THE DESCENDANT COLORED JONES POLYNOMIALS

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In memory of our friend, Vaughan Jones.

Abstract. We discuss two realizations of the colored Jones polynomials of a knot, one appearing in an unnoticed work of the second author in 1994 on quantum R-matrices at roots of unity obtained from solutions of the pentagon identity, and another formulated in terms of a sequence of elements of the Habiro ring appearing in recent work of D. Zagier and the first author on the Refined Quantum Modularity Conjecture.

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1. Introduction

The Jones polynomial [Jon87] is a fascinating polynomial invariant of knots with deep connections to the topology and geometry in dimension 3. Its discovery in 1984, apart from revolutionizing Knot Theory, lead to a new area of research, Quantum Topology, with applications and challenges that we will not attempt to summarize here. The Jones polynomial and its colored versions (a sequence of polynomials, one for every irreducible finite dimensional representation of the Lie algebra $\mathfrak{sl}_2$) are versatile invariants with numerous

Date: 19 June 2022.

Key words and phrases: knots, Jones polynomial, colored Jones polynomials, Kashaev invariant, Habiro ring, Habiro polynomials, cyclotomic expansion, ADO invariants, descendants, holomorphic quantum modular forms, Volume Conjecture, Quantum Modularity Conjecture, q-holonomic functions, q-hypergeometric functions.

interpretations, among them as partition functions of a 3-dimensional Chern–Simons gauge theory [Wit89]. This sequence of Laurent polynomials with integer coefficients (the so-called colored Jones polynomials of a knot), introduced by Kirillov–Reshetikhin [KR89] and Turaev [Tur88], is not a random sequence of polynomials. Indeed, it was shown in [GL05] that it is q-holonomic, i.e., that it satisfies a nontrivial recursion relation with coefficients in $\mathbb{Z}[q, q^n]$. Moreover, a canonically chosen recursion is a knot invariant (the so-called $\hat{A}$-polynomial of a knot) which is conjectured to determine the $\mathrm{SL}_2(\mathbb{C})$-character variety of the knot, viewed from the boundary [Gar04]. The colored Jones polynomial of a knot has appeared in equivalent forms in several occasions, such as the sequence of Habiro polynomials of a knot [Hab02, Hab08] and the $\mathfrak{sl}_2$ Akutsu–Deguchi–Ohtsuki invariants [ADO92] which only recently were shown to be equivalent to the colored Jones polynomials [Wil22].

An unexpected connection of the colored Jones polynomials and hyperbolic geometry [Thu77] arose from the Volume Conjecture [Kas97] for the Kashaev invariant of a knot whose exponential growth rate is conjectured to be the suitably normalized Volume of a hyperbolic knot. This conjecture, combined with the identification of the Kashaev invariant with an evaluation of the colored Jones polynomials by Murakami–Murakami [MM01], linked the Jones polynomials with hyperbolic geometry.

In this paper we discuss two equivalent reformulations (or rather, realizations) of the colored Jones polynomials of a knot: one is a sequence $\mathrm{DJ}^{(m)}(q)$ of elements of the Habiro ring defined in Equation (6) below and shown to be equivalent to the colored Jones polynomial of a knot in Proposition 2.2. This sequence of elements of the Habiro ring defines the top row of a matrix of knot invariants that arises in recent work of D. Zagier and the first author regarding the Refined Quantum Modularity Conjecture [GZb, GZa].

A second reformulation of the colored Jones polynomials comes from unnoticed work of the second author in 1994 regarding quantum R-matrices at roots of unity obtained from solutions of the pentagon identity [Kas96]. These R-matrix invariants $\langle K\rangle_{N,n}$ of a knot K are defined for a primitive N-th root of unity ζ (of strictly positive order), and for a number $n \in \mathbb{Z}/N\mathbb{Z}$ in Theorem 3.1 and are conjectured to be equivalent to the colored Jones polynomials of a knot; see Conjecture 3.2 below.

Acknowledgements. The authors wish to thank Jie Gu, Marcos Mariño, Campbell Wheeler and Don Zagier for many enlightening conversations, and the International Math Center in SUSTech and the University of Geneva for their hospitality. The work of R. K. is partially supported by the Swiss National Science Foundation, the subsidies no 200020_192081 and no 200020_200400, and the Russian Science Foundation, the subsidy no 21-41-00018.

2. The descendant colored Jones polynomials of a knot

2.1. A conjectured matrix-valued invariant from [GZb].

We begin by explaining the most recent reappearance of the colored Jones polynomials. An extension of the Volume Conjecture is the Quantum Modularity Conjecture of Zagier [Zag10] which, roughly speaking, discusses the asymptotics of the Kashaev invariant at roots of unity. More recently, a Refined Quantum Modularity Conjecture was proposed in [GZb] which suggests the existence of a matrix-valued invariant defined at (and near) roots of unity. These conjectured matrix-valued invariants (denoted by $\mathbf{J}(x)$ for $x \in \mathbb{Q}/\mathbb{Z}$) were studied in [GZb, Sec.5] with explicit examples given for the 4_1 and the 5_2 knots in Sections 7.1 and 7.2 of [GZb], respectively.

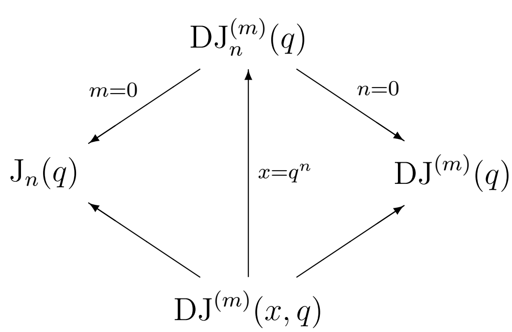
The conjectured matrix-valued knot invariant $\mathbf{J}$ (defined as a 1-periodic function at the rationals, or alternatively as a function at the complex roots of unity), has many interesting analytic, arithmetic properties and geometric properties which are listed in detail in the introduction (Part 0) of [GZb]. The rows and columns of the matrix $\mathbf{J}$ are parametrized by the set of boundary parabolic $\mathrm{SL}_2(\mathbb{C})$-representations of the knot (assuming the latter set is finite), and this set always contains two distinguished elements, the trivial representation σ_0, and the geometric representation σ_1 of a hyperbolic knot. The σ_0-column of $\mathbf{J}$ is $(1, 0, \dots, 0)^t$ but the σ_0-row is very interesting. Its (σ_0, σ_1)-entry is the Kashaev invariant of a knot, whereas the remaining entries are expected to be elements of the Habiro ring (whose definition is recalled below), tensored with the rational numbers. Moreover, the entries of each row of the matrix are expected to extend to sequences that satisfy a homogeneous recursion relation (if not the σ_0-row), or an inhomogeneous recursion (if the σ_0-row). In other words, the matrix $\mathbf{J}$ is expected to be a fundamental matrix solution of a q-holonomic module.

In [Zag10, Sec.7.3,7.4] a method was proposed for defining all but the first row of this matrix, given a suitable ideal triangulation of the knot. But the first row of the above matrix remained elusive. The goal of this paper is to propose a definition for this first row and verify some of its conjectured properties. After a linear change of variables, the first row of the matrix $\mathbf{J}$ is conjecturally equal to one of the best-known quantum knot invariants, namely the colored Jones polynomials; see Conjecture 2.4 below.

2.2. **Definition of the descendants.** In this section we extend the 2-parameter colored Jones function $\mathrm{J}_n(q)$ of a knot in 3-space into a 3-parameter *descendant colored Jones* function $\mathrm{DJ}_n^{(m)}(q)$ for $n \geq 0$, $m \in \mathbb{Z}$, which

- specializes to the colored Jones polynomials $\mathrm{J}_n(q)$ when $m = 0$ and $n \geq 1$;
- specializes to a sequence of elements of the Habiro ring $\mathrm{DJ}^{(m)}(q)$ when $n = 0$;
- is determined by either of the above specializations;
- is determined by a 3-parameter function $\mathrm{DJ}^{(m)}(x, q)$ by $\mathrm{DJ}_n^{(m)}(q) = \mathrm{DJ}^{(m)}(q^n, q)$ for all $n \geq 1$.

