A MEROMORPHIC EXTENSION OF THE 3D INDEX

STAVROS GAROUFALIDIS AND RINAT KASHAEV

To Don Zagier, on the occasion of his 65th birthday

ABSTRACT. Using the locally compact abelian group $\mathbb{T} \times \mathbb{Z}$, we assign a meromorphic function to each ideal triangulation of a 3-manifold with torus boundary components. The function is invariant under all 2–3 Pachner moves, and thus is a topological invariant of the underlying manifold. If the ideal triangulation has a strict angle structure, our meromorphic function can be expanded into a Laurent power series whose coefficients are formal power series in q with integer coefficients that coincide with the 3D index of [DGG13]. Our meromorphic function can be computed explicitly from the matrix of the gluing equations of a triangulation, and we illustrate this with several examples.

CONTENTS

Date: June 24, 2017.

Key words and phrases: ideal triangulations, 3-manifolds, ideal triangulations, 3D index, locally compact abelian groups, quantum dilogarithm, state-integrals, meromorphic functions.

1. Introduction

1.1. **The 3D-index of Dimofte-Gaiotto-Gukov.** In a recent breakthrough in mathematical physics, the physicists Dimofte, Gaiotto and Gukov [DGG14, DGG13] introduced the *3D-index*, a powerful new invariant of an ideal triangulation $\mathcal{T}$ of a compact orientable 3-manifold M with non-empty boundary consisting of tori. The 3D-index was motivated by the study of the low energy limit of a famous 6-dimensional $(2,0)$ superconformal field theory, and seems to contain a great deal of information about the geometry and topology of the ambient manifold. For suitable ideal triangulations, the 3D-index is a collection of formal Laurent power series in a variable q, parametrized by a choice of peripheral homology class, i.e., an element of $H_1(\partial M, \mathbb{Z})$.

Physics predicts that the 3D-index is independent of the triangulation $\mathcal{T}$ and that it is a topological invariant of the ambient manifold. However, there is a subtlety: the 3D-index itself (which is a sum over some q-series over a lattice) is only defined for suitable triangulations, and it is invariant under 2–3 moves of such triangulations. It is not known whether suitable triangulations are connected under 2–3 moves, and it is known that some 3-manifolds (for instance, the unknot) have no suitable triangulation. It was shown in [Gar16, GHRS15] that a triangulation is suitable if and only if it is 1-efficient, i.e., has no normal surfaces which are topologically 2-spheres or tori. Thus, the connected sum of two nontrivial knots, or the Whitehead double of a nontrivial knot has no 1-efficient triangulations. With some additional work, one can extract from the 3D-index a topological invariant of hyperbolic 3-manifolds [GHRS15].

This partial success in constructing a topological invariant suggests the existence of an invariant of ideal triangulations unchanged under all 2–3 Pachner moves. The construction of such an invariant is the goal of our paper. Indeed, to any ideal triangulation, we associate an invariant which is a meromorphic function of the peripheral variables, and, for triangulations with strict angle structures, the coefficients of its expansion into Laurent series coincides with the 3D-index of [DGG13]. Our meromorphic function is an example of a topological invariant associated to the self-dual locally compact abelian group (abbreviated LCA group) $\mathbb{T} \times \mathbb{Z}$. A more detailed formulation of our results follows.

In a sense, our paper does the opposite from that of [GK17]. In the latter paper we expressed state integral invariants (which are analytic functions in a cut plane) in terms of q-series, whereas in the present paper we assemble q-series into meromorphic state-integral invariants. Our work illustrates the principle that some state-integrals can be formulated in terms of q-series and vice-versa.

1.2. **Our results.** Fix an ideal triangulation $\mathcal{T}$ of an oriented 3-manifold M whose boundary consists of r tori, and choose peripheral curves that form a basis of $H_1(\partial M, \mathbb{Z})$. To simplify notation, we will present our results only in the case when M has a single torus boundary though our statements and proofs remain valid in the general case.

After a choice of a meridian and longitude, we can identify the complex torus $\mathbb{T}_M = H^1(\partial M, \mathbb{C}^*)$ with $(\mathbb{C}^*)^2$ where the latter is given by the coordinates (e_μ, e_λ). Throughout the paper, q will denote a complex number inside the unit disk: $|q| < 1$. When $q = -e^h$ with $\mathrm{Re}(h) < 0$ and $z \in \mathbb{C}$, we define $(-q)^z = e^{zh}$. For $r, s \in \mathbb{Q}$, we define the associated q-rays of the complex torus by

$$\Sigma_{r,s} = \{(e_\mu, e_\lambda) \in (\mathbb{C}^*)^2 \mid e_\mu^r e_\lambda^s \in q^{\mathbb{N}}\} \tag{1}$$

where $\mathbb{N} = \mathbb{Z}_{\geq 0}$ is the set of nonnegative integers. A shifted q-ray is a subset of the complex torus of the form $\varepsilon q^t \Sigma_{r,s}$ for some $t \in \mathbb{Q}$ and $\varepsilon = \pm 1$.

Theorem 1.1. *With the above assumptions, there exists a meromorphic function*

$$I_{\mathcal{T}}(q) \colon (\mathbb{C}^*)^2 \ni (e_\mu, e_\lambda) \mapsto I_{\mathcal{T},e_\mu,e_\lambda}(q) \in \mathbb{C} \cup \{\infty\}$$

with the following properties:

(a) *$I_{\mathcal{T}}(q)$ is invariant under 2–3 Pachner moves.*
(b) *$I_{\mathcal{T}}(q)$ is given by a balanced state-integral depending only on the Neumann–Zagier matrices of the gluing equations of $\mathcal{T}$.*
(c) *The singularities of $I_{\mathcal{T}}(q)$ are contained the union of finitely many shifted q-rays.*

Theorem 1.2. *When $\mathcal{T}$ has a strict angle structure, we have a Laurent series expansion*

$$I_{\mathcal{T},e_\mu,e_\lambda}(q) = \sum_{(m,e)\in\mathbb{Z}^2} e_\mu^m e_\lambda^e I_{\mathcal{T}}(m,e)(q^2) \tag{2}$$

convergent on the unit torus $|e_\mu| = |e_\lambda| = 1$, where $I_{\mathcal{T}}(m,e)$ is the 3D-index of [DGG14].

Fix a 3-manifold M as above and consider the set $\mathcal{S}_M$ of all ideal triangulations of M that admit a strict angle structure.

Corollary 1.3. Although it is not known yet if $\mathcal{S}_M$ is connected or not by 2–3 Pachner moves, the 3D-index of [DGG14] is constant on $\mathcal{S}_M$.

1.3. **Discussion.** In a series of papers [AK14, AKb, AKa, Kasb], topological invariants of (ideally triangulated) 3-manifolds have been constructed from certain self-dual LCA groups equipped with quantum dilogarithm functions. The main idea of those constructions is the following. Fixing a self-dual LCA group with a gaussian exponential and a quantum dilogarithm function, one assigns a state-integral invariant to an ideal triangulation decorated by a pre-angle structure (in the cited papers this is called *shape structure*), that is a choice of a strict angle structure within each ideal tetrahedron, but the angles do not have to add up to 2π around the edges edges of the ideal triangulation. The resulting state-integral is often the germ of a meromorphic function on the (affine vector) space of real-valued pre-angle structures. This affine space has an (affine vector) subspace of complex-valued angle structures (the pre-angle structures that add up to 2π around each geometric edge of the triangulation). The above meromorphic function is either infinity or restricts to a meromorphic function on the space of complex-valued angle structures. When the latter happens, the state-integral depends only on the peripheral angle monodromy. This way we obtain an invariant of ideal triangulations which depends on the peripheral angle monodromy.

The above construction is general and, in particular, it applies to the invariants constructed in [AK14, AKa, AKb] and [KLV16]. Our goal is to give a self-contained presentation in the case of the self-dual LCA group $\mathbb{T} \times \mathbb{Z}$ with a quantum dilogarithm first found by Woronowicz in [Wor92] and to relate this invariant to the 3D-index of Dimofte–Gaiotto–Gukov [DGG13].

