset W$ with $h(W') \geq h$, and either W' is a level induced subwall of W_1 and has the cross property, $\omega(W') < \omega(W)$, or both $\omega(W') \leq \omega(W)$ and $\pi(W') > \pi(W)$. In the latter two cases, we are immediately done. In the first case we are also done, since W_1 satisfies (*) and W' is a level induced subwall of W_1. $\square$

LEMMA 2.9. *There exists a function $e(h, \omega)$ such that if W is a wall in a graph G with $G \in \mathrm{Forb}(P_{5,1}), h(W) \geq e(h)$, and $\omega(G) \leq \omega$, then there exists an induced subwall $W' \subset W$ such that $h(W') \geq h$ and W' has the strong cross property.*

Proof. The proof is essentially the same as the proof of Theorem 2.1, with Lemma 2.6 replaced by Lemma 2.8 and observation (2) replaced by the following remark: (2′) if W is a wall in a graph G with $\pi(W) \geq 3$, then W contains a subwall W_0 such that $h(W_0) = h(W) - 1$ and $\omega(W_0) < \omega(W)$. Let $P = (x_1, x_2, x_3)$ be a path that witnesses that $\pi(W) \geq 3$. If $W_0 = W \cap N(x)$ is a wall, we are clearly done; otherwise, there exist

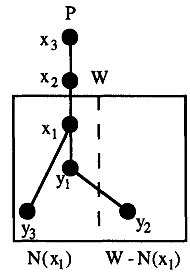

FIG. 9.

$y_1 \in [\alpha]_W$ with $x_1 \sim y_1$ and $y_2, y_3 \in [\beta]_W$, such that $y_1 \sim y_2, x_1 \not\sim y_2$, and $x_1 \sim y_3$, where $\alpha > \beta$. But then $\{x_3, x_2, x_1, y_1, y_2, y_3\}$ induces $P_{5,1}$, which is a contradiction. See Fig. 9. $\square$

We note that Lemma 2.9 holds with $P_{5,1}$ replaced by $P_{n,1}$ or $P^2_{n,k}$. However, we as yet have no application for such results. We need one more lemma for the proof of Theorem 2.3.

LEMMA 2.10. *Let G be a graph in* $\text{Forb}(P_{5,1})$ *with $\omega(G) \leq \omega$, let x be a vertex of G, and let W be a wall in G such that both $h(W) \geq R(\omega + 1, t)$ and W has the strong cross property with respect to x. Then G induces B_t.*

Proof. Let $r = R(\omega + 1, t)$, and for $1 \leq \alpha \leq r$, let y_α and y'_α denote the left and right $*$-witnesses for α. First observe that for $1 \leq \beta < \alpha \leq r, y'_\alpha$ is adjacent to y_β. Otherwise, since y'_α is a right $*$-witness point, there exists $z \in N(x) \cap [\beta]_W$ such that $z \sim y'_\alpha$. Since y_α is a left $*$-witness point, there exists $z' \in [\beta]_W - N(x)$ such that $y_\alpha \sim z'$. But then $\{z', y_\alpha, x, z, y'_\alpha, y_\beta\}$ induces $P_{5,1}$, which is a contradiction. In particular, if $\alpha > \beta$, then the left $*$-witness for β supports some vertex in $[\alpha]_W - N(x)$. See Fig. 10.

We call a vertex $z' \in [\gamma]_W - N(x)$ *special* for γ if, for all $\alpha \neq \gamma, y_\alpha \sim z'$. We next show that for every color $\gamma \in C(W)$, there exists a vertex z'_γ that is special for γ. For each $\alpha \neq \gamma$, let $N_\alpha = \{z' \in [\gamma]_W - N(x) \colon y_\alpha \sim z'\}$. We must show that $\cap_{\alpha \neq \gamma} N_\alpha \neq \varnothing$. Each N_α is nonempty. If $\alpha > \gamma$, then this follows from the definition of y_α, and if $\alpha < \gamma$, then it follows from the observation above. Thus it suffices to show that for all $\alpha, \beta \in C(W) - \{\gamma\}, N_\alpha \subset N_\beta$ or $N_\beta \subset N_\alpha$. Suppose not. Then there exist $z', w' \in [\gamma]_W - N(x)$ such that $y_\alpha \sim z' \not\sim y_\beta$ and $y_\alpha \not\sim w' \sim y_\beta$. But then $\{z', y_\alpha, x, y_\beta, w', y_\gamma\}$ induces $P_{5,1}$, which is a contradiction. See Fig. 11.