In other words, we have a commutative diagram



(1)

To define the functions appearing in this diagram, we start with the colored Jones polynomials $\mathrm{J}_n^K(q) \in \mathbb{Z}[q^{\pm}]$ of a knot K (mostly omitted from the notation), colored by the n-dimensional irreducible representation of $\mathfrak{sl}_2$ for $n \geq 1$ and normalized to be 1 at the unknot [Tur94]. In [Hab08] Habiro proved that the colored Jones polynomials can be written in the form

$$\mathrm{J}_n(q) = \sum_{k=0}^{\infty} c_{n,k}(q)\mathrm{H}_k(q), \qquad c_{n,k}(q) = q^{-kn}(q^{n+1};q)_k(q^{n-1};q^{-1})_k \tag{2}$$

where $\mathrm{H}_k(q) \in \mathbb{Z}[q^{\pm}]$ for all $k \geq 0$ and $c_{n,k}(q)$ is the cyclotomic kernel which is independent of the knot and vanishes when $k \geq n \geq 1$. Equation (2) can be inverted [Hab08, Lem. 6.1]

$$\mathrm{H}_k(q) = \sum_{n=1}^{k+1} \gamma_{k,n}(q)\mathrm{J}_n(q), \qquad \gamma_{k,n}(q) = (-1)^{k-n-1}\frac{q^{\frac{k(k+3)+n(n-3)}{2}+1}(1-q^n)(1-q^{2n})}{(q;q)_{k+n+1}(q;q)_{k-n+1}} \tag{3}$$

where $\gamma_{k,n}(q) = 0$ for $n \geq k+2$ and $k \geq 0$. There are three important features in Habiro's expansion (2) of the colored Jones polynomials.

(i) The Habiro polynomials $\mathrm{H}_k(q)$ which a priori lie in $\mathbb{Q}(q)$ (as follows from (3)), actually lie in $\mathbb{Z}[q^{\pm}]$.

(ii) Setting $n = 0$ in the right hand side of (2) gives a well-defined element of the Habiro ring $\widehat{\mathbb{Z}[q]} = \varprojlim_n \mathbb{Z}[q]/((q;q)_n)$.

(ii) The dependence on the color comes only through the cyclotomic kernel $c_{n,k}(q)$. Thus, one can replace q^n by a variable x and define an element $\mathrm{J}(x,q)$ of the colored Habiro ring $\Lambda = \varprojlim_n \mathbb{Z}[q^{\pm}][x+x^{-1}-2]/(c_n(x,q))$, so that $\mathrm{J}(q^n,q) = \mathrm{J}_n(q)$ for $n \geq 1$, where

$$\mathrm{J}(x,q) = \sum_{k=0}^{\infty} c_k(x,q)\mathrm{H}_k(q), \qquad c_k(x,q) = x^{-k}(qx;q)_k(q^{-1}x;q^{-1})_k\,. \tag{4}$$

We now have all the ingredients to define the descendant colored Jones function.

Definition 2.1. For an integer m and for $n \geq 0$, we define

$$\mathrm{DJ}_n^{(m)}(q) = \sum_{k=0}^{\infty} c_{n,k}(q)\mathrm{H}_k(q)\,q^{km}, \qquad \mathrm{DJ}^{(m)}(x,q) = \sum_{k=0}^{\infty} c_k(x,q)\mathrm{H}_k(q)q^{km} \tag{5}$$

and

$$\mathrm{DJ}^{(m)}(q) = \sum_{k=0}^{\infty} (q;q)_k(q^{-1};q^{-1})_k\mathrm{H}_k(q)\,q^{km}\,. \tag{6}$$

2.3. **Properties of the descendants.** From their very definition, the 0-th descendants $\mathrm{DJ}_n^{(0)}(q)$ and $\mathrm{DJ}^{(0)}(q)$ are nothing but the colored Jones polynomial $\mathrm{J}_n(q)$ and the Kashaev invariant of the knot. Thus, one may call $\mathrm{DJ}_n^{(m)}(q)$ and $\mathrm{DJ}^{(m)}(q)$ the descendant colored Jones and the descendant Kashaev invariant of a knot, respectively.

Note further that for all integers m, the 3-variable invariant $\mathrm{DJ}^{(m)}(x,q)$ is an element of the colored Habiro ring, and, for all $n \geq 0$, it satisfies $\mathrm{DJ}^{(m)}(q^n,q) = \mathrm{DJ}_n^{(m)}(q)$.

Note moreover that $\mathrm{DJ}_n^{(m)}(q) \in \mathbb{Z}[q^{\pm}]$ for $n \geq 1$ whereas $\mathrm{DJ}^{(m)}(q) \in \widehat{\mathbb{Z}[q]}$. In a sense, $\mathrm{DJ}^{(m)}(q)$ is a twisted Fourier transform of $\mathrm{J}_n(q)$.

The next proposition describes the equivalence among these knot invariants. Let μ denote the set of complex roots of unity.

Proposition 2.2. Each family of invariants $\{\mathrm{J}_n(q)|n \in \mathbb{Z}_{\geq 0}\}$, $\{\mathrm{DJ}_n^{(m)}(q)|n \in \mathbb{Z}_{\geq 0}, m \in \mathbb{Z}\}$, $\{\mathrm{DJ}^{(m)}(x,q)|m \in \mathbb{Z}\}$, $\{\mathrm{DJ}^{(m)}(q)|m \in \mathbb{Z}\}$, $\{\mathrm{J}_n(\zeta)|n \in \mathbb{Z}_{\geq 0}, \zeta \in \mu\}$, $\{\mathrm{DJ}_n^{(m)}(\zeta)|n \in \mathbb{Z}_{\geq 0}, \zeta \in \mu\}$, $\{\mathrm{DJ}^{(m)}(x,\zeta) m \in \mathbb{Z}, \zeta \in \mu\}$, $\{\mathrm{DJ}^{(m)}(\zeta)|m \in \mathbb{Z}, \zeta \in \mu\}$ determines all other families.

Proof. We first discuss the equivalence of invariants in the left triangle of Diagram (1). Clearly, the descendant colored Jones polynomials $\mathrm{DJ}_n^{(m)}(q)$ specialize to the colored Jones polynomials $\mathrm{J}_n(q)$ when $m = 0$. Conversely, $\mathrm{J}_n(q)$ determines $\mathrm{H}_k(q)$ by Equation (2) and hence determines $\mathrm{DJ}_n^{(m)}(q)$ by Equation (5).

The evaluation maps $\Lambda \to \widehat{\mathbb{Z}[q]}$ given by $x \mapsto q^n$ for all n give an injection of Λ to $\widehat{\mathbb{Z}[q]}^{\mathbb{N}}$ as was shown in [Hab08]. It follows that $\mathrm{DJ}_n^{(m)}(q)$ determines $\mathrm{DJ}^{(m)}(x,q)$.

This shows that each one of the invariants $\mathrm{J}_n(q)$, $\mathrm{DJ}_n^{(m)}(q)$, $\mathrm{DJ}^{(m)}(x,q)$ determines all others. Moreover, since the collection of evaluation maps $\widehat{\mathbb{Z}[q]} \to \overline{\mathbb{Q}}$ given by $q \mapsto \zeta$ uniquely determines an element of the Habiro ring [Hab04], and the same holds for the colored Habiro ring [Hab08], it follows that each one of the invariants $\mathrm{J}_n(q)$, $\mathrm{DJ}_n^{(m)}(q)$, $\mathrm{DJ}^{(m)}(x,q)$, $\mathrm{J}_n(\zeta)$, $\mathrm{DJ}_n^{(m)}(\zeta)$, $\mathrm{DJ}^{(m)}(x,\zeta)$ determines all others.

We now discuss the equivalence of invariants in the right triangle of Diagram (1). The descendant colored Jones polynomials $\mathrm{DJ}_n^{(m)}(q)$ specialize to $\mathrm{DJ}^{(m)}(q)$ when $n = 0$, which specialize to $\mathrm{DJ}^{(m)}(\zeta)$ for any complex root of unity ζ. Moreover, by Equation (5) and the inversion of the discrete Fourier transform, it follows that $\mathrm{DJ}^{(m)}(\zeta)$ determines $(\zeta;\zeta)_k(\zeta^{-1};\zeta^{-1})_k \,\mathrm{H}_k(\zeta)$, and hence, $(q;q)_k(q^{-1};q^{-1})_k \,\mathrm{H}_k(q)$ (since the latter are in $\mathbb{Z}[q^{\pm}]$), and hence $\mathrm{H}_k(q)$, and hence $\mathrm{DJ}_n^{(m)}(q)$. □

The next proposition concerns the q-holonomic properties of the descendant colored Jones functions. For a detailed definition of q-holonomic functions in several variables and their closure properties, we refer the reader to [Zei90, PWZ96] and also to [Sab93, GL16]. Roughly speaking, q-holonomic functions have annihilating ideals (i.e., ideals of linear q-difference equations) of maximal dimension.