2. Building blocks

2.1. **The tetrahedral weight.** In this section we define the tetrahedral weight which is the building block of our state-integral. We give a self-contained treatment of the symmetries and identities that it satisfies.

Below, we will often consider expansions of meromorphic functions defined on open annuli or punctured disks, examples of which are given in Equations (20), (23), (24). These Laurent expansions (not to be confused with the formal Laurent series which involve only finitely many negative powers and arbitrarily many positive powers) are well-known in complex analysis and their existence, convergence and manipulation follows from Cauchy's theorem. A detailed discussion of this can be found, for example, in [Ahl78].

As a warm up, recall the Pochhammer symbol

$$(x;q)_m := \prod_{i=0}^{m-1}(1-q^i x), \quad m \in \mathbb{N}\cup\{\infty\}, \tag{3}$$

where $\mathbb{N} := \mathbb{Z}_{\geq 0}$ and we always assume that $|q| < 1$. The next lemma summarizes the well-known properties of the Pochhammer symbol $(x;q)_\infty$.

Lemma 2.1. The Pochhammer symbol $(x;q)_\infty$ has the following properties.
(a) It is an entire function of x with simple zeros $x \in q^{-\mathbb{N}}$.
(b) It satisfies the q-difference equation

$$(x;q)_\infty = (1-x)(qx;q)_\infty\,. \tag{4}$$

(c) It has convergent power series expansions

$$\frac{1}{(x;q)_\infty} = \sum_{n=0}^{\infty}\frac{x^n}{(q;q)_n}, \quad |x|<1 \tag{5a}$$

$$(x;q)_\infty = \sum_{n=0}^{\infty}(-1)^n\frac{q^{\frac{1}{2}n(n-1)}x^n}{(q;q)_n}, \quad \forall x \in \mathbb{C}. \tag{5b}$$

For the proof of part (c), see for instance [Zag07, Prop. 2].

Consider the function

$$G_q(z) = \frac{(-qz^{-1};q)_\infty}{(z;q)_\infty} = \frac{\left(1+\frac{q}{z}\right)\left(1+\frac{q^2}{z}\right)\left(1+\frac{q^3}{z}\right)\dots}{(1-z)(1-qz)(1-q^2z)\dots}\,, \tag{6}$$

The next lemma summarizes its properties.

Lemma 2.2. The function $G_q(z)$ defined in (6) has the following properties.
(a) It is a meromorphic function of $z \in \mathbb{C}^* := \mathbb{C}\setminus\{0\}$ with simple zeros and poles in $-q^{1+\mathbb{N}}$ and $q^{-\mathbb{N}}$ respectively, and with essential singularities at $z=0$ and $z=\infty$.
(b) It satisfies the q-difference equation

$$G_q(qz) = (1-z)(1+z^{-1})G_q(z) \tag{7}$$

and the involution equation

$$G_q(-qz) = \frac{1}{G_q(z^{-1})}\,. \tag{8}$$

(c) It has a convergent Laurent series expansion in the punctured unit disk $0 < |z| < 1$:

$$G_q(z) = \sum_{n \in \mathbb{Z}} J(n)(q) z^n \tag{9}$$

where

$$J(n)(q) := \sum_{k=(-n)_+}^{\infty} \frac{q^{\frac{k(k+1)}{2}}}{(q)_k (q)_{n+k}}, \quad (n)_+ := \max\{n, 0\}, \tag{10}$$

is a well-defined element of $\mathbb{Z}[[q]]$, analytic in the disk $|q| < 1$.

Parts (a) and (b) follow easily from the product expansion of the Pochhammer symbol and part (c) follows from (5a)-(5b).

The *tetrahedral weight* is a function $\psi^0(z, w)$ defined by:

$$\psi^0(z, w) := c(q)\, G_q(-qz) G_q(w^{-1}) G_q(wz^{-1}) \tag{11}$$

where

$$c(q) := \frac{(q; q)_\infty^2}{(q^2; q^2)_\infty}\,. \tag{12}$$

The properties of this function are summarized in the following lemma.

Lemma 2.3. The function $\psi^0(z, w)$ defined in (11) has the following properties.
(a) It is a meromorphic function of $(z, w) \in (\mathbb{C}^*)^2$ with zeros in

$$z \in q^{\mathbb{N}}, \quad \text{or} \quad w^{-1} \in -q^{1+\mathbb{N}}, \quad \text{or} \quad z^{-1} w \in -q^{1+\mathbb{N}} \tag{13}$$

and poles in

$$z \in -q^{-1-\mathbb{N}}, \quad \text{or} \quad w \in q^{\mathbb{N}}, \quad \text{or} \quad zw^{-1} \in q^{\mathbb{N}}\,. \tag{14}$$

(b) It satisfies the q-difference equations

$$w^{-1}\psi^0(qz, w) - \psi^0(z, w) - q^{-1}z^{-1}\psi^0(z, w) = 0, \tag{15a}$$

$$w\psi^0(qz, w) - \psi^0(z, w) - qz\psi^0(z, w) = 0\,. \tag{15b}$$

(c) It satisfies the $\mathbb{Z}/2$ and $\mathbb{Z}/3$-invariance equations

$$\psi^0(z, w) = \psi^0(-q^{-1}w^{-1}, -q^{-1}z^{-1}) \tag{16a}$$

$$\psi^0(z, w) = \psi^0(-q^{-1}z^{-1}w, -q^{-1}z^{-1}) = \psi^0(-q^{-1}w^{-1}, zw^{-1})\,. \tag{16b}$$

(d) In the domain

$$1 < |w| < |z| < |q|^{-1}, \tag{17}$$

we have the absolutely convergent expansion

$$\psi^0(z, w) = c(q) \sum_{e, m \in \mathbb{Z}} z^e w^m \sum_{\substack{k_1, k_2, k_3 \in \mathbb{Z} \\ k_1 - k_3 = e;\ k_3 - k_2 = m}} (-q)^{k_1} J(k_1)(q) J(k_2)(q) J(k_3)(q) \tag{18}$$

where the interior sum is a well-defined element of $\mathbb{Z}[[q]]$, analytic in the disk $|q| < 1$.

These properties follow from the definition of ψ^0 and the properties of G_q listed in Lemma 2.2.

2.2. **The quantum dilogarithm.** In this sub-section we identify the tetrahedral weight function $\psi^0(z,w)$ with (the reciprocal of) the quantum dilogarithm function $\psi(z,m)$ on the self-dual LCA group $\mathbb{T}\times\mathbb{Z}$, given by [Kasb, Eqn.97]

$$\psi(z,m) = \frac{(-q^{1-m}/z;q^2)_\infty}{(-q^{1-m}z;q^2)_\infty}\,. \tag{19}$$

In the context of quantum groups, this function first appeared in [Wor92]. In Appendix A, we explain how formula (19) fits the general definition of a quantum dilogarithm over the LCA group $\mathbb{T}\times\mathbb{Z}$.

Lemma 2.1 and the above definition imply the following properties of the function $\psi(z,w)$.

Lemma 2.4. The function $\psi(z,m)$ defined in (19) has the following properties.
(a) It is a meromorphic function of z with simple poles and zeros at $z\in -q^{-1-|m|-2\mathbb{N}}$ and $z\in -q^{1+|m|+2\mathbb{N}}$, respectively.
(b) It is analytic in the annulus $0<|z|<|q|^{-1-|m|}$ (which always includes the unit circle $|z|=1$) where it has an absolutely convergent Laurent series expansion

$$\psi(z,m) = \sum_{e\in\mathbb{Z}} I^\Delta(m,e)(q)z^e \tag{20}$$

where

$$I^\Delta(m,e)(q) = (-q)^e I_\Delta(m,e)(q^2) \tag{21}$$

is related to the tetrahedron index I_Δ of [DGG13] given by

$$I_\Delta(m,e)(q) = \sum_{n=(-e)_+}^{\infty}(-1)^n\frac{q^{\frac12 n(n+1)-\left(n+\frac12 e\right)m}}{(q)_n(q)_{n+e}} \in \mathbb{Z}[[q^{\frac12}]] \tag{22}$$

and $(e)_+ := \max\{0,e\}$ and $(q)_n := (q;q)_n = \prod_{i=1}^n(1-q^i)$.