Finally, by the choice of r, there exists a subset $H \subset C(W)$ such that $|I| = t$ and $\{z'_\gamma \colon \gamma \in I\}$ is independent. Then the set $\{y_\gamma \colon \gamma \in I\} \cup \{z'_\gamma \colon \gamma \in I\}$ induces B_t. $\square$

Theorem 2.3 now follows easily from Lemmas 2.5, 2.9, and 2.10.

Proof of Theorem 2.3. Fix t. We claim that $f(\omega) = e(R(\omega + 1, t), \omega)$ is a χ-binding function for $\text{Forb}(P_{5,1}, B_t)$. Suppose not. Then there exists a graph G in $\text{Forb}(P_{5,1}, B_t)$ such that $\chi_{\text{FF}}(G) \geq f(\omega(G))$. By Lemma 2.5, there is a wall W in G such that $h(W) \geq f(\omega(G))$. Thus by Lemma 2.9, there exists a wall W in G such that $h(W) \geq R(\omega + 1, t)$ and W has the strong cross property. Thus by Lemma 2.10, G induces B_t, which is a contradiction. $\square$

Proof of Theorem 2.4. Since we are not concerned with finding an optimal binding function, we may assume that $k = t$. Let f be defined recursively by $f(1) = 1$ and $f(\omega + 1) = j \circ R(1 + f(\omega), 1 + R_{16}(\max\{2t, \omega + 1\}))$, where j is the function from Lemma 2.7. We shall show by induction on ω that, if $G \in \text{Forb}(D_t, B_t)$ and $\omega(G) \leq \omega$, then $\chi_{\text{FF}}(G) \leq f(\omega(G))$. The base step is trivial, so suppose the result holds for ω and suppose both $\omega(G) = \omega + 1$ and $\chi_{\text{FF}}(G) > f(\omega + 1)$. Then, by Lemma 2.5, there

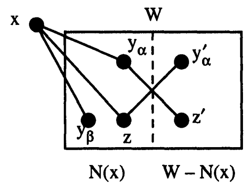

FIG. 10

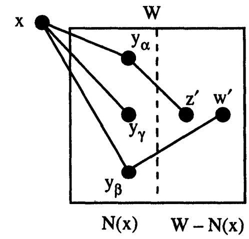

FIG. 11

exists a wall W in G of height $f(\omega+1)$ that supports a vertex x. We shall obtain a contradiction in two steps. We first show (1) there exists a set of vertices $X = \{x, y, a_1, \ldots, a_s, b_1, \ldots, b_s\}$ such that $s = R_{16}(\max\{2t, \omega+1\}), x \sim y, \{a_1, \ldots, a_s\} \subset N(x) - N(y), \{b_1, \ldots, b_s\} \subset N(y) - N(x)$, and $a_i \not\sim b_i$ for all i. We then show (2) there exists a subset of X that induces either D_t or B_t.

By Lemma 2.7 there exists an induced subwall $W_0 \subset W$ such that W_0 supports $x, h(W_0) \geq R(1+f(\omega), 1+R_{16}(\max\{2t, k+1\})$, and (*) holds. Define a coloring q on the two element subsets of $C(W_0)$ by $q(\alpha > \beta) = c$, where

$c = 2$ iff $\exists y \in ([\alpha]_{W_0} \cap N(x))\ \forall z \in ([\beta]_{W_0} \cap N(x))(y \not\sim z)$ and

$c = 1$ otherwise.

By Ramsey's theorem, there exists a homogeneous subset $H \subset C(W_0)$ such that either $q(\alpha > \beta) = 1$ for all $\alpha, \beta \in H$, and $|H| = 1 + f(\omega)$ or $q(\alpha > \beta) = 2$ for all $\alpha, \beta \in H$, and $|H| = 1 + R_{16}(\max\{2t, \omega+1\})$. In the first case, $W_1 = [H]_{W_0}$ is a wall such that $\omega(W_1) \leq \omega$ and $h(W_1) \geq 1 + f(\omega)$, which by Lemma 2.5 contradicts the induction hypothesis. In the second case, for each $\gamma \in C(W_1)$, there exists a left *-witness y_γ for γ. Let $y = y_\alpha$, where α is the largest color in $C(W_1)$, and let $a_i = y_{\gamma_i}$, where γ_i is the ith smallest color of $C(W_1)$. Finally choose $b_i \in [\gamma_i]_{W_1}$, so that $y_\alpha \sim b_i$. It is now easy to check that $X = \{x, y, a_1, \ldots, a_s, b_1, \ldots, b_s\}$ has the desired properties for (1).