Proposition 2.3. $\mathrm{DJ}_n^{(m)}(q)$, $\mathrm{DJ}^{(m)}(x,q)$ and $\mathrm{DJ}^{(m)}(q)$ are q-holonomic functions of (m,n), (m,x) and m, respectively. Linear q-difference equations for these functions can be obtained from one for the Habiro polynomials $\mathrm{H}_k(q)$.

Proof. The first part follows from the closure properties of q-holonomic functions explained in detail in [GL16], and the fact that the colored Jones polynomial (and hence, the sequence of Habiro polynomials) is q-holonomic [GL05] and $c_{n,k}(q)$ and $g_{k,n}(q)$ are q-holonomic (in fact q-proper hypergeometric) functions. Using a computer implementation given by Koutschan [Kou09, Kou10], we can obtain a linear q-difference equation for $\mathrm{DJ}^{(m)}(q)$ given one for $\mathrm{H}_k(q)$. The obtained linear q-difference equations may not be of the smallest order. □

The above proposition implies that the annihilating ideal of the 3-variable invariant $\mathrm{DJ}^{(m)}(x,q)$ is a knot invariant, and so is a minimal recursion of $\mathrm{DJ}^{(m)}(x,q)$ with respect to x or m. Moreover, a minimal recursion with respect to m can be computed from a recursion for the Habiro polynomials H_k as well as some of their initial values. This is illustrated in Section 2.5 below.

The annihilating polynomial of $\mathrm{DJ}^{(m)}(x,q)$ with respect to m has an excess degree whose relation to the character variety of $\mathrm{SL}_2(\mathbb{C})$-representations is a challenging question discussed in Section 2.5 below.

To effectively apply Proposition 2.3 we need to know a recursion for the Habiro polynomials of a knot. Such a recursion can be obtained from a q-hypergeometric formula for the Habiro polynomials, as was done in [GS06] for all twist knots using a formula for the Habiro polynomials given in [Hab02, Mas03]. Alternatively, one can use a recursion for $\mathrm{J}_n(q)$, Equation (3) and the methods of [Kou10] to obtain a recursion for the Habiro polynomials. Although the above mentioned computer implementation in theory succeeds (and does so with a certified computation), in practice it may not terminate. As a concrete example, the colored Jones polynomials of the $(-2,3,7)$ pretzel knot satisfy a guessed inhomogeneous sixth order q-difference equation [GK12], but a direct attempt to find a recursion for its Habiro polynomials did not terminate. Instead, one can use the recursion for the colored Jones to compute several thousand values of them (modulo a prime) and Equation (3) to do the same for the Habiro polynomials, and with some hints for the shape of the Newton polygon to guess a recursion for the Habiro polynomials. This is explained in detail for the $(-2,3,7)$ pretzel knot and for several other knots in forthcoming work [GK].

Let us comment on the chirality properties of the descendant colored Jones functions. It is well-known that the Jones polynomial (and its colored versions) are chiral invariants of knots. Explicitly, if K^* denotes the mirror of a knot K, then $\mathrm{J}_n^{K^*}(q)=\mathrm{J}_n(q^{-1})$ which implies that $\mathrm{H}_k^{K^*}(q)=\mathrm{H}_k(q^{-1})$ and $\mathrm{DJ}^{K^*,(m)}(x,q)=\mathrm{DJ}^{K,(-m)}(x^{-1},q^{-1})$.

2.4. **A conjecture.** We end this section with a conjecture. Let us denote by $r+1$ the size of the matrix $\mathbf{J}$.

Conjecture 2.4. The top row of the matrix $\mathbf{J}$ of the conjectured knot invariants [GZb] is a rational linear combination of $r+1$ consecutive terms of the descendant invariant $\mathrm{DJ}^{(m)}(q)$.

2.5. **Examples.** In this section we give explicit formulas for the descendant colored Jones polynomials and their recursions for the trefoil (a non-hyperbolic knot) and for the two simplest hyperbolic knots, the 4_1 and 5_2 knots.

For the 3_1 knot, we have $\mathrm{H}_k^{3_1}(q)=(-1)^k q^{\frac{1}{2}k(k+3)}$ for all $k\geq 0$ [Hab02, Mas03]. It follows from Equation (5) that the descendant colored Jones polynomials $\mathrm{DJ}^{3_1,(m)}(x,q)$ satisfy the inhomogeneous linear q-difference equation

$$-q^{m+3}\,\mathrm{DJ}^{3_1,(m+2)}(x,q)+(x+x^{-1})q^{m+2}\,\mathrm{DJ}^{3_1,(m+1)}(x,q)+(1-q^{m+1})\,\mathrm{DJ}^{3_1,(m)}(x,q)=1 \quad (7)$$

for $m\in\mathbb{Z}$. We can write the above q-difference equation in operator form by introducing the operators S_m and Q_m which act on $f(m)$ by $(S_m f)(m)=f(m+1)$, $(Q_m f)(m)=q^m f(m)$ and satisfy the q-commutation relation $S_m Q_m=qQ_m S_m$. Then, Equation (7) takes the form $B^{3_1}\mathrm{DJ}^{3_1,(m)}=1$ where

$$B^{3_1}(S_m,Q_m,x,q)=-q^3 Q_m S_m^2+(x+x^{-1})q^2 Q_m S_m+(1-qQ_m)\,. \quad (8)$$

We next consider the 4_1 knot, whose Habiro polynomials are given by $\mathrm{H}_k^{4_1}(q) = 1$ for all $k \geq 0$, hence

$$\mathrm{DJ}^{4_1,(m)}(x,q) = \sum_{k=0}^{\infty} x^{-k}(qx;q)_k(q^{-1}x;q^{-1})_k\, q^{km}\,. \tag{9}$$

When $x = 1$, these coincide (after a suitable $\mathbb{Q}[q^{\pm}]$ linear combination) with the elements in the top row of an experimentally found matrix in [GZb]. The descendants satisfy the inhomogeneous linear q-difference equation

$$q^{m+1}\,\mathrm{DJ}^{4_1,(m+1)}(x,q) + (1-(x+x^{-1})q^m)\,\mathrm{DJ}^{4_1,(m)}(x,q) + q^{m-1}\mathrm{DJ}^{4_1,(m-1)}(x,q) = 1 \tag{10}$$

for $m \in \mathbb{Z}$. Equation (10) takes the form $B^{4_1}\mathrm{DJ}^{4_1,(m)} = 1$ where

$$B^{4_1}(S_m, Q_m, x, q) = qQ_mS_m^2 + (1-(x+x^{-1})Q_m)S_m + Q_m^{-1}S_m^{-1}\,. \tag{11}$$

Note that after setting $q = 1$ and $x = 1$ and $Q_m = 1$ and renaming S_m to be L, we have $B^{4_1}(L,1,1,1) = L^2 - L + 1$ has roots generating the two embeddings of the trace field $\mathbb{Q}(\sqrt{-3})$ of the 4_1 knot.

A more interesting example is the 5_2 knot, where the descendant colored Jones polynomials (or their evaluation at $q = 1$) is a one-parameter q-holonomic module, whereas the two-dimensional state-sum formula for the Kashaev invariant leads to a two-parameter q-holonomic module considered in [GZb] in relation to the Refined Quantum Modularity Conjecture. The coincidence of the spanning space of the two modules is no longer obvious, but appears to be true. Another feature of this knot is that the recursion for the descendant colored Jones polynomials has higher degree (precisely, two more) than the degree of the recursion of the colored Jones polynomials, and the degree of the A-polynomial of the knot. This excess degree and its relation to the character variety of $\mathrm{SL}_2(\mathbb{C})$-representations is a challenging question discussed further below.

