APPLICATIONS OF TQFT INVARIANTS IN LOW DIMENSIONAL TOPOLOGY

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ABSTRACT. In this short note we give *lower* bounds for the Heegaard genus of 3-manifolds using various TQFT in 2+1 dimensions. We also study the large k limit and the large G limit of our lower bounds, using a conjecture relating the various combinatorial and physical TQFTs. We also prove, assuming this conjecture, that the HOMFLY polynomial of a framed knot in S^3 along with all of its cables distinguishes the knot from the unknot!

CONTENTS

1. INTRODUCTION

In recent years a remarkable relation between physics and low dimensional topology has emerged, under the name of *topological quantum field theory* (TQFT for short).

An axiomatic definition of a TQFT in $d+1$ dimensions has been provided by Atiyah-Segal in [At]. We briefly recall it:

- To an oriented d dimensional manifold X, one associates a complex vector space $Z(X)$.
- To an oriented $d+1$ dimensional manifold M with boundary ∂M, one associates an element $Z(M) \in Z(\partial M)$.

This (functor) Z usually satisfies extra compatibility conditions (depending on the dimension d), some of which are:

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- For a disjoint union of d dimensional manifolds X, Y

$$Z(X \sqcup Y) = Z(X) \otimes Z(Y)$$

- For a change of orientation of a (unitary) TQFT we have:

$$Z(\overline{X}) = Z(X)^{\star}$$

(where V^* is the dual vector space of V.)

- For $M = M_1 \cup_X M_2$ where $\partial M_1 = X_1 \sqcup X$, $\partial M_2 = X_2 \sqcup \overline{X}$, one has

$$Z(M) = \langle Z(M_1), Z(M_2) \rangle \in Im(Z(X_1 \sqcup X_2 \sqcup X \sqcup \overline{X}) \to Z(X_1 \sqcup X_2))$$

The above mentioned axioms for a TQFT in $d+1$ dimensions come from an attempt to axiomatize the path integral (nonperturbative) and the Hamiltonian approach to a quantum field theory.

An axiomatic definition of a *perturbative* TQFT in $d+1$ dimensions is still missing, but in the case of the Chern-Simons theory in $2+1$ dimensions there are some attempts [AxS], [BN], and especially [Ko].

From now on, we will concentrate on topological quantum field theories in $2+1$ dimensions. For a precise definition of them, the reader is refered to [Ga1], and [Wa].

Any such theory gives invariants of closed 3-manifolds (with values in C), invariants of framed (labeled) links in 3-manifolds (with values in C), as well as finite dimensional representations of the mapping class groups.

The first such theory was constructed using path integrals in the seminal paper of Witten [Wi]. We briefly recall the definition, fixing some notation:

Let G be a compact simple simply connected group, and k an integer. Let M be a (2-framed) closed 3-manifold with a framed colored link L. A coloring of the link is the assignment of a representation of the loop group ΩG at level k [Ka]. Let $G \hookrightarrow P \to M$ be the trivial principal G- bundle. We consider the space $\mathcal{A}$ of all G-connections on P.

Let

$$CS : \mathcal{A} \to R/Z$$

be the Chern-Simons action The gauge group $\mathcal{G} = Map(M, G)$ of G-automorphisms of P acts on $\mathcal{A}$, and for any framed colored link L, the holonomy around it gives

$$\mathcal{O}_L : \mathcal{A} \to C$$

The invariant of the framed colored link L is the following partition function:

$$Z_{ph,G,k}(M, L) = \int_{\mathcal{A}} \mathcal{D}A e^{2\pi i k CS(A)} \mathcal{O}_L(A)$$

The subscript *ph* stands for physics. Needless to say, the above path integrals have not yet been defined.

Shortly afterwards, a number of topological (combinatorial) definitions appeared in [RT], [TW]. They depended on a simple Lie algebra $\mathcal{G}$ of a group G, and a primitive root of unity $exp(2\pi i/r)$. They will be denoted by $Z_{G,exp(2\pi i/r)}$.