The next theorem connects the tetrahedral weight function $\psi^0(z,w)$ with the above function $\psi(z,m)$.

Theorem 2.5. (a) In the domain (17) we have the identity

$$\psi^0(z,w) = \sum_{m\in\mathbb{Z}}\psi(z,m)w^m \tag{23}$$

where the sum is absolutely convergent.

(b) In the domain (17) we have an absolutely convergent double Laurent series expansion

$$\psi^0(z,w) = \sum_{e,m\in\mathbb{Z}} I^\Delta(m,e)(q)z^e w^m\,. \tag{24}$$

Proof. We let RHS denote the sum in the right hand side of Equation (23). RHS is absolutely convergent in the domain (17) and it can be explicitly calculated by using Ramanujan's ${}_1\psi_1$-summation formula. The detailed computation appears in [Kasb, Eqn. (98)] and the result reads

$$\text{RHS} = \frac{(q^2;q^2)_\infty\,(z^{-2};q^2)_\infty}{(w^{-2};q^2)_\infty\,(w^2z^{-2};q^2)_\infty}\left(\frac{\theta_q(zw^{-2})}{\theta_q(z)} + w\frac{\theta_q(qw^2/z)}{\theta_q(q/z)}\right) \tag{25}$$

where

$$\theta_q(x) := \sum_{k\in\mathbb{Z}} q^{k^2} x^k = \left(q^2;q^2\right)_\infty \left(-qx;q^2\right)_\infty \left(-q/x;q^2\right)_\infty \tag{26}$$

is the Jacobi theta function, and the second equality in (26) is the Jacobi triple product identity. By using Lemma 2.6 (see below) and the Jacobi triple product identity we can further simplify the right hand side of (25), thus getting

$$\begin{aligned}\mathrm{RHS} &= \frac{(q;q)_\infty^2 \, (z^{-1};q)_\infty \, (-qw;q)_\infty \, (-qz/w;q)_\infty}{(q^2;q^2)_\infty \, (-qz;q)_\infty \, (w^{-1};q)_\infty \, (w/z;q)_\infty} \\ &= c(q)\, G_q(-qz) G_q(w^{-1}) G_q(wz^{-1}) \\ &= \psi^0(z,w)\,.\end{aligned}$$

This concludes the proof of the part (a). Part (b) follows from (23) combined with (20). ☐

Lemma 2.6. For any choice of the square root $p := \sqrt{q}$, we have the identity

$$\frac{\theta_q(zw^{-2})}{\theta_q(z)} + w\frac{\theta_q(qw^2/z)}{\theta_q(q/z)} = \frac{(q;q)_\infty}{(q^2;q^2)_\infty^2} \frac{\theta_p\left(pw/z\right)\theta_p\left(w/p\right)}{\theta_p\left(z/p\right)}. \tag{27}$$

Proof. Denoting the left hand side of (27) as $f(z^{-1},w)$, we have

$$f(u,w) = h(u,w)/g(u)\,,$$

where

$$g(u) := \theta_q(u)\theta_q(qu), \qquad h(u,w) := \theta_q(qu)\theta_q(uw^2) + w\theta_q(u)\theta_q(quw^2)\,.$$

First, we rewrite $g(u)$ as follows:

$$\begin{aligned} g(u) &= \left(q^2;q^2\right)_\infty^2 \left(-qu;q^2\right)_\infty \left(-q^2u;q^2\right)_\infty \left(-q/u;q^2\right)_\infty \left(-1/u;q^2\right)_\infty \\ &= \left(q^2;q^2\right)_\infty^2 \left(-qu;q\right)_\infty \left(-1/u;q\right)_\infty = \frac{\left(q^2;q^2\right)_\infty^2}{\left(q;q\right)_\infty}\theta_p(pu). \end{aligned} \tag{28}$$

Then, we transform $h(u/w,w)$ as follows:

$$h\left(\frac{u}{w},w\right) = \sum_{k,l\in\mathbb{Z}} q^{k^2+l^2} u^{k+l} w^{l-k}(q^k+q^l w) = \sum_{k,l\in\mathbb{Z}} q^{k^2-2kl+2l^2} u^k w^{2l-k}(q^{k-l}+q^l w) \tag{29}$$

where in the second equality we have shifted the summation variable $k \mapsto k-l$. Next, we write out separately the sum over even and odd k:

$$\begin{aligned} h\left(\frac{u}{w},w\right) &= \sum_{k,l\in\mathbb{Z}} q^{2k^2+2(l-k)^2} u^{2k} w^{2l-2k} \left(q^{2k-l}+q^l w + q^{4k+1-2l} u w^{-1} (q^{2k+1-l}+q^l w)\right) \\ &= \sum_{k,l\in\mathbb{Z}} q^{2k^2+2l^2+k} u^{2k} w^{2l} \left(q^{-l}+q^l w + q^{2k+1-2l} u w^{-1} (q^{1-l}+q^l w)\right) \end{aligned} \tag{30}$$

where, this time, in the second equality we have shifted the summation variable $l \mapsto l+k$. Now, we can absorb both summations by using the definition of the θ-function:

$$\begin{aligned} h\left(\frac{u}{w}, w\right) & \\ &= \theta_{q^2}(qu^2)\left(\theta_{q^2}(q^{-1}w^2) + w\theta_{q^2}(qw^2)\right) + qu\theta_{q^2}(q^3u^2)\left(qw^{-1}\theta_{q^2}(q^{-3}w^2) + \theta_{q^2}(q^{-1}w^2)\right). \end{aligned} \tag{31}$$

Finally, by using the functional equation

$$\theta_q(q^{2l}x) = q^{-l^2}x^{-l}\theta_q(x), \qquad \text{for all} \qquad l \in \mathbb{Z}, \tag{32}$$

with $q \mapsto q^2$, $x = w^2$ and $l = -1$ in the second term, we arrive to the following factorized formula

$$h\left(\frac{u}{w}, w\right) = \left(\theta_{q^2}(qu^2) + qu\theta_{q^2}(q^3u^2)\right)\left(\theta_{q^2}(q^{-1}w^2) + w\theta_{q^2}(qw^2)\right) = s(qu)s(w) \tag{33}$$

where

$$\begin{aligned} s(w) &:= \theta_{q^2}(q^{-1}w^2) + w\theta_{q^2}(qw^2) = \sum_{k\in\mathbb{Z}} q^{2k^2}w^{2k}\left(q^{-k} + q^k w\right) \\ &= \sum_{k\in\mathbb{Z}} p^{4k^2}w^{2k}\left(p^{-2k} + p^{2k}w\right) = \sum_{k\in\mathbb{Z}}\left(p^{(2k)^2-2k}w^{2k} + p^{(2k+1)^2-2k-1}w^{2k+1}\right) \\ &= \sum_{k\in\mathbb{Z}} p^{k^2-k}w^k = \theta_p(w/p). \end{aligned} \tag{34}$$

Thus, we have obtained the equality

$$h(u,w) = \theta_p(puw)\theta_p(w/p), \tag{35}$$

and formula (27) follows straightforwardly. □

In summary, the tetrahedral weight $\psi^0(z,w)$ is given by two sum formulas and a product formula, and the last of which implies the meromorphicity of the function and the location of its zeros and poles:

$$\psi^0(z,w) = \sum_{m\in\mathbb{Z}} \psi(z,m)w^m \tag{36}$$

$$= \sum_{e,m\in\mathbb{Z}} I^\Delta(m,e)(q)z^e w^m \tag{37}$$

$$= c(q)\, G_q(-qz)G_q(w^{-1})G_q(wz^{-1}). \tag{38}$$

For completeness, the next lemma summarizes the q-difference equations and the symmetries of $\psi(z,w)$.

Lemma 2.7. The function $\psi(z,m)$ defined in (19) satisfies the following q-difference equations

$$\psi(qz, m+1) - \psi(z,m) - q^{-m-1}z^{-1}\psi(z,m) = 0 \tag{39a}$$

$$\psi(qz, m-1) - \psi(z,m) - q^{-m+1}z\psi(z,m) = 0. \tag{39b}$$

The above equations characterize the meromorphic function $\psi(z,m)$ up to multiplication by a function of q, analytic in the unit disk $|q| < 1$.