We first recall the Habiro polynomials of the 5_2. The latter is the twist knot K_2 and its Habiro polynomials were given by Habiro [Hab02]; see also Masbaum [Mas03] for a detailed discussion. In fact, the Habiro polynomials of 5_2 are given explicitly by

$$\mathrm{H}_k^{5_2}(q) = (-1)^k q^{\frac{1}{2}k(k+3)} \sum_{s=0}^{k} q^{s(s+1)} \binom{k}{s}_q \tag{12}$$

where $\binom{a}{b}_q = (q;q)_a/((q;q)_b(q;q)_{b-a})$ is the q-binomial function. In [GS06], it was shown that $\mathrm{H}_k = \mathrm{H}_k^{5_2}(q)$ satisfies the linear q-difference equation

$$\mathrm{H}_{k+2}^{5_2}(q) + q^{3+k}(1+q-q^{2+k}+q^{4+2k})\mathrm{H}_{k+1}^{5_2}(q) - q^{6+2k}(-1+q^{1+k})\mathrm{H}_k^{5_2}(q) = 0, \qquad (k \geq 0) \tag{13}$$

with initial conditions $\mathrm{H}_k^{5_2}(q) = 0$ for $k < 0$ and $\mathrm{H}_0^{5_2}(q) = 1$. Actually, the above recursion is valid for all integers if we replace the right hand side of it by $\delta_{k+2,0}$. This, combined with Equation (5) and [Kou10], gives that $\mathrm{DJ}^{(m)} = \mathrm{DJ}^{5_2,(m)}(x,q)$ satisfies the linear q-difference equation

$$
\begin{aligned}
&(-1+q^{1+m})(-1+q^{2+m})x^2\mathrm{DJ}^{(m)} - q^{2+m}(-1+q^{2+m})x(1+q+x+(1+q)x^2)\mathrm{DJ}^{(1+m)} \\
&+ q^{3+m}(q^{3+m}+(-1+q^{2+m}+q^{3+m})x+(-2-q+q^{2+m}+2q^{3+m}+q^{4+m})x^2+(-1+q^{2+m}+q^{3+m})x^3+q^{3+m}x^4)\mathrm{DJ}^{(2+m)} \\
&- q^{4+m}(q^{3+m}+(-1+q^{3+m}+q^{4+m})x+(-1+q^{2+m}+2q^{3+m}+q^{4+m})x^2+(-1+q^{3+m}+q^{4+m})x^3+q^{3+m}x^4)\mathrm{DJ}^{(3+m)} \\
&+ q^{5+m}x(q^{3+m}+q^{4+m}+(-1+q^{4+m})x+(q^{3+m}+q^{4+m})x^2)\mathrm{DJ}^{(4+m)} - q^{10+2m}x^2\mathrm{DJ}^{(5+m)} \\
&= x(q^{2+m}+q^{4+m}+(1-q^{1+m}-2q^{3+m}-q^{5+m})x+(q^{2+m}+q^{4+m})x^2)\mathrm{H}_0(q) + q^m x(1-xq^{-1})(1-qx)\mathrm{H}_1(q)\,. \qquad (14)
\end{aligned}
$$

Using the values $\mathrm{H}_0(q)=1$, $\mathrm{H}_1(q)=-q^2-q^4$, it follows that the right hand side of the above recursion is x^2 for all m. Writing the above equation in operator form $B^{5_2}\mathrm{DJ}^{5_2,(m)}=x^2$, where $B^{5_2}=B^{5_2}(S_m,Q_m,x,q)$, we obtain that $B^{5_2}(L,1,1,1)=-L^2(-5+7L-4L^2+L^3)$, and three nonzero roots of the above polynomial generate the three embeddings of the trace field of the 5_2 knot. The latter is a cubic field of discriminant -23.

Although Equation (14) is fifth order and inhomogeneous, we claim that the $\mathbb{Z}[q^{\pm}]$-module of the colored Habiro ring generated by $\mathrm{DJ}^{(m)}(x,q)$ for all $m\geq 0$ equals to the module $\mathcal{M}$ spanned by $\{1,\mathrm{DJ}^{(0)}(x,q),\mathrm{DJ}^{(1)}(x,q),\mathrm{DJ}^{(2)}(x,q)\}$. Indeed, since the coefficient of $\mathrm{DJ}^{(m)}$ (resp. $\mathrm{DJ}^{(m+1)}$) in (14) vanishes when $m=-2$ and $m=-1$, (resp. $m=-2$) the specialization of (14) to $m=-2$ and to $m=-1$ gives that $\mathrm{DJ}^{(3)}(x,q)$ (resp. $\mathrm{DJ}^{(4)}(x,q)$) is in $\mathcal{M}$. Given this, Equation (14) then implies that $\mathrm{DJ}^{(m)}(x,q)\in\mathcal{M}$ for all natural numbers m. On the other hand, $\mathrm{DJ}^{(m)}(x,q)$ for $m=-1$ and $m=-2$ do not seem to lie in $\mathcal{M}$.

We now recall a second q-holonomic module which is motivated by the (projection-dependent) formula for the Kashaev invariant of 5_2 given in [Kas97, Eqn. (2.3)]

$$
\mathrm{J}(q)=\sum_{k,l\geq 0} q^{-(k+l+1)l}\frac{(q;q)_{k+l}^2}{(q^{-1};q^{-1})_l} = \sum_{k,l\geq 0}(-1)^l q^{-\frac{1}{2}(2k+l+1)l}\frac{(q;q)_{k+l}^2}{(q;q)_l}\,. \qquad (15)
$$

(Warning: the above formula is q times the evaluation of the n-th colored Jones polynomial at the n-th root of unity.) Given this formula, we can define a 2-parameter family of descendants (elements of the Habiro ring) by

$$
\mathrm{DJ}_{a,b}(q)=\sum_{k,l\geq 0}(-1)^l q^{-\frac{1}{2}(2k+l+1)l+ak+bl}\frac{(q;q)_{k+l}^2}{(q;q)_l} \qquad (a,b\in\mathbb{Z})\,. \qquad (16)
$$

Then, $\mathrm{DJ}_{a,b}(q)$ is a q-holonomic module in two variables. Experimentally, it appears that the span over $\mathbb{Q}(q)$ of $\mathrm{DJ}_{a,b}(q)$ for (a,b) in some cone coincides with the span of $\{1,\mathrm{DJ}^{(0)},\mathrm{DJ}^{(1)},\mathrm{DJ}^{(2)}\}$. For instance, we have:

$$
\begin{aligned}
\mathrm{DJ}_{1,0} &= (3-q^{-1})\mathrm{DJ}^{(0)}+(1-3q)\mathrm{DJ}^{(1)}+q^2\mathrm{DJ}^{(2)} \\
\mathrm{DJ}_{-1,0} &= 3q\mathrm{DJ}^{(0)}-3q^2\mathrm{DJ}^{(1)}+q^3\mathrm{DJ}^{(2)} \\
\mathrm{DJ}_{1,-1} &= (3+q^{-2}-q^{-1})\mathrm{DJ}^{(0)}+(1-q^{-1}-3q)\mathrm{DJ}^{(1)}+q^2\mathrm{DJ}^{(2)}-q^{-2}+q^{-1} \\
\mathrm{DJ}_{0,-1} &= -\mathrm{DJ}^{(0)}+q\mathrm{DJ}^{(1)}+1 \\
\mathrm{DJ}_{-1,-1} &= 2q\mathrm{DJ}^{(0)}-q^2\mathrm{DJ}^{(1)}\,.
\end{aligned}
$$

In the examples given above for the 3_1, 4_1 and 5_2 knots, we observed two phenomena for the annihilating polynomial $B(S_m, Q_m, x, q)$ with respect to m of the descendant colored Jones polynomials, namely:

(a) The degree of $B(S_m, Q_m, x, q)$ is greater than or equal to the degree of the A-polynomial of a knot [CCG+94] (which conjecturally coincides with the degree of the minimal inhomogeneous recursion of the colored Jones polynomials). Let δ denote this difference.

(b) $B(S_m, Q_m, x, q)$ is monic and the coefficient of S_m^j in $B(S_m, Q_m, x, q)$ vanishes when $m = -\delta, -\delta+1, \dots, -j-1$ for $j = 0, \dots, \delta-1$. This implies that the $\mathbb{Z}[q^{\pm}]$-span of $\mathrm{DJ}^{(m)}(x, q)$ for $m \geq 0$ equals to the span of $\mathrm{DJ}^{(m)}(x, q)$ for $m = 0, \dots, \deg(A) - 1$.

The above observations hold for all twist knots K_p considered in [Mas03, Fig. 3], where $K_1 = 3_1$, $K_{-1} = 4_1$ and $K_2 = 5_2$. Their A-polynomials were computed by Hoste–Shanahan [HS04] and the $\widehat{A}$ polynomials (i.e. a minimal recursion for the colored Jones polynomials) was computed in [LÔ6]. The recursions for the Habiro polynomials of the twist knots was given in [GS06], and, using this, one can obtain recursions for the descendant colored Jones polynomials with respect to m. An explicit computation shows that the B-polynomials of twist knots satisfy the above mentioned two properties. Moreover, the degrees of the A-polynomials, the B-polynomials and the excess numbers of twist knots are given by

$$\begin{aligned} &\deg(A_{K_p}) = 2p-1, \quad \deg(B_{K_p}) = 3p-1, \quad \delta(K_p) = p, \qquad (p>0) \\ &\deg(A_{K_p}) = 2|p|, \quad \deg(B_{K_p}) = 3|p|-1, \quad \delta(K_p) = |p|-1 \quad (p<0)\,. \end{aligned} \tag{17}$$

We mention parenthetically that the degree of the A-polynomials of the twist knots coincide with the degrees of their trace fields [HS01]. The excess degree of a knot and its relation to quantum invariants, their asymptotics, and their connection with the work of Gukov et al [GM21] is studied in detail by Campbell Wheeler in his upcoming thesis [Whe] and in forthcoming work [GGMnW].