Define a coloring r on the two element subsets of $[s]$ by $r(\beta > \gamma) = Y$, where Y is the image of the graph $G_{\beta,\gamma} = G[\{a_\beta, b_\beta, a_\gamma, b_\gamma\}]$ under the graph isomorphism that maps $a_\beta, b_\beta, a_\gamma, b_\gamma$ to 1, 2, 3, 4, respectively. There are 16 possibilities for such graphs depending on which of four possible edges are present. Thus, by Ramsey's theorem, there exists a homogeneous subset H' such that $|H'| \geq \max\{2t, \omega+1\}$ and $r(\beta > \gamma) = Y$, for all $\beta, \gamma \in H'$. Let $A = \{a_i : i \in H'\}$ and $B = \{b_i : i \in H'\}$. Since $|H| \geq \omega + 1$ and $A \subset N(y), 1 \not\sim 3$ in Y, i.e., A is an independent set. Similarly $2 \not\sim 4$ in Y and B is an independent set. This leaves four possibilities, which are illustrated in Figs. 12–15, for Y. If Y has no edges, then $G[X]$ contains an induced D_{2t}; if Y has one edge, then $G[X]$ contains an induced D_t; and if Y has two edges, then $G[X]$ contains an induced B_{2t}. Each possibility is a contradiction, so we are done. □

3. Open problems. The problem of determining whether $\mathrm{Forb}(P_5)$ has a polynomial on-line χ-binding function remains open. In fact, this problem is open even in the off-line case; all that is known is that if f is a χ-binding function for $\mathrm{Forb}(P_5)$, then f satisfies $c(\omega/\log\omega)^2 \leq f(\omega) \leq 2^\omega$. The lower bound follows from an observation of Gyárfás [4]: if $\alpha(G) < 3$, then $G \in \mathrm{Forb}(P_5)$ and $\chi(G) \geq \nu(G)/2$, and thus $(R(\omega,3)-1)/2 \leq f(\omega)$. The result then follows from a well-known lower bound on $R(3,\omega)$. The upper bound is only a small improvement on the on-line χ-binding function presented here.

For trees T for which $\mathrm{Forb}(T)$ is not χ_{FF}-bounded, it may be possible to determine the reason why. We have previously noted that $\mathrm{Forb}(T, K_{t,t})$ is χ_{FF}-bounded for any

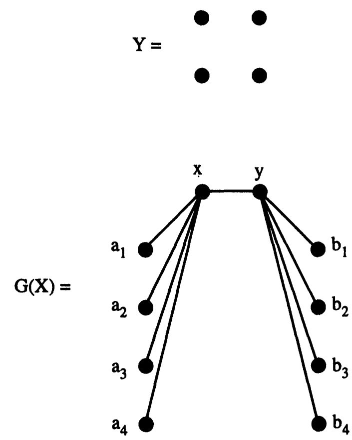

FIG. 12

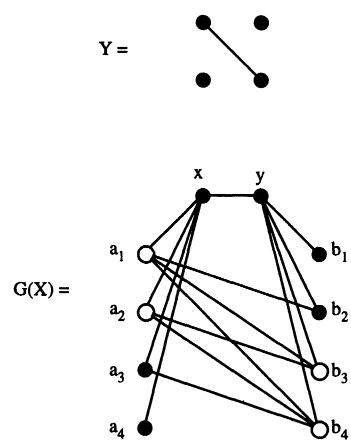

FIG. 13

tree. However, this does not tell us why Forb(T) is not χ-bounded. Our result that Forb(T, B_t) is χ-bounded for $T = D_k$ or $P_{k,1}$ is more informative, since $\chi_{\mathrm{FF}}(B_t) = t$. It would be interesting to prove similar results for other trees. However, the following two negative examples show that some caution is in order.

Gyárfás and Lehel's proof [7] that Forb(P_6) is not on-line χ-bounded actually shows more. Since the graphs they constructed do not induce B_3, their arguments show that Forb(P_6, B_3) is not on-line χ-bounded. Thus if T is a tree with radius greater than two, then Forb(T, B_t) is not on-line χ-bounded. In particular, it is not χ_{FF}-bounded.

We next present an example which provides a general construction for graphs which force the First-Fit algorithm to use a large number of colors. This example includes B_t as a special case.