The main conjecture is that:

Conjecture 1.1. If h is the dual Coxeter number of G, then

$$Z_{ph,G,k} = Z_{G,exp(2\pi i/(k+h))}$$

The above conjecture seems ill-defined, as the left hand side has not yet been defined. However, taking the large k limit (as $k \to \infty$) and using stationary phase approximation of the path integral, we arrive at the following conjecture:

Conjecture 1.2. If M is a closed 3-manifold, as $k \to \infty$ we have:

$$Z_{G,exp(2\pi i/(k+h))}(M) \sim_{k\to\infty} k^{\theta_G(M) dim(G)/2}$$

where $f(k) \sim_{k\to\infty} g(k)$ means that

$$0 < a_1 \leq |f(k)/g(k)| \leq a_2 \text{ as } k \to \infty$$

and $\theta_G(M)$ is as in the following definition:

Definition 1.3. For a closed 3-manifold M, and a compact Lie group G, let

$$\theta_G(M) := max_{\alpha \in hom(\pi_1(M),G)/G} \frac{h^1(M,\alpha) - h^0(M,\alpha)}{dim(G)}$$

where $h^k(M,\alpha)$ is the k^{th} cohomology of M with twisted coefficients.

Remark 1.3.1. We will actually only use the above conjecture 1.2 in the case of a subsequence of k approaching infinity.

Remark 1.3.2. The normalization of $\theta_G(M)$ used in conjecture 1.2 is chosen so that corollary 2.4 has a simple form.

Let us give one more definition that we will need in the next section:

Definition 1.4. For a closed 3-manifold M let

$$\theta(M) = \overline{\lim}_{N\to\infty} \theta_{SU(N)}(M)$$

2. Lower bounds for the Heegaard genus of 3-manifolds

We first begin with a lemma:

Lemma 2.1. *If Z is a TQFT in 2+1 dimensions, and M, N are closed 3-manifolds, then*

- $Z(M\sharp N)Z(S^3) = Z(M)Z(N)$
- $Z(S^2 \times S^1) = 1$

Proof. It follows easily from the glueing axioms, as in [Wi]. □

Now we are ready to state the following theorem:

Theorem 2.2. *If Z is any unitary TQFT in $2+1$ dimensions, and M is a closed 3-manifold, then*

$$|Z(M)| \leq Z(S^3)^{-g(M)+1}$$

where $g(M)$ is the Heegaard genus of M, i.e. the genus of a minimal Heegaard splitting.

Proof. Let $M = H \cup_f H$ be a Heegaard splitting of M, where H is a handelbody of genus g ($\partial(H) = \Sigma_g$), and $f \in Diff^+(\Sigma_g)$. Let $u := Z(H) \in Z(\Sigma_g)$. Then, we have

$$\begin{aligned}
|Z(M)| &= |\langle u, f_\star(u)\rangle| \\
&\leq \sqrt{\langle u, u\rangle\langle f_\star(u), f_\star(u)\rangle} \text{ (by Cauchy Schwarz)} \\
&= \langle u, u\rangle \text{ (since } Z \text{ is unitary)} \\
&= Z(\sharp_{i=1}^g S^2 \times S^1) \text{ (by topology)} \\
&= Z(S^2 \times S^1)^g Z(S^3)^{-g+1} \text{ (by lemma 2.1)} \\
&= Z(S^3)^{-g+1} \; \square
\end{aligned}$$

Corollary 2.3. *For any unitary TQFT Z and a closed 3-manifold M we have:*

- $0 < Z(S^3) < 1$
- $g(M) - 1 \geq -\dfrac{\log |Z(M)|}{\log Z(S^3)}$

We also have the following:

Corollary 2.4 (depending on conjecture 1.2). *For a closed 3-manifold M, and a compact simple simply connected group G we have:*

- $g(M) - 1 \geq \theta_G(M)$
- $\theta_G(M\sharp N) = \theta_G(M) + \theta_G(N) + 1$

Proof. For the first part use the previous corollary for the TQFT $Z = Z_{G,exp(2\pi i/(k+h))}$ and the fact that $Z(S^3)$ is given by an explicit expression of [KW]. For the second part use the TQFT $Z = Z_{G,exp(2\pi i/(k+h))}$, and lemma 2.1 and the fact that $Z(S^3)$ is given by [KW]. □

Remark 2.4.1. In the following table we calculate a list of values of $\theta_G(M)$ for certain classes of 3-manifolds M for which conjecture 1.2 has been verified by direct calculation [Ga1], [Ga2].

manifold M	$g(M)-1$	$\theta_G(M)$	$\theta(M)$
S^3	-1	-1	-1
$L_{p,q}$	0	$-l_G/d_G$	0
$S(a_1,...a_n)$	$n-2$	$2\mu_G/d_G n-2$	$n-2$
$S^1\times\Sigma_g$	$2g$	$2g-2$	$2g-2$
$S(0;e_0;1/2,...,1/2,b_m/(2k+1))$	?	?	?

where d_G, l_G, μ_G are the dimension, rank and number of positive roots of the Lie group G. $L_{p,q}$ is the Lens space with $\pi_1(L_{p,q}) = Z/p$, $S(a_1,\dots,a_n)$ is the Seifert fibered integral homology sphere (a_i are coprime), and $S(0;e_0;1/2,...,1/2,b_m/(2k+1))$ is the Seifert fibered manifold over S^2 with m many exceptional fibers and rational euler class e_0.