3. The colored Jones polynomials at roots of unity

This section fills a historical gap where we discuss a previously unnoticed conjectural relationship between the colored Jones polynomials and some quantum R-matrices at roots of unity considered by the second author in [Kas96]. Namely, we show that the R-matrices over the vector space $\mathbb{C}^N$ depending on a primitive N-th root of unity ζ and indexed by an integer parameter $0 < m < N$ actually give knot invariants which conjecturally coincide with the $(m+1)$-colored Jones polynomial evaluated at $q = \zeta$. In particular, we show that a specific choice of the discrete parameter $m = N-1$ gives the Kashaev invariant $\langle K \rangle_N$ of a knot, which, in its turn, appears to be equal to the N-colored Jones polynomial $J_N(\zeta)$ [MM01]. The other choices of m lead to knot invariants $\langle K \rangle_{N,m}$ for $m \in \{0, \dots, N-1\}$ (or even links, with each component colored by its own integer parameter).

In [Kas94], the second author assigned a state-sum to a root of unity and to a pair consisting of a triangulated compact closed oriented 3-manifold and a link represented by a Hamiltonian sub-complex, and showed that the state-sum is invariant under some natural Pachner moves. The state-sum depended on a tetrahedral weight function, from which one can obtain an R-matrix that satisfies the Yang–Baxter equation. This led to an invariant of knots in the 3-sphere that depends to a root of unity, where the knot in question is a Hamiltonian path of a triangulation of the 3-sphere. This is how the R-matrix of [Kas95] was found and explained (without pictures), see [Kas95, Sec. 4]. The R-matrices of [Kas96], which depend on discrete and continuous spectral parameters, are expected to generalize the R-matrix of [Kas95]. The continuous spectral parameter apparently plays no role in constructing knot invariants, while the discrete spectral parameter seems to take care of the color in the colored Jones polynomials.

We should remark that for all values of the colors, this R-matrix of [Kas96] is over the same vector space $\mathbb{C}^N$ and appears nontrivial even when the color is trivial. This is in contrast to the standard (quantum group) definition of the colored Jones polynomials, where the size of the R-matrix depends on the color and is trivial for the trivial color.

3.1. **R-matrices with a spectral parameter.** We begin our discussion of R-matrices by introducing some useful notation. For a positive integer $n \in \mathbb{Z}_{>0}$, we define

$$(x)_n := \prod_{k=1}^{n}(1-x^k), \qquad \langle\!\langle x\rangle\!\rangle_n := \frac{1-x^n}{(1-x)n} \tag{18}$$

with the convention $(x)_0 := 1$.

For a fixed root of unity ζ of order $N = \mathrm{ord}(\zeta) \in \mathbb{Z}_{>0}$, we define rational functions $w(x|n) \in \mathbb{C}(x)$ for all integers $n \in \mathbb{Z}$ by the recurrence relation

$$w(x|n)(1-x\zeta^n) = w(x|n-1), \quad w(x|0) = 1. \tag{19}$$

One has the addition property

$$w(x|m+n) = w(x|m)w(x\zeta^m|n) \qquad (m,n \in \mathbb{Z}) \tag{20}$$

which, by taking into account the equality $w(x|N) = 1/(1-x^N)$, gives the quasi-periodicity property

$$(1-x^N)w(x|n+N) = w(x|n) \qquad (n \in \mathbb{Z}). \tag{21}$$

Notice the relation to the q-Pochhammer symbols with $q = \zeta$

$$w(x|n) = \frac{1}{(x\zeta;\zeta)_n} \qquad (n \in \mathbb{Z}_{\geq 0}). \tag{22}$$

It follows that $w(x|n)$ is a rational function of x with coefficients in the cyclotomic field $F_N := \mathbb{Q}(\zeta)$ whose poles and zeros (if any) are integer powers of ζ. We will call such rational functions, and more generally matrices of such rational functions, N-cyclotomic.

In [Kas96], the second author considered the N-cyclotomic matrix $r(x;m,n) \in \mathrm{Mat}_{N^2}(\mathbb{C}(x))$ defined by the formula

$$\langle i,j|r(x;m,n)|k,l\rangle = \langle\!\langle x\rangle\!\rangle_N \zeta^{(i-k+n)(l-j)} \frac{w(x/\zeta|j-i-m)w(x|l-k+n)}{w(x/\zeta|j-k+n-m)w(x/\zeta|l-i)} \tag{23}$$

(where the left hand-side is the bra-ket notation for the entries of a matrix) and proved that it satisfies the Yang–Baxter equation

$$\begin{aligned} &r_{1,2}(x;m_1,m_2)r_{1,3}(xy;m_1,m_3)r_{2,3}(y;m_2,m_3) \\ &\qquad = r_{2,3}(y;m_2,m_3)r_{1,3}(xy;m_1,m_3)r_{1,2}(x;m_1,m_2) \end{aligned} \tag{24}$$

where, along with continuous spectral parameters x and y, the integers m_1, m_2, m_3 play the role of discrete spectral parameters. Here, $r_{1,2}$, $r_{1,3}$ and $r_{2,3}$ denote the N-cyclotomic matrices in $\mathrm{Mat}_{N^3}(\mathbb{C}(x))$ obtained by interpreting r as an endomorphism of an N-dimensional space $V^{\otimes 2}$, equipped with a basis, and $r_{i,j}$ as the corresponding endomorphisms of $V^{\otimes 3}$, acting nontrivially on the i, j copies of V in $V^{\otimes 3}$ and trivially on the remaining copy.

Aside from being a solution to the Yang–Baxter equation with a spectral parameter, a somehow surprising fact is that $r(x; m, n)$ is regular (i.e. it has a finite limit) at $x = 1$. This follows from the existence of the gauge conjugating matrix $h(y, m) \in \mathrm{Mat}_N(\mathbb{C}(y))$ defined by

$$\langle i|h(y,m)|j\rangle = \zeta^{(j-i)m} \langle\!\langle y\,\zeta^{j-i}\rangle\!\rangle_N \,. \tag{25}$$

The entries of the matrix $h(y, m)$ are, in fact, polynomials in y, it satisfies the multiplicative property

$$h(x,m)h(y,m) = h(xy,m) \tag{26}$$

and it enters the important gauge symmetry transformation formulae for the r-matrix (23):

$$h_1(y,m+1)r(x;m,n)h_1(y^{-1},m+1) = r(xy;m,n) = h_2(y^{-1},0)r(x;m,n)h_2(y,0)\,. \tag{27}$$

Here, h_1 and h_2 denote matrices in $\mathrm{Mat}_{N^2}(\mathbb{C}(y))$ obtained by interpreting h as an endomorphism of an N dimensional space V with a basis, and h_j as corresponding endomorphisms of $V^{\otimes 2}$ obtained by acting on the j-th copy of V in $V^{\otimes 2}$ by h and on the remaining copy of V by identity.

The above gauge symmetry transformation formulae, together with the fact that $h(y, m)$ has polynomial entries in y, imply easily that $r(x; m, n)$ is regular at $x = 1$. An additional calculation shows that

$$\langle i,j|r(1;m,n)|k,l\rangle = V_{i,j-m,k-n,l}(\zeta)\zeta^{k-l-n+(k-i-n)m} \tag{28}$$

where

$$V_{i,j,k,l}(\zeta) := \frac{N\,\theta_N(\{j-i-1\}_N + \{l-k\}_N)\,\theta_N(\{i-l\}_N + \{k-j\}_N)}{(\bar\zeta)_{\{j-i-1\}_N}(\zeta)_{\{i-l\}_N}(\bar\zeta)_{\{l-k\}_N}(\zeta)_{\{k-j\}_N}} \tag{29}$$

and where $\bar\zeta$ denotes the complex conjugate of the complex number ζ and for an integer k, $\{k\}_N$ denotes the unique integer such that

$$\{k\}_N \equiv k \pmod N, \quad 0 \le \{k\}_N < N\,, \tag{30}$$

and

$$\theta_N(k) := \delta_{k,\{k\}_N} \tag{31}$$

where δ is the standard Kronecker delta symbol. The regularity of the matrix $r(x; m, n)$ is reminiscent to the p-adic valuation of factorials in Landau's work, see for instance [RV, Sou20] and references therein.