These follow from the definition of $\psi(z,w)$ and equations (4), (23) and (15a)-(15b). Alternatively, they can be derived from equation (20) and the symmetries of the tetrahedron index I_Δ given in [Gar16, Thm.3.2].

For completeness, the next lemma summarizes the symmetries of $I^\Delta(m,e)$.

Lemma 2.8. For all integers m and e we have the $\mathbb{Z}/2$ and $\mathbb{Z}/3$-invariance equations:

$$I^\Delta(m,e)(q) = (-q)^{e+m} I^\Delta(-e,-m)(q) \tag{40a}$$

$$I^\Delta(m,e)(q) = (-q)^{e} I^\Delta(-e-m,m)(q) = (-q)^{e+m} I^\Delta(e,-e-m)(q)\,. \tag{40b}$$

As a consequence, we have another $\mathbb{Z}/2$-invariance equation

$$I^\Delta(m,e) = I^\Delta(-m,m+e)\,. \tag{41}$$

These follow from equations (24) and (16a)-(16b). Additionally, they follow from equations (21) and the symmetries of the tetrahedron index $I_\Delta(m,e)$ given in [Gar16, Thm.3.2].

2.3. **The pentagon identity and the Pachner 2–3 move.** Let us recall the pentagon identity for the tetrahedron index I_Δ from [Gar16, Thm.3.7].

$$I_\Delta(m_1-e_2,e_1)I_\Delta(m_2-e_1,e_2) = \sum_{e_3\in\mathbb{Z}} q^{e_3} I_\Delta(m_1,e_1+e_3)I_\Delta(m_2,e_2+e_3)I_\Delta(m_1+m_2,e_3) \tag{42}$$

for all integers m_1, m_2, e_1, e_2. Replacing q by q^2 in (42) and using equation (21), it follows that the tetrahedron index I^Δ satisfies the equation

$$\begin{aligned} I^\Delta(m_1-e_2,e_1)&I^\Delta(m_2-e_1,e_2) \\ &= \sum_{e_3\in\mathbb{Z}} (-q)^{-e_3} I^\Delta(m_1,e_1+e_3)I^\Delta(m_2,e_2+e_3)I^\Delta(m_1+m_2,e_3)\,. \end{aligned} \tag{43}$$

To remove the factor $(-q)^{e_3}$ in the above equation, we apply equation (40b) to the two terms of the left hand side and to the last term in the right hand side, and obtain

$$\begin{aligned} I^\Delta(e_1,-e_1+e_2-m_1)&I^\Delta(e_2,e_1-e_2-m_2) \\ &= \sum_{e_3\in\mathbb{Z}} I^\Delta(m_1,e_1+e_3)I^\Delta(m_2,e_2+e_3)I^\Delta(e_3,-e_3-m_1-m_2)\,. \end{aligned}$$

Setting

$$x=e_1, \quad y=-e_1+e_2-m_1, \quad u=e_2, \quad v=e_1-e_2-m_2, \quad e_3=-z,$$

we rewrite the latter equality as

$$I^\Delta(x,y)I^\Delta(u,v) = \sum_{z\in\mathbb{Z}} I^\Delta(u-x-y,x-z)I^\Delta(-z,z+v+y)I^\Delta(-u-v+x,u-z)\,.$$

Applying equation (41) to both functions on the left hand side of the above equation, and after a linear change of variables we obtain

$$I^\Delta(x,y)I^\Delta(u,v) = \sum_{z\in\mathbb{Z}} I^\Delta(-u-y,-x-z)I^\Delta(-z,z+x+y+u+v)I^\Delta(-x-v,-u-z)\,.$$

Applying equation (41) to the first and third terms of the right hand side of the above equation, we obtain

$$I^\Delta(x,y)I^\Delta(u,v)=\sum_{z\in\mathbb{Z}} I^\Delta(u+y,-x-y-u-z)I^\Delta(-z,z+x+y+u+v)I^\Delta(x+v,-x-u-v-z)\,.$$

Finally, changing the summation variable $z\mapsto z-x-y-u-v$, we obtain equation

$$I^\Delta(x,y)I^\Delta(u,v)=\sum_{z\in\mathbb{Z}} I^\Delta(u+y,v-z)I^\Delta(x+y+u+v-z,z)I^\Delta(x+v,y-z)\,. \tag{44}$$

which coincides with a special (constant) form of the beta pentagon equation [Kas14, Eqn.(2)] for the LCA group $\mathbb{Z}$. Using the fact that the beta pentagon relation is stable under Fourier transformation, see [Kas14, Sec.2], we also conclude that the function

$$\phi(x,y):=\sum_{m,n\in\mathbb{Z}} x^m y^{-n} I^\Delta(m,n)=\psi^0(x,1/y) \tag{45}$$

satisfies the constant beta pentagon equation for the circle LCA group $\mathbb{T}=\{z\in\mathbb{C}\mid |z|=1\}$:

$$\phi(x,y)\phi(u,v)=\int_{\mathbb{T}}\phi(uy,v/z)\phi(xyuv/z,z)\phi(xv,y/z)\frac{\mathrm{d}z}{2\pi i z}\,. \tag{46}$$

Equivalently, the function $\psi^0(z,w)$, thought of as a distribution on $\mathbb{T}^2$, satisfies the integral identity

$$\psi^0(x,y)\,\psi^0(u,v)=\int_{\mathbb{T}}\psi^0\left(\frac{u}{y},\frac{v}{z}\right)\psi^0\left(\frac{xuz}{yv},z\right)\psi^0\left(\frac{x}{v},\frac{y}{z}\right)\frac{\mathrm{d}z}{2\pi i z} \tag{47}$$

where all variables belong to the unit circle $\mathbb{T}$. The distributional interpretation of $\psi^0(z,w)$ means that its restriction to $\mathbb{T}^2$ should be obtained by approaching $\mathbb{T}^2$ from the domain (17), which is the domain of absolute convergence of the double series (24), i.e.,

$$\psi^0(z,w)=\psi^0_{+0,+0}(z,w):=\lim_{\substack{\alpha,\gamma\to 0\\ \alpha>0,\gamma>0,\alpha+\gamma<\pi}}\psi^0_{\alpha,\gamma}(z,w),\quad |z|=|w|=1, \tag{48}$$

where

$$\psi^0_{\alpha,\gamma}(z,w):=\psi^0\left(z(-q)^{-\frac{\alpha+\gamma}{\pi}},w(-q)^{-\frac{\alpha}{\pi}}\right) \tag{49a}$$

$$=c(q)\,G_q(z(-q)^{\frac{\beta}{\pi}})\,G_q(w^{-1}(-q)^{\frac{\alpha}{\pi}})\,G_q(z^{-1}w(-q)^{\frac{\gamma}{\pi}}) \tag{49b}$$

where $\alpha+\beta+\gamma=\pi$. The positive parameters α,β,γ satisfying the condition $\alpha+\beta+\gamma=\pi$ (which we call pre-angle structure) can be identified with the dihedral angles of a positively oriented ideal hyperbolic tetrahedron. These angles are placed on the edges of a tetrahedron with ordered vertices according to Figure 1.