Question 2.4.1. In the case of the Seifert fibered manifolds $M_m = S(0;e_0;1/2,...,1/2,b/(2k+1))$ with $e_0 \neq 1/(4k+2)$ and m even, $m \geq 4$ it would be interesting to know the lower bounds for the Heegaard genus thus obtained. Numerical calculations (using $Z_{SU(2),exp(2\pi i/(k+2))}$) suggest that $g(M_4) \geq 2$, but from the work of Bolieau and Zieschang [BZ] it follows that

$$m-2 \leq g(M_m) \leq m-1$$

with the above restrictions on the euler class e_0 and the parity of m, and in addition $g(M_4) = 3$. The interesting fact about these manifolds is that at least when $m = 4$ Bolieau and Zieschang [BZ] have shown that the minimum number of generators of the fundamental group is *strictly smaller* than the Heegaard genus (thus giving a negative answer to a question of Waldhausen [Wd] which was asserting the opposite).

3. Detecting the unknot

In this section we use conjecture 1.2 to show how TQFT invariants can detect the unknot.

Theorem 3.1 (depending on conjecture 1.2). *Let $K \subseteq S^3$ be a framed oriented knot, and $K^{(n)}$ the n^{th} parallel of it ($n \geq 1$). Let $\mathcal{O}$ be the unknot. If*

$$HOMFLY(K^{(n)}) = HOMFLY(\mathcal{O}^{(n)}) \text{ for all } n \geq 1$$

then $K = \mathcal{O}$!

Proof. Let $K^{a/b}$ denote a/b surgery on K, with the convention $H_1(K^{a/b}, Z) = Z/a$. Using the fact that (i) $Z_{SU(N),exp(2\pi i/k+N)}(K^m)$ (where $m \in Z$) is a linear combination of values of the HOMFLY polynomial of parallels of K, evaluated at roots of unity, and $(ii) Z_{SU(N),exp(2\pi i/k+N)}(K^{a/b})$ is a linear combination of $Z_{SU(N),exp(2\pi i/k+N)}(K^m)$ (for suitable m), we deduce that:

$$Z_{SU(N),exp(2\pi i/k+N)}(K^{1/n}) = Z_{SU(N),exp(2\pi i/k+N)}(\mathcal{O}^{1/n}) = Z_{SU(N),exp(2\pi i/k+N)}(S^3)$$

for all $n \in Z, N \geq 2$.

Now use the conjecture 1.2 and the precise knowledge of $Z_{SU(N),exp(2\pi i/k+N)}(S^3)$ as in [KW], to deduce that:

$$\theta_{SU(N)}(K^{1/n}) = -1 \text{ for all } N \geq 2,\ n \in N$$

Using the fact that $\pi_1(K^{1/n})$ is a residually finite group for $n >> 0$ (as follows by Thurston [Th]) we obtain that:

$$hom(\pi_1(K^{1/n}), SU(N)) = 0$$

and consequently (by residual finiteness)

$$\pi_1(K^{1/n}) = 0$$

for all $n >> 0$.

Using the cyclic surgery theorem of Gordon-Luecke [CGLS] we are done! □

An equivalent formulation of the result above is the following:

Corollary 3.2 (depending on conjecture 1.2). *If $K \subseteq S^3$ is a framed oriented knot, and $Z(K) = Z(\mathcal{O})$ for all TQFT Z in $2+1$ dimensions, then K is the unknot.*

Remark 3.2.1. The previous theorem should be compared with the following inclusions of [BN], [Ko] (with the definitions given there):

perturbative Chern-Simons oriented knot invariants	$\subseteq$	Vassiliev invariants of oriented knots	$\subseteq$	invariants of oriented knots

One knows that strict inequality must hold in some of the two inclusions (because the purturbative Chern-Simons knot invariants are insensitive to the orientation of the knot). Nevertheless, the perturbative Chern-Simons oriented knot invariants (such as the various derivatives of the HOMFLY polynomial of parallels of a knot, evaluated at 1), can distinguish the knot from the unknot.

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