Example 3.1. Let $t \geq 2$ and let $H = (V, E)$ be a graph such that

1. $V = A_1 \cup A_2 \cup \cdots \cup A_t$;
2. $A_j = \{a_{1j}, a_{2j}, \ldots, a_{jj}\}$ is a set of j independent vertices for $j = 1, 2, \ldots, t$;
3. $A_j \cap A_{j+1} = \varnothing$ for $j = 1, 2, \ldots, t-1$;
4. $a_{ij} \not\sim a_{ij+1}$ whenever $1 \leq i \leq j \leq t-1$; and
5. $a_{ij} \sim a_{kj+1}$ whenever $1 \leq i < k \leq j+1 \leq t$.

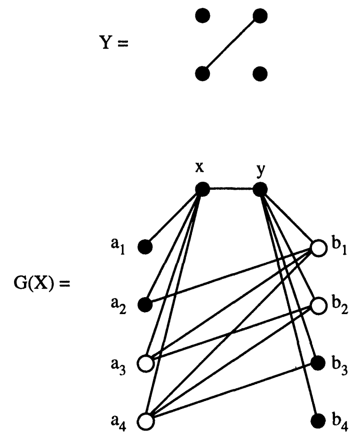

FIG. 14

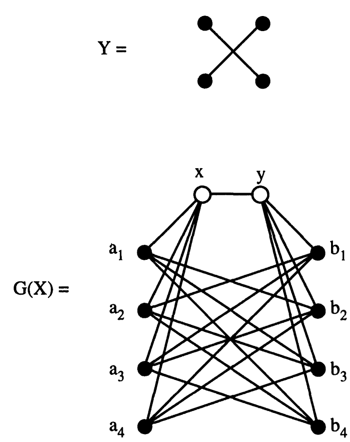

FIG. 15

Note that we do not require that $A_j \cap A_{j'} = \varnothing$ when $|j' - j| \geq 2$. Now let $|V| = n$ and let $v_1 < v_2 < \cdots < v_n$ be a linear order on V so that $\alpha < \beta$ whenever $v_\alpha = a_{ij}, v_\beta = v_{kj+1}$, and $1 \leq i < k \leq j+1 \leq t$. Then an easy inductive argument shows that the First-Fit algorithm will color H with t colors when the vertices are presented in this order; in fact, FF will color a vertex $v_\alpha = a_{ij}$ with color i.

The graph B_t (actually B_t with a single vertex removed) is obtained if $a_{ij} = a_{ij+2}$ whenever $1 \leq i \leq j \leq t-2$. More generally, suppose there is some $k \geq 2$ so that $a_{ij} = a_{ij+k}$ whenever $1 \leq i \leq j \leq t-k$, and the only adjacencies in G are those required by property 5 above. Then the chromatic number of the graph is three if k is odd and two if k is even.

On the other hand, if the sets $A_1, A_2, \ldots, A_t$ are pairwise disjoint and independent whenever $|j - i| \geq 2$, then we obtain a bipartite graph H which is the complement of a comparability graph (a *cocomparability* graph). In [9], Kierstead used this example to show that First-Fit can be forced to use arbitrarily many cliques to cover a comparability graph with independence number two. Of course this implies that First-Fit can be forced to use arbitrarily many chains to cover a width-two ordered set.

Since covering a comparability graph with cliques is equivalent to coloring a cocomparability graph, and cocomparability graphs induce neither LS_3 nor B_3, it follows that $\mathrm{Forb}(LS_3, B_3)$ is not χ_{FF}-bounded.

Motivated by the results presented previously and the examples discussed above, we suggest the following problems.

Problem 1. Given a tree T, do there exist a function $g(\omega, \chi)$ and an integer r such that if $G \in \mathrm{Forb}(T)$ and $\chi_{\mathrm{FF}}(G) > g(\omega(G), \chi)$, then there exists an induced subgraph H of G with $\chi(H) \leq r$ and $\chi_{\mathrm{FF}}(H) \geq \chi$?

Problem 2. Given a tree T, does there exist a function $h(\omega, \chi)$ such that if $G \in \mathrm{Forb}(T)$ and $\chi_{\mathrm{FF}}(G) > g(\omega(G), \chi)$, then G contains an induced subgraph H of the type constructed in Example 3.1 with $\chi_{\mathrm{FF}}(H) \geq \chi$?

Problem 3. Is $\mathrm{Forb}(L_k, B_t)$ χ_{FF}-bounded?

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