A consequence of formula (28) is that the rows and columns of $r(1; m, n)$ are naturally indexed by elements of the additive group $\mathbb{Z}/N\mathbb{Z}$, as well as the discrete spectral parameters are in $\mathbb{Z}/N\mathbb{Z}$.

3.2. **Proof of the gauge symmetry equations.** In this section we give the proofs of equations (26) and (27) which were stated in [Kas96], but proofs were ommitted.

We begin with Equation (26). Writing out its left hand side in matrix coefficients (indexed by $i,j \in \{0,\dots,N-1\}$), we have

$$\langle i|h(x,m)h(y,m)|j\rangle = \sum_{k=0}^{N-1} \langle i|h(x,m)|k\rangle\langle k|h(y,m)|j\rangle = \zeta^{(j-i)m} \sum_{k=0}^{N-1} \langle\langle x\zeta^{k-i}\rangle\rangle_N \langle\langle y\zeta^{j-k}\rangle\rangle_N \tag{32}$$

so that Equation (26) is equivalent to

$$\sum_{k=0}^{N-1} \langle\langle x\zeta^{k-i}\rangle\rangle_N \langle\langle y\zeta^{j-k}\rangle\rangle_N = \langle\langle xy\zeta^{j-i}\rangle\rangle_N \Leftrightarrow \sum_{k=0}^{N-1} \langle\langle x\zeta^{k}\rangle\rangle_N \langle\langle y\zeta^{j-k}\rangle\rangle_N = \langle\langle xy\zeta^{j}\rangle\rangle_N \,. \tag{33}$$

The definition (18) gives $\langle\langle x\rangle\rangle_N = N^{-1}\sum_{a=0}^{N-1} x^a$. Using this equality, we obtain

$$\begin{aligned} \sum_{k=0}^{N-1} \langle\langle x\zeta^{k}\rangle\rangle_N \langle\langle y\zeta^{j-k}\rangle\rangle_N &= N^{-2} \sum_{k,a,b=0}^{N-1} (x\zeta^k)^a (y\zeta^{j-k})^b = N^{-1} \sum_{a,b=0}^{N-1} x^a y^b \zeta^{jb} \delta_{0,\{a-b\}_N} \\ &= N^{-1} \sum_{b=0}^{N-1} (xy\zeta^j)^b = \langle\langle xy\zeta^{j}\rangle\rangle_N \,. \end{aligned} \tag{34}$$

We can diagonalize the matrices $h(x,m)$ by conjugating them by the (discrete) Fourier transformation operator $\langle i|F|j\rangle = \zeta^{ij}$. Indeed, assuming $i,j \in \{0,\dots,N-1\}$, we have

$$\begin{aligned} \langle i|Fh(x,m)F^{-1}|j\rangle &= N^{-1} \sum_{i',j'} \zeta^{ii'-jj'} \langle i'|h(x,m)|j'\rangle = N^{-1} \sum_{i',j'} \zeta^{ii'-jj'+(j'-i')m} \langle\langle x\zeta^{j'-i'}\rangle\rangle_N \\ &= N^{-1} \sum_{i',j'} \zeta^{ii'-j(j'+i')+j'm} \langle\langle x\zeta^{j'}\rangle\rangle_N = \delta_{i,j} \sum_{j'} \zeta^{(m-j)j'} \langle\langle x\zeta^{j'}\rangle\rangle_N \\ &= N^{-1}\delta_{i,j} \sum_{j'} \zeta^{(m-j)j'} \sum_{a=0}^{N-1} x^a \zeta^{aj'} = \delta_{i,j} \sum_{a=0}^{N-1} \delta_{0,\{m-j+a\}_N} x^a = \delta_{i,j} x^{\{j-m\}_N} \,. \end{aligned} \tag{35}$$

Thus, conjugating the r-matrix (23) by F, we can prove Equations (27) by explicit computation. Indeed, assuming that $i,j,k,l \in \{0,\dots,N-1\}$, we have

$$\begin{aligned} \langle i,j|(F\otimes F)r(x;m,n)(F^{-1}\otimes F^{-1})|k,l\rangle &= N^{-2} \sum_{i',j',k',l'} \zeta^{ii'+jj'-kk'-ll'} \langle i',j'|r(x;m,n)|k',l'\rangle \\ &= \frac{\langle\langle x\rangle\rangle_N}{N^2} \sum_{i',j',k',l'} \zeta^{ii'+jj'-kk'-ll'+(i'-k'+n)(l'-j')} \frac{w(x/\zeta|j'-i'-m)w(x|l'-k'+n)}{w(x/\zeta|j'-k'+n-m)w(x/\zeta|l'-i')} \\ &= \frac{\langle\langle x\rangle\rangle_N}{N^2} \sum_{i',j',k',l'} \zeta^{i(i'+k')+j(j'+k')-kk'-l(l'+k')+(i'+n)(l'-j')} \frac{w(x/\zeta|j'-i'-m)w(x|l'+n)}{w(x/\zeta|j'+n-m)w(x/\zeta|l'-i')} \\ &= \frac{\langle\langle x\rangle\rangle_N \delta_{0,\{i+j-k-l\}_N}}{N} \sum_{i',j',l'} \zeta^{ii'+jj'-ll'+(i'+n)(l'-j')} \frac{w(x/\zeta|j'-i'-m)w(x|l'+n)}{w(x/\zeta|j'+n-m)w(x/\zeta|l'-i')} \end{aligned} \tag{36}$$

where, in third equality, we have shifted the summation variables i', j', l' by k' and, in the last equality, performed the k'-summation by using the formula

$$\sum_{a=0}^{N-1} \zeta^{ab} = N\delta_{0,\{b\}_N}. \tag{37}$$

We continue the calculation in (36) by shifting the summation variables $l' \mapsto l' + i'$ and $j' \mapsto j' + m - n$ followed by the shift $i' \mapsto i' - n$:

$$\begin{aligned}
(36) &= \frac{\langle\!\langle x\rangle\!\rangle_N \delta_{0,\{i+j-k-l\}_N}}{N} \sum_{i',j',l'} \zeta^{i(i'-n)+j(j'+m-n)-l(l'+i'-n)+i'(l'+i'-j'-m)} \frac{w(x/\zeta|j'-i')w(x|l'+i')}{w(x/\zeta|j')w(x/\zeta|l')} \\
&= \frac{\langle\!\langle x\rangle\!\rangle_N \delta_{0,\{i+j-k-l\}_N}}{N\zeta^{kn-jm}} \sum_{i',j',l'} \zeta^{(i-l-m+i')i'+(j-i')j'+(i'-l)l'} \frac{w(x/\zeta|j'-i')w(x|l'+i')}{w(x/\zeta|j')w(x/\zeta|l')} \\
&= \frac{\langle\!\langle x\rangle\!\rangle_N \delta_{0,\{i+j-k-l\}_N}}{N\zeta^{kn-jm}} \sum_{i'=0}^{N-1} \frac{w(x/\zeta|-i')w(x|i')}{\zeta^{(l+m-i-i')i'}} \sum_{j',l'} \zeta^{(j-i')j'+(i'-l)l'} \frac{w(x\zeta^{-i'-1}|j')w(x\zeta^{i'}|l')}{w(x/\zeta|j')w(x/\zeta|l')} \\
&= \frac{\langle\!\langle x\rangle\!\rangle_N \delta_{0,\{i+j-k-l\}_N}}{N\zeta^{kn-jm}} \sum_{i'=0}^{N-1} \frac{w(x/\zeta|-i')w(x|i')}{\zeta^{(l+m-i-i')i'}} f(x\zeta^{-i'-1}, x/\zeta|\zeta^{j-i'}) f(x\zeta^{i'}, x/\zeta|\zeta^{i'-l})
\end{aligned} \tag{38}$$

where, in third equality, we used the addition formula (20), and in the last equality, we use function $f(x,y|z)$ defined by

$$f(x,y|z) := \sum_a \frac{w(x|a)}{w(y|a)} z^a, \qquad (1-y^N)z^N = 1 - x^N, \tag{39}$$

whose automorphic properties are described in [KMS93]. In particular, we have the equality

$$f(x\zeta^a, x/\zeta|\zeta^{-b}) = \frac{x^{\{b\}_N}}{\langle\!\langle x\rangle\!\rangle_N w(x|\{a\}_N)} \frac{(\zeta)_{\{a\}_N+\{b\}_N}}{(\zeta)_{\{a\}_N}(\zeta)_{\{b\}_N}} \qquad (a,b\in\mathbb{Z}) \tag{40}$$