Moreover, the constant distributional beta pentagon identity (47) is a constant limit of the analytically continued non-constant identity

$$\psi^0_{\alpha_3,\gamma_3}(x,y)\,\psi^0_{\alpha_1,\gamma_1}(u,v)=\int_{\mathbb{T}}\psi^0_{\alpha_0,\gamma_0}\left(\frac{u}{y},\frac{v}{z}\right)\psi^0_{\alpha_2,\gamma_2}\left(\frac{xuz}{yv},z\right)\psi^0_{\alpha_4,\gamma_4}\left(\frac{x}{v},\frac{y}{z}\right)\frac{\mathrm{d}z}{2\pi i z} \tag{50}$$

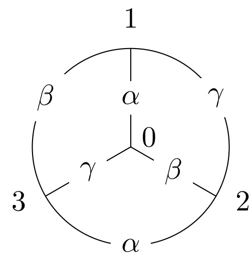

FIGURE 1. The angles of an ideal tetrahedron with ordered vertices.

where $(\alpha_i, \beta_i, \gamma_i)$ are pre-angle structures on five tetrahedra T_i for $i = 0, \dots, 4$ which are compatible, i.e., satisfy the linear relations

$$\alpha_1 = \alpha_0 + \alpha_2, \quad \alpha_3 = \alpha_2 + \alpha_4, \quad \gamma_1 = \gamma_0 + \alpha_4, \quad \gamma_3 = \alpha_0 + \gamma_4, \quad \gamma_2 = \gamma_1 + \gamma_3 \,. \tag{51}$$

Notice that a compatible angle structure satisfies the balancing condition for the interior edge 13: $\beta_0 + \gamma_2 + \beta_4 = 2\pi$. Such an identity for Faddeev's quantum dilogarithm appeared in [AKb, Prop.1].

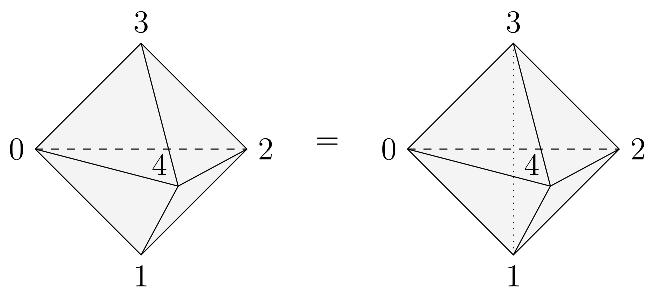

FIGURE 2. The ordered 2–3 Pachner move.

3. The state-integral

3.1. **Definition of the state-integral.** Fix an ideal triangulation $\mathcal{T}$ of an oriented 3-manifold with N tetrahedra T_i for $i = 1, \dots, N$. The invariant is defined as follows:

(a) Assign variables x_i for $i = 1, \dots, N$ to N edges of $\mathcal{T}$.
(b) Choose a strictly positive pre-angle structure $\theta = (\alpha, \beta, \gamma)$ at each tetrahedron. Here, α is the angle of the 01 and 23 edges, β is the angle of the 02 and 13 edges, and γ is the angle of the 03 and 12 edges. The angles are normalized so that at each tetrahedron, their sum is π.
(c) The weight of a tetrahedron T is given by

$$B(T, x, \theta) = \psi^0 \left((-q)^{-\frac{\alpha+\gamma}{\pi}} \frac{X_\alpha}{X_\gamma}, (-q)^{-\frac{\alpha}{\pi}} \frac{X_\beta}{X_\gamma} \right) \tag{52a}$$

$$= c(q)\, G_q \left((-q)^{\frac{\alpha}{\pi}} \frac{X_\gamma}{X_\beta} \right) G_q \left((-q)^{\frac{\beta}{\pi}} \frac{X_\alpha}{X_\gamma} \right) G_q \left((-q)^{\frac{\gamma}{\pi}} \frac{X_\beta}{X_\alpha} \right) \tag{52b}$$

where

$$X_\alpha := x_{01}x_{23}, \quad X_\beta := x_{02}x_{13}, \quad X_\gamma := x_{03}x_{12},$$

and x_{ij} is the variable at edge ij of the tetrahedron.

(d) Define

$$I^{\mathrm{pre}}_{\mathcal{T},\theta}(q) = \int_{\mathbb{T}^N} \prod_{i=1}^{N} B(T_i, x, \theta) d\mu(x) \tag{53}$$

where $d\mu(x) = (2\pi \mathsf{i})^{-N} \prod_{i=1}^N dx_i/x_i$ is the normalized Haar measure on $\mathbb{T}^N$.

Recall the exponent matrices $(\overline{A}|\overline{B}|\overline{C})$ of the edge gluing equations of $\mathcal{T}$ [Thu77, NZ85, CDW]. These are $N \times N$ matrices with integer entries which determine the gluing equations

$$\prod_{j=1}^{N} z_j^{\overline{A}_{i,j}} (z'_j)^{\overline{B}_{i,j}} (z''_j)^{\overline{C}_{i,j}} = 1, \qquad i = 1, \dots, N \tag{54}$$

where $z' = 1/(1-z)$ and $z'' = 1 - 1/z$, and as usual we have $zz'z'' = -1$.

For a vector $x = (x_1, \dots, x_N)$ of nonzero complex numbers, and a vector $v = (v_1, \dots, v_N)$ of integers, define $x^v = \prod_{i=1}^N x_i^{v_i}$. Also, for a matrix A, let A_i denote its ith column.

The next proposition implies that the integral (53) (and even the integrand) depends on only the Neumann–Zagier matrices of the gluing equations of the triangulation $\mathcal{T}$.

Proposition 3.1. With the above notation and for $i = 1, \dots, N$, we have:

$$B(T_i, x, \theta) = c(q)\, G_q\left((-q)^{\frac{\alpha_i}{\pi}} x^{(\overline{C}-\overline{B})_i}\right) G_q\left((-q)^{\frac{\beta_i}{\pi}} x^{(\overline{A}-\overline{C})_i}\right) G_q\left((-q)^{\frac{\gamma_i}{\pi}} x^{(\overline{B}-\overline{A})_i}\right) \tag{55}$$

It follows that $I_{\mathcal{T},\theta}(q)$ depends on only the matrices $\overline{A}$, $\overline{B}$, $\overline{C}$ and θ.

Proof. Let $e_i \in \mathbb{Z}^N$ denote the ith coordinate vector (1 in position i and 0 otherwise). It suffices to show that

$$\left(\frac{X_\gamma}{X_\beta}\right)_i = x^{(\overline{C}-\overline{B})e_i}, \qquad \left(\frac{X_\alpha}{X_\gamma}\right)_i = x^{(\overline{A}-\overline{C})e_i}, \qquad \left(\frac{X_\beta}{X_\alpha}\right)_i = x^{(\overline{B}-\overline{A})e_i}. \tag{56}$$

This follows from the fact that the matrices $\overline{A}$, $\overline{B}$ and $\overline{C}$ indexed by edges × tetrahedra record the number of times a shape z_j (resp., z'_j, z''_j) of the jth tetrahedron appears around an edge e_i. □

Our choice of θ and equation (17) imply that the integrand in (53) is an analytic function of $x \in \mathbb{T}^N$, therefore the integral converges. We are interested in two affine vector subspaces of $\mathbb{C}^{3N}$:

- $\mathcal{A}_\mathcal{T}$, the space of complexified pre-angle structures, i.e., $\theta \in \mathbb{C}^{3N}$ such that $\alpha_i + \beta_i + \gamma_i = \pi$ for $i = 1, \dots, N$.
- $\mathcal{B}_\mathcal{T} \subset \mathcal{A}_\mathcal{T}$, the affine subspace of $\mathcal{A}_\mathcal{T}$ that consists of balanced complexified pre-angle structures, that is the sum of the angles around each edge of $\mathcal{T}$ is 2π. The points of $\mathcal{B}$ are also known as complex-valued angle structures on $\mathcal{T}$.

$\mathcal{A}_\mathcal{T}$ and $\mathcal{B}_\mathcal{T}$ are complex affine subspaces of $\mathbb{C}^{3N}$ of dimension $2N$ and $N+1$ respectively [LT08].

The integral in (53) extends to a meromorphic function of $\theta \in \mathcal{A}_\mathcal{T}$, regular when $\mathrm{Re}(\theta) > 0$. Our task is to show that this extension restricts to a meromorphic function on $\mathcal{B}_\mathcal{T}$. To do so, it will be convenient to parametrize $\mathcal{A}_\mathcal{T}$. This breaks the symmetry in the definition of the integral, however it is a useful gauge to draw conclusions.

Consider a vector $\varepsilon = (\varepsilon_1, \dots, \varepsilon_N) \in \mathbb{C}^N$ and complex numbers μ, λ defined by

$$\overline{A}\alpha + \overline{B}\beta + \overline{C}\gamma = \pi\mathbf{2} + \varepsilon, \qquad \begin{array}{l} \nu_\mu \cdot \alpha + \nu'_\mu \cdot \beta + \nu''_\mu \cdot \gamma = \mu \\ \nu_\lambda \cdot \alpha + \nu'_\lambda \cdot \beta + \nu''_\lambda \cdot \gamma = \lambda \end{array} \tag{57}$$

where μ and λ are the sums of the angles along the meridian and the longitude curves, $\mathbf{2} \in \mathbb{C}^N$ is the vector with all coordinates 2 and ν_μ, ν_λ and their primed versions are the vectors of the meridian and longitude cusp equations. Note that $\varepsilon_1 + \dots + \varepsilon_N = 0$. A *quad* is a choice of a pair of opposite edges in a tetrahedron. Choosing a quad, allows one to eliminate the angle variable of those edges using the equation $\alpha_i + \beta_i + \gamma_i = \pi$. Note that each tetrahedron has 3 quads, hence $\mathcal{T}$ has 3^N quads. If Q is a system of N quads obtained by choosing one quad from each tetrahedron of $\mathcal{T}$, let $Q(\theta) \in \mathbb{C}^N$ be defined so that its ith component is the angle associated to the corresponding quad in T_i, i.e. $Q(\theta)_i = \alpha_i$ if, for example, the quad of the ith tetrahedron is $(01), (23)$. In effect, $Q(\theta)$ chooses one of the 3 angles for every tetrahedron of $\mathcal{T}$. Together with equations (57), we get a linear map

$$\mathcal{A}_\mathcal{T} \to \mathbb{C}^N \times \mathbb{C}^N \times \mathbb{C}^2, \qquad \theta \mapsto (Q(\theta), \varepsilon, \mu, \lambda)\,.$$

For $i, j \in \{1, \dots, N\}$, let $\pi_{i,j} : \mathbb{C}^N \times \mathbb{C}^N \times \mathbb{C}^2 \to \mathbb{C}^{N-1} \times \mathbb{C}^{N-1} \times \mathbb{C}^2$ denote the projection that removes the ith entry of the first copy of $\mathbb{C}^N$ and the jth entry of the second copy of $\mathbb{C}^N$. The next proposition describes a parametrization of $\mathcal{A}_\mathcal{T}$.

Proposition 3.2. For every j, there exists a system of quads Q of $\mathcal{T}$ and an i such that the composition $\mathcal{A}_\mathcal{T} \to \mathbb{C}^N \times \mathbb{C}^N \times \mathbb{C}^2 \xrightarrow{\pi_{i,j}} \mathbb{C}^{N-1} \times \mathbb{C}^{N-1} \times \mathbb{C}^2$ is an affine linear isomorphism.

Proof. Consider the standard system of quads Q of $\mathcal{T}$, i.e., the one that chooses the $(01), (23)$ edges of each tetrahedron of $\mathcal{T}$. It follows that $Q(\theta) = \alpha$. Eliminating β from Equation (57) we obtain that

$$A\alpha + B\gamma = \pi\mathbf{2} + \varepsilon + \nu \tag{58}$$

where

$$A = \overline{A} - \overline{B}, \qquad B = \overline{C} - \overline{B}, \qquad \nu = -\overline{B}\mathbf{1}\pi\,.$$

Consider the matrix $(A'|B')$ obtained from $(A|B)$ by replacing the jth row of $(A|B)$ with the peripheral cusp equation corresponding to the meridian. Neumann–Zagier prove that $(A'|B')$ is the upper half of a symplectic $2N \times 2N$ matrix [NZ85]. In fact we can take the first row of the bottom half of the symplectic matrix to be the peripheral cusp equation corresponding to the longitude. If B' is invertible, then we can solve for γ from Equation (58) and deduce that θ is determined by $(\alpha, \pi_j(\varepsilon), \mu)$. Using the longitude cusp equation, it follows that θ is determined by $(\pi_i(\alpha), \pi_j(\varepsilon), \mu, \lambda)$.

When B' is not invertible, using the fact that $(A'|B')$ is the upper half of a symplectic matrix and [DG13, Lem.A.3], it follows that we can always find a system of quads Q for which the corresponding matrix B' is invertible. The result follows. $\square$

Without loss of generality, we can assume that Proposition 3.2 holds for the standard system of quads, and that $i = j = N$. After a change of variables $x_i \to x_i/x_N$ for $i = 1, \dots, N-1$ the integral $I_{\mathcal{T},\theta}(q)$ reduces to an $N-1$ dimensional integral since the integrand is independent of x_N.

The next lemma shows that after a change of variables, $I_{\mathcal{T},\theta}(q)$ is expressed as an integral whose contour (a product of tori, with radi $|q|$ raised to linear forms of α) depends only on α and whose integrand depends on only $(\varepsilon, \mu, \lambda)$.

Proposition 3.3. With the above assumptions, there exists an edge column vector η whose coordinates are affine linear forms in α, a contour C_α, and affine linear forms ν_i, ν_i', ν_i'' in the variables $\varepsilon = (\varepsilon_1, \dots, \varepsilon_{N-1})$ and μ, λ such that after the change of variables $x_i = (-q)^{\eta_i} y_i$, we have

$$(59)\quad I^{\mathrm{pre}}_{\mathcal{T},\theta}(q) = c(q)^N \int_{C_\alpha} d\mu(y) \prod_{i=1}^N G_q\left((-q)^{\nu_i(\varepsilon,\mu,\lambda)} y^{(\overline{C}-\overline{B})_i}\right) G_q\left((-q)^{\nu_i'(\varepsilon,\mu,\lambda)} y^{(\overline{A}-\overline{C})_i}\right) G_q\left((-q)^{\nu_i''(\varepsilon,\mu,\lambda)} y^{(\overline{B}-\overline{A})_i}\right)$$

Proof. Recall equations (56) and (58). If $x = (-q)^\eta y$ then

$$x^{(\overline{C}-\overline{B})e_i} = (-q)^{\alpha_i + e_i^T (\overline{C}-\overline{B})^T \eta} y^{(\overline{C}-\overline{B})e_i} = (-q)^{\alpha_i + e_i^T B^T \eta} y^{Be_i}$$

and likewise for the cyclic permutations. Using equation (55), and the above equalities, we are looking for an edge vector η such that $\alpha + B^T\eta$, $\beta + (A-B)^T\eta$ and $\gamma - A^T\eta$ depend on only $\varepsilon, \mu, \lambda$. Since the sum of these three vectors is constant, it suffices to find η such that $\alpha - B^T\eta$ and $\gamma - A^T\eta$ depend on only $\varepsilon, \mu, \lambda$. Using the fact that B' is invertible, we can take $\eta = (B'^T)^{-1}\alpha$. Solving for γ from equation (58), we obtain that

$$\gamma - A'^T\eta = B'^{-1}(\pi\mathbf{2} + \varepsilon + \nu) - (B')^{-1}A'\alpha - (A')^T((B')^T)^{-1}\alpha = B'^{-1}(\pi\mathbf{2} + \varepsilon + \nu)\,.$$

The last equality follows from the fact that $(A'|B')$ is the upper half of a symplectic matrix, hence $(B')^{-1}A'$ is symmetric. Finally observe that $|x_i| = 1$, hence $|y_i| = |q|^{\eta_i}$ where η_i are linear forms in α. Hence, the y-contour C_α is a product of tori whose radii depend linearly on α. This completes the proof. $\square$

Remark 3.4. With the assumptions of Proposition 3.3, there exists an edge column vector η' whose coordinates are affine linear forms in α, μ, λ, a contour C, and affine linear forms ξ_i, ξ_i', ξ_i'' in the variables $\varepsilon = (\varepsilon_1, \dots, \varepsilon_{N-1})$, μ and λ such that after the change of variables $y_i = (-q)^{\eta_i'} z_i$, we have

$$(60)\quad I^{\mathrm{pre}}_{\mathcal{T},\theta}(q) = c(q)^N \int_{C} d\mu(z) \prod_{i=1}^N G_q\left((-q)^{\xi_i(\varepsilon,\mu,\lambda)} z^{(\overline{C}-\overline{B})_i}\right) G_q\left((-q)^{\xi_i'(\varepsilon,\mu,\lambda)} z^{(\overline{A}-\overline{C})_i}\right) G_q\left((-q)^{\xi_i''(\varepsilon,\mu,\lambda)} z^{(\overline{B}-\overline{A})_i}\right)$$

where $\xi_i(0,\mu,\lambda)$, $\xi_i'(0,\mu,\lambda)$ and $\xi_i''(0,\mu,\lambda)$ are $\mathbb{Z}$-linear combinations of $1, \mu/\pi, \lambda/(2\pi)$. This follows from the symplectic properties of the Neumann–Zagier matrices [NZ85] (compare also with the matrices $(A'|B')$ in equation (67) below).

The contours in equations (59) and (60) depend on a positive pre-angle structure, but the integrals are independent of the choice of the pre-angle structure. When we balance, i.e., set $\varepsilon = 0$, there are two posibilities: either we can move the contour by a small isotopy in order to avoid the singularities of the integrand, or we cannot do so. In the former case, the new contour is canonically defined from the old contour. In the latter case, we apply the residue theorem to change the integration contour, and the residue contribution is either finite or infinite. In the latter case, by definition, our meromorphic function is infinity.

Remark 3.5. An example of an integral where the latter case occurs is the following integral

$$I_\varepsilon(C) := \int_C \frac{dz}{z} G_q(z) G_q((-q)^\varepsilon z^{-1}) \tag{61}$$

where C is the contour $|z| = 1 - \delta$ for $\delta > 0$ small. The singularities of the integrand are $z \in q^{-\mathbb{N}}$ and $z \in (-q)^\varepsilon q^{\mathbb{N}}$. When ε approaches zero, the contour is pinched from two sides by the singularities at $z = (-q)^\varepsilon$ and $z = 1$. To avoid this pinching, by applying the residue theorem, we move the contour of integration C to the other side of 1, i.e. the contour C' given by $|z| = 1 + \delta$:

$$I_\varepsilon(C) = I_\varepsilon(C') - 2\pi i \operatorname{Res}_{z=1}\left(\frac{1}{z} G_q(z) G_q((-q)^\varepsilon z^{-1})\right) = I_\varepsilon(C') + 2\pi i \frac{(-q;q)_\infty}{(q;q)_\infty} G_q((-q)^\varepsilon)\,. \tag{62}$$

Even though $I_\varepsilon(C')$ is regular at $\varepsilon = 0$, the contribution from the residue is singular because of the simple pole of $G_q((-q)^\varepsilon)$ at $\varepsilon = 0$. Thus, we conclude that $I_0(C) = \infty$.

The next proposition defines the meromorphic function that appears in Theorem 1.1.

Proposition 3.6. Setting $\varepsilon = 0$ (i.e., balancing the edges), and assuming that we find a contour C, we obtain a meromorphic function of $(e_\mu, e_\lambda) := ((-q)^{\mu/\pi}, (-q)^{\lambda/(2\pi)}) \in (\mathbb{C}^*)^2$

$$\begin{aligned} I_{\mathcal{T},e_\mu,e_\lambda}(q) = c(q)^N \int_C d\mu(z) \\ \prod_{i=1}^N G_q\left((-q)^{\xi_i(0,\mu,\lambda)} z^{(\overline{C}-\overline{B})_i}\right) G_q\left((-q)^{\xi'_i(0,\mu,\lambda)} z^{(\overline{A}-\overline{C})_i}\right) G_q\left((-q)^{\xi''_i(0,\mu,\lambda)} z^{(\overline{B}-\overline{A})_i}\right)\,. \end{aligned}$$

The above integral is absolutely convergent.

Let us phrase our previous proposition in more invariant language. Recall the complex torus $\mathbb{T}_M = H^1(\partial M, \mathbb{C}^*)$. Define the map

$$\varpi_{\mathcal{T}} : \mathcal{B}_{\mathcal{T}} \to \mathbb{T}_M, \qquad \delta \mapsto (-q)^{\mathrm{hol}(\delta)/\pi} \tag{63}$$

for a simple closed curve δ of ∂M, where $\mathrm{hol}(\delta)$ denotes the angle holonomy along δ. Proposition 3.6 states that the restriction $I^{\mathrm{bal}}_{\mathcal{T}}$ of the meromorphic function $I^{\mathrm{pre}}_{\mathcal{T}}$ on $\mathcal{B}$ is the pullback of a meromorphic function $I_{\mathcal{T}}$ on $\mathbb{T}_M$. In other words, the following diagram commutes:

$$\begin{tikzcd} \mathcal{A}_{\mathcal{T}} \arrow[dr, "I^{\mathrm{pre}}_{\mathcal{T}}"'] & \mathcal{B}_{\mathcal{T}} \arrow[l, hook'] \arrow[r, two heads, "\varpi_{\mathcal{T}}"] \arrow[d, "I^{\mathrm{bal}}_{\mathcal{T}}"] & \mathbb{T}_M \arrow[dl, "I_{\mathcal{T}}"] \\ & \mathbb{C} & \end{tikzcd} \tag{64}$$

3.2. **Identification with the 3D-Index of Dimofte–Gaiotto–Gukov.** In this sub-section we discuss the Laurent expansion of the meromorphic function $I_{\mathcal{T},e_\mu,e_\lambda}(q)$ on the real torus $|e_\mu| = |e_\lambda| = 1$, under the assumption that $\mathcal{T}$ supports a strict angle structure. As we will show, the coefficients of the Laurent series are the 3D-index of Dimofte–Gaiotto–Gukov. This will conclude the proof of Theorem 1.2.

The next lemma (for $h = 0$) is based on the idea that the upper half part of a symplectic matrix with integer entries is a pair of coprime matrices.

Lemma 3.7. Suppose $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is a symplectic matrix with integer entries where A, B, C, D are $N \times N$ matrices. Let $(A'|B')$ denote the upper $(N + h) \times 2N$ part of M and consider $r, s \in \mathbb{Z}^N$ that satisfy the equation

$$A'r + B's = \begin{pmatrix} 0 \\ \nu \end{pmatrix}$$

for a vector $\nu \in \mathbb{Z}^h$. Then, there exists $k \in \mathbb{Z}^{N-h}$ such that

$$r = -B'^T \begin{pmatrix} \nu \\ k \end{pmatrix}, \qquad s = A'^T \begin{pmatrix} \nu \\ k \end{pmatrix}.$$

Proof. We apply the symplectic matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ to the vector $\begin{pmatrix} r \\ s \end{pmatrix}$ and define $k \in \mathbb{Z}^{N-h}$ so that $k' := \begin{pmatrix} \nu \\ k \end{pmatrix} \in \mathbb{Z}^N$ satisfies $M \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} 0 \\ k' \end{pmatrix}$. Then,

$$\begin{pmatrix} r \\ s \end{pmatrix} = M^{-1} \begin{pmatrix} 0 \\ k' \end{pmatrix} = \begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix} \begin{pmatrix} 0 \\ k' \end{pmatrix} = \begin{pmatrix} -B^T k' \\ A^T k' \end{pmatrix}.$$

The result follows. $\square$

Proof. (of Theorem 1.2) Fix an ideal triangulation $\mathcal{T}$ of M with N tetrahedra and let $\overline{A}, \overline{B}$ and $\overline{C}$ denote the Neumann–Zagier matrices describing gluing equations (54). If we eliminate the shape $z_i' = -1/(z_i z_i'')$, we obtain the gluing equations in the form:

$$\prod_{j=1}^{N} z_j^{A_{i,j}} (z_j'')^{B_{i,j}} = (-1)^{\nu_i}, \qquad i = 1, \dots, N \tag{65}$$

where

$$A = \overline{A} - \overline{B}, \qquad B = \overline{C} - \overline{B}.$$

Notce that $\overline{A}-\overline{C} = A-B$, $\overline{B}-\overline{C} = -B$. Fix a strict angle structure θ and use Proposition 3.1 to write the integrand of $I^{\mathrm{pre}}_{\mathcal{T},\theta}$ as follows:

$$\begin{aligned}
\prod_{i=1}^{N} B(T_i, x, \theta) &= \prod_{i=1}^{N} \psi^0\left((-q)^{-\frac{\alpha_i+\gamma_i}{\pi}} \frac{X_{\alpha_i}}{X_{\gamma_i}}, (-q)^{-\frac{\alpha_i}{\pi}} \frac{X_{\beta_i}}{X_{\gamma_i}}\right) \\
&= \prod_{i=1}^{N} \psi^0\left((-q)^{-\frac{\alpha_i+\gamma_i}{\pi}} x^{(\overline{A}-\overline{C})_i}, (-q)^{-\frac{\alpha_i}{\pi}} x^{(\overline{B}-\overline{C})_i}\right) \\
&= \prod_{i=1}^{N} \psi^0\left((-q)^{-\frac{\alpha_i+\gamma_i}{\pi}} x^{(A-B)_i}, (-q)^{-\frac{\alpha_i}{\pi}} x^{-(B)_i}\right).
\end{aligned}$$

Use equation (24) to expand the integrand of $I^{\mathrm{pre}}_{\mathcal{T},\theta}$ as a convergent series on the torus $|x_i| = 1$ (for $i = 1, \dots, N$):

$$\prod_{i=1}^{N} B(T_i, x, \theta) = \sum_{r,s\in\mathbb{Z}^N} I^{\Delta}(r_1, s_1) \dots I^{\Delta}(r_N, s_N) x^{\sum_i (A-B)_i s_i} x^{-\sum_i (B)_i r_i} (-q)^{-\frac{1}{\pi}\sum_i (\alpha_i+\gamma_i)s_i + \alpha_i r_i}$$

where $r = (r_1, \dots, r_N) \in \mathbb{Z}^N$ and $s = (s_1, \dots, s_N) \in \mathbb{Z}^N$. Now, we can compute the absolutely convergent integral $I^{\mathrm{pre}}_{\mathcal{T},\theta}$ by integrating over the torus. After interchanging summation and integration (justified by uniform convergence) and applying the residue theorem we obtain a sum over $r, s \in \mathbb{Z}^N$ such that $(A - B)s - Br = 0$:

$$\begin{aligned}
I^{\mathrm{pre}}_{\mathcal{T},\theta} &= \int_{\mathbb{T}^N} d\mu(x) \prod_{i=1}^{N} B(T_i, x, \theta) \\
&= \sum_{r,s} I^{\Delta}(r_1, s_1) \dots I^{\Delta}(r_N, s_N) (-q)^{-\frac{1}{\pi}\sum_i (\alpha_i+\gamma_i)s_i + \alpha_i r_i} \int_{\mathbb{T}^N} d\mu(x) x^{\sum_i (A-B)_i s_i - \sum_i (B)_i r_i} \\
&= \sum_{r,s:(A-B)s-Br=0} I^{\Delta}(r_1, s_1) \dots I^{\Delta}(r_N, s_N) (-q)^{-\frac{1}{\pi}\sum_i (\alpha_i+\gamma_i)s_i + \alpha_i r_i}.
\end{aligned}$$

Collecting further terms whose meridian and (half) longitude holonomy is a fixed integer, we obtain the uniformly convergent sum:

$$I^{\mathrm{pre}}_{\mathcal{T},\theta}(q) = \sum_{m,e\in\mathbb{Z}} (-q)^{\frac{m\mu}{\pi}} (-q)^{\frac{e\lambda}{2\pi}} \sum_{r,s} I^{\Delta}(r_1, s_1) \dots I^{\Delta}(r_N, s_N) \tag{66}$$

where $r, s \in \mathbb{Z}^N$ satisfy the equation

$$A's + B'(-r - s) = \begin{pmatrix} 0 \\ \nu \end{pmatrix}, \qquad \nu = \begin{pmatrix} m \\ e \end{pmatrix} \tag{67}$$

for the matrix $(A'|B')$ (where A', B' are $(N + 1) \times N$ matrices) obtained from $(A|B)$ after we remove any one row of it and replace it with two rows of the meridian and half-longitude monodromy. Neumann–Zagier [NZ85] prove that $(A'|B')$ can be completed to a symplectic matrix. Using this, and Lemma 3.7, it follows that there exists $k \in \mathbb{Z}^{N-1}$ such that

$$s = -B'^T \begin{pmatrix} \nu \\ k \end{pmatrix}, \qquad -r - s = A'^T \begin{pmatrix} \nu \\ k \end{pmatrix}.$$

Let a_i and b_i for $i = 1, \dots, N$ denote the ith column of A' and B' respectively and let $k' = \begin{pmatrix} \nu \\ k \end{pmatrix}$. Using the above and equations (40a), (41), we obtain that

$$I^\Delta(r_i, s_i) = I^\Delta(-a_i \cdot k' + b_i \cdot k', -b_i \cdot k') = I^\Delta(-b_i \cdot k', a_i \cdot k')$$

for $i = 1, \dots, N$. Combined with equation (66), this gives

$$\begin{aligned} I^{\mathrm{pre}}_{\mathcal{T},\theta}(q) &= \sum_{m,e \in \mathbb{Z}} (-q)^{\frac{m\mu}{\pi}} (-q)^{\frac{e\lambda}{2\pi}} \sum_{k \in \mathbb{Z}^{N-1}} \prod_{i=1}^{N} I^\Delta(r_i, s_i)(q) \\ &= \sum_{m,e \in \mathbb{Z}} (-q)^{\frac{m\mu}{\pi}} (-q)^{\frac{e\lambda}{2\pi}} \sum_{k \in \mathbb{Z}^{N-1}} \prod_{i=1}^{N} I^\Delta(-b_i \cdot k', a_i \cdot k')(q) \\ &= \sum_{m,e \in \mathbb{Z}} (-q)^{\frac{m\mu}{\pi}} (-q)^{\frac{e\lambda}{2\pi}} \sum_{k \in \mathbb{Z}^{N-1}} \prod_{i=1}^{N} (-q)^{v \cdot k'} I_\Delta(-b_i \cdot k', a_i \cdot k')(q^2). \end{aligned}$$

where the last equality follows from equation (21) and $v = (1, \dots, 1, m, e) \in \mathbb{Z}^{N+1}$. The latter sum coincides with the 3Dindex of [DGG14]; see also [GHRS15, Sec.4.5]. This completes the proof of Theorem 1.2. $\square$

3.3. **Invariance under 2–3 Pachner moves.** Next we prove the invariance of the meromorphic function $I_\mathcal{T}$ under 2–3 Pachner moves. Consider two ideal triangulations $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ with N and $N+1$ tetrahedra that are related by a 2–3 Pachner move as in Figure 2 and choose θ and $\widetilde{\theta}$ compatible positive angle structures on $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ that satisfy equations (51). In particular, this means that the sum of angles around the interior edge of the 3 tetrahedra is 2π.

Note that $\widetilde{\theta}$ determines θ, but not vice-versa. Let us denote the linear map $\widetilde{\theta} \mapsto \theta$ by $\theta = m_3^2(\widetilde{\theta})$. The commutative diagram (64) gives a commutative diagram

$$\begin{tikzcd} \mathcal{A}_{\widetilde{\mathcal{T}}} \arrow[d, two heads, "m_3^2"'] & \mathcal{B}_{\widetilde{\mathcal{T}}} \arrow[l, hook'] \arrow[r, two heads, "\varpi_{\widetilde{\mathcal{T}}}"] \arrow[d, two heads, "m_3^2"'] & \mathbb{T}_M \arrow[d, equal] \\ \mathcal{A}_{\mathcal{T}} & \mathcal{B}_{\mathcal{T}} \arrow[l, hook'] \arrow[r, two heads, "\varpi_{\mathcal{T}}"] & \mathbb{T}_M \end{tikzcd} \tag{68}$$

Let $\widetilde{\theta}$ and θ be positive pre-angle structures. We claim that

$$I^{\mathrm{pre}}_{\mathcal{T},\theta}(q) = I^{\mathrm{pre}}_{\widetilde{\mathcal{T}},\widetilde{\theta}}(q). \tag