which we can use to proceed in (38) as follows:

$$\begin{aligned}
(38) &= \frac{\langle\!\langle x\rangle\!\rangle_N \delta_{0,\{i+j-k-l\}_N}}{\zeta^{kn-jm}} \sum_{i'=j}^{N-1} \frac{w(x|i')x^{i'-j}}{\zeta^{(l+m-i-i')i'}} \frac{(\zeta)_{N-1-j}}{(\zeta)_{N-1-i'}(\zeta)_{i'-j}} f(x\zeta^{i'}, x/\zeta|\zeta^{i'-l}) \\
&= \frac{\langle\!\langle x\rangle\!\rangle_N \delta_{0,\{i+j-k-l\}_N}}{\zeta^{kn-jm}} \sum_{i'=j}^{N-1} \frac{w(x|i')x^{i'-j}}{\zeta^{(l+m-i-i')i'}} \frac{(\bar\zeta)_{i'}}{(\bar\zeta)_j(\zeta)_{i'-j}} f(x\zeta^{i'}, x/\zeta|\zeta^{i'-l})
\end{aligned} \tag{41}$$

where, in first equality, we used the addition formula (20) for simplification, and in the last equality, the property

$$(\zeta)_k(\bar\zeta)_{N-1-k} = N \qquad (k\in\mathbb{Z},\ 0\le k\le N-1). \tag{42}$$

Continuing with the second f-function in (41), we have

$$
\begin{aligned}
&= \frac{\delta_{0,\{i+j-k-l\}_N}}{\zeta^{kn-jm}} \sum_{i'=j}^{l} \frac{x^{l-j}}{\zeta^{(l+m-i-i')i'}} \frac{(\bar{\zeta})_{i'}}{(\bar{\zeta})_j(\zeta)_{i'-j}} \frac{(\zeta)_l}{(\zeta)_{i'}(\zeta)_{l-i'}} \\
&= \frac{\delta_{0,\{i+j-k-l\}_N}(\zeta)_l x^l}{\zeta^{(n-j)k}(\zeta)_j x^j} \sum_{i'=0}^{l-j} \frac{(-1)^{i'}\zeta^{(i'-1)i'/2}}{\zeta^{(l+m-i-j)i'}(\zeta)_{i'}(\zeta)_{l-j-i'}} \\
&= \frac{\delta_{0,\{i+j-k-l\}_N}(\zeta)_l x^l(\zeta^{k-m};\zeta)_{l-j}}{\zeta^{(n-j)k}(\zeta)_j x^j(\zeta)_{l-j}} = \frac{\delta_{0,\{i+j-k-l\}_N}(\zeta)_l x^l(\zeta)_{\{k-m-1\}_N+l-j}}{\zeta^{(n-j)k}(\zeta)_j x^j(\zeta)_{l-j}(\zeta)_{\{k-m-1\}_N}} \\
&= \delta_{\{i-m-1\}_N+j,\{k-m-1\}_N+l}\, \zeta^{(j-n)k} x^{l-j} \frac{(\zeta)_l(\zeta)_{\{i-m-1\}_N}}{(\zeta)_j(\zeta)_{l-j}(\zeta)_{\{k-m-1\}_N}}
\end{aligned}
\tag{43}
$$

where, in third equality, we used the well-known q-binomial formula

$$
(z;q)_s = \sum_{t=0}^{s} (-z)^t q^{(t-1)t/2} \frac{(q)_s}{(q)_t(q)_{s-t}}, \qquad (s \in \mathbb{N}), \tag{44}
$$

with $q = \zeta$, $s = l - j$ and $z = \zeta^{k-m}$; in fourth equality, the identity

$$
(\zeta^a;\zeta)_b = \frac{(\zeta)_{\{a-1\}_N+b}}{(\zeta)_{\{a-1\}_N}} \qquad (a \in \mathbb{Z},\, b \in \mathbb{N}), \tag{45}
$$

and, in the last equality, the equivalence

$$
\begin{cases} i+j \equiv k+l \pmod N \\ 0 \le \{k-m-1\}_N + l - j \le N-1 \end{cases} \Leftrightarrow \ \{i-m-1\}_N + j = \{k-m-1\}_N + l\,. \tag{46}
$$

Equation (43), combined with (35), straightforwardly implies relations (27).

We remark that the R-matrix coefficients in (43), when specialized to $m = n = -1$, coincide with the standard colored Jones R-matrix coefficients. This means that in this case, the associated knot invariant is the N-colored Jones polynomial specialized to a primitive N-th root of unity.

3.3. **Knot invariants.** We can use the r-matrix given in Equation (28) to construct knot invariants from their planar projections. This is a standard construction explained in several places that include [Jon89, RT90, Tur88, Tur94]. We will follow the presentation of [Kas21] which does not require neither the theory of Hopf algebras nor the existence of ribbon elements, and uses as a combinatorial input a generic planar projection of a long knot with no local extrema oriented from left to right and equal numbers of positive and negative crossings. Locally, such a planar projection has eight types of crossings (four positive and four negative ones)

(47)

to which one assigns the suitably "rotated" r-matrices shown in Equations (16)-(19) of [Kas21], two types of vertical segments, and two types of local extrema

(48)

to which ones assigns respectively the identity operators and ε and η maps of Equation (7) of [Kas21]. In our case, we fix a primitive root of unity ζ of order N, a (discrete spectral parameter) $n \in \mathbb{Z}/N\mathbb{Z}$ and a planar projection D of a long knot K. We color each arc of D by an element of $\mathbb{Z}/N\mathbb{Z}$, place r-matrices at the crossings (using the fixed spectral parameter at all crossings), according to the rules described below, and sum over all indices. The result of this contraction is a complex number that depends on the colors i and j of the outgoing and incoming arcs of the diagram of the long knot. These complex numbers arrange in a matrix

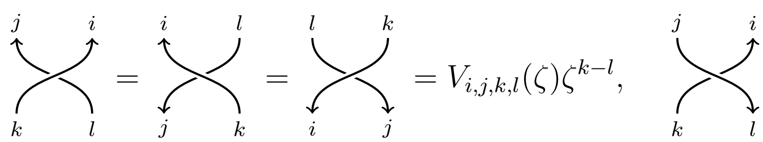
$$\langle D \rangle_{N,n} \in \mathrm{Mat}_N(\mathbb{Z}[N^{-1}, \zeta]) \tag{49}$$

The r-matrices given below are rigid, i.e., satisfy the Yang-Baxter equation (9) and the inverse matrix equation (10) of [Kas21]. The next theorem follows from the results of [Kas21].

Theorem 3.1. *The state-sum $\langle D \rangle_{N,n}$ depends on the long knot K and not on the planar projection D used.*

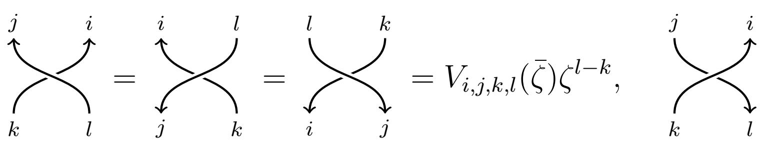
It remains to explain the rules for the r-matrices.

Starting with the case with $m = n = 0$, we get the following rules for the four types of positive crossings



$$= V_{i,j,k,l}(\zeta)\zeta^{k-l}, \qquad = V_{k,l,i,j-1}(\bar{\zeta})\zeta^{j-1-k} \tag{50}$$

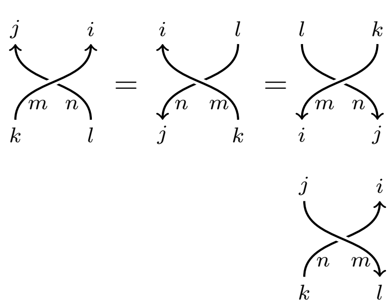
and for the four types of negative crossings



$$= V_{i,j,k,l}(\bar{\zeta})\zeta^{l-k}, \qquad = V_{k,l,i,j-1}(\zeta)\zeta^{k-j+1}. \tag{51}$$

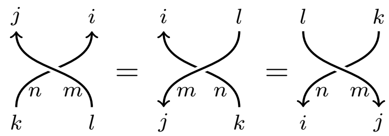
Note that the weights of the negative crossings are the complex conjugates of the corresponding weights of the positive crossings.

Switching in now the general colors m and n, we get the rules for the positive crossings

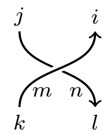


$$\begin{aligned} &= V_{i,j-m,k-n,l}(\zeta)\zeta^{k-l-n+(k-i-n)m}, \\ &= V_{k,l,i-m,j-n-1}(\bar{\zeta})\zeta^{j-1-k-n+(j-l-n)m} \end{aligned} \tag{52}$$

and the negative crossings



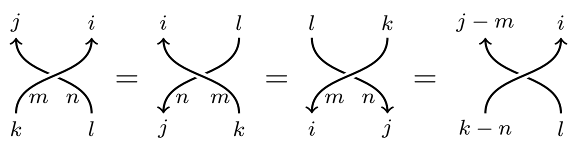
$$= V_{i,j-n,k-m,l}(\bar{\zeta})\zeta^{l-k+(l-j+1+n)m},$$



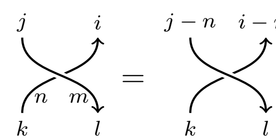
$$= V_{k,l,i-n,j-m-1}(\zeta)\zeta^{k-j+1+(k-i+1+n)m}\,. \tag{53}$$

Note that for general m, n, the weights of the negative crossings are not the complex conjugates of the corresponding weights of the positive crossings. Note also that the eight r-matrices needed in Equations (16)-(19) of [Kas21], up to powers of ζ, are all expressed in terms of the symbols $V_{i,j,k,l}(\zeta)$ and their complex conjugates $V_{i,j,k,l}(\bar{\zeta})$.

Additionally, we have relations between colored and uncolored weights

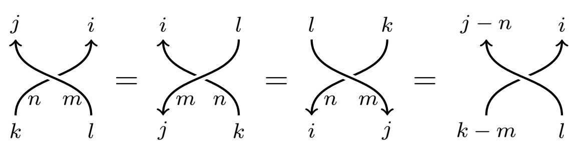


$$\zeta^{(k-i-n)m},$$

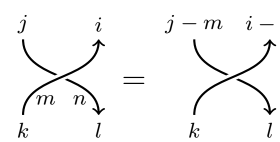


$$\zeta^{(j-l-n)m} \tag{54}$$

in the case of positive crossings, and



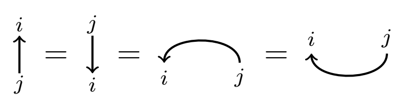
$$\zeta^{(l-j+n)m},$$



$$\zeta^{(k-i+n)m} \tag{55}$$

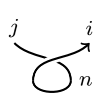
in the case of negative crossings.

We complement the rules by the weights on the four types of segments

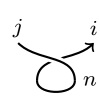


$$= \delta_{i,j}. \tag{56}$$

With our rules, one can calculate weights of few simplest composed diagrams in the colored case

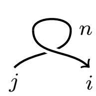


$$= \delta_{j,\{i+1\}_N},$$

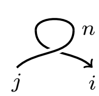


$$= \delta_{j,\{i+1\}_N}\zeta^n \tag{57}$$

and

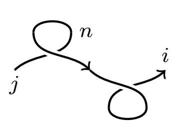


$$= \delta_{i,\{j+1\}_N},$$

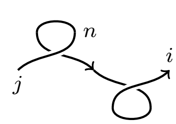


$$= \delta_{i,\{j+1\}_N}\zeta^{-n} \tag{58}$$

which imply that

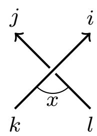


$$= \delta_{i,j}\zeta^n,$$

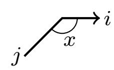


$$= \delta_{i,j}\zeta^{-n}\,. \tag{59}$$

It is interesting to note, that the continuous spectral parameter can be used in an alternative formulation of knot and link invariants along the idea of Jones [Jon89] of considering a "statistical mechanics" model on piecewise linear diagrams with straight segments where the multiplicative angles are encoded in the continuous spectral parameters. In our case, such an approach might be applicable only in the case with color $n \equiv -1 \pmod N$ but not for other values of n. In the case $n \equiv -1 \pmod N$, the starting graphical rules would look like



$$= \langle i, j | r(x; -1, -1) | k, l \rangle,$$

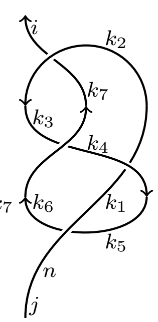


$$= \langle i | h(-1/x, 0) | j \rangle \,, \tag{60}$$

where the continuous variables x's are thought of as exponentiated angles $e^{i\alpha}$, whose product around each vertex is 1.

3.4. **An example.** In this sub-section we write an explicit expression for the invariant $\langle 4_1 \rangle_{N,n}$ of the simplest hyperbolic knot 4_1 in the case of a primitive N-th root of unity ζ and a color $n \in \mathbb{Z}/N\mathbb{Z}$. The invariant is an $N \times N$ matrix whose entries $M_{i,j} := \langle i | \langle 4_1 \rangle_{N,n} | j \rangle$ are given by

$$M_{i,j} = \sum_{k_1,\dots,k_7}$$



$$\begin{aligned}
&= \sum_{k_1,\dots,k_7} \begin{matrix} V_{i,k_3-n,k_7-n,k_2}(\bar\zeta)\zeta^{k_2-k_7+(k_2-k_3+1+n)n} \\ V_{k_6,k_4,k_7-n,k_3-n-1}(\zeta)\zeta^{k_6-k_3+1+(k_6-k_7+1+n)n} \\ V_{k_1,k_5,k_2-n,k_4-n-1}(\bar\zeta)\zeta^{k_4-k_1-1+(k_4-k_5-1-n)n} \\ V_{k_1,k_6-n,j-n,k_5}(\zeta)\zeta^{j-k_5+(j-k_1-1-n)n} \end{matrix} \\
&= \sum_{k_1,\dots,k_7} V_{i,k_3-n,k_7-n,k_2}(\bar\zeta)V_{k_6,k_4,k_7-n,k_3-n-1}(\zeta)V_{k_1,k_5,k_2-n,k_4-n-1}(\bar\zeta)V_{k_1,k_6-n,j-n,k_5}(\zeta) \\
&\qquad \times \zeta^{(j-k_1+k_2-k_3+k_4-k_5+k_6-k_7)(n+1)} \\
&= \sum_{k_1,\dots,k_7} V_{i,k_3,k_7,k_2}(\bar\zeta)V_{k_6,k_4,k_7,k_3-1}(\zeta)V_{k_1,k_5,k_2-n,k_4-n-1}(\bar\zeta)V_{k_1,k_6-n,j-n,k_5}(\zeta) \\
&\qquad \times \zeta^{(j-k_1+k_2-k_3+k_4-k_5+k_6-k_7-2n)(n+1)} \\
&= \sum_{k_1,\dots,k_7} V_{0,k_3,k_7,k_2}(\bar\zeta)V_{k_6,k_4,k_7,k_3-1}(\zeta)V_{k_1,k_5,k_2-n,k_4-n-1}(\bar\zeta)V_{k_1,k_6-n,j-i-n,k_5}(\zeta) \\
&\qquad \times \zeta^{(j-i-k_1+k_2-k_3+k_4-k_5+k_6-k_7-2n)(n+1)} \\
&= \sum_{k_1,\dots,k_7} V_{0,k_3,k_7,k_2}(\bar\zeta)V_{0,k_4,k_7-k_6,k_3-k_6-1}(\zeta)V_{0,k_5,k_2+k_1-k_6,k_4+k_1-1}(\bar\zeta) \\
&\qquad \times V_{0,k_1,j-i-k_6+k_1,k_5}(\zeta)\zeta^{(j-i+2k_1+k_2-k_3+k_4-k_5-k_7)(n+1)} \,.
\end{aligned} \tag{61}$$

The above is a sum over $(\mathbb{Z}/N\mathbb{Z})^7$ and it is not clear how to simplify to sum in fewer variables. Numerical calculations suggest that $\langle K\rangle_{N,n}$ is a multiple of the identity matrix.

3.5. **A conjecture.** When $n=-1$, Equation (43) implies that the invariant of a long knot K is a multiple of the identity $N\times N$ matrix, and the multiple is the specialization of the N-colored Jones polynomial that enters the Volume Conjecture [Kas95, Kas97]. When $n\in\mathbb{Z}/N\mathbb{Z}$ is arbitrary, we conjecture the following.

Conjecture 3.2. For all knots K, strictly positive integers N and n, and a primitive N-th root of unity ζ, we have

$$\langle K\rangle_{N,n} = J^K_{n+1}(\zeta) 1_N \tag{62}$$

where 1_N denotes the identity $N\times N$ matrix.

In particular, $\langle K\rangle_{N,0} = 1_N$ is a trivial invariant coming from a nontrivial r-matrix over an N-dimensional vector space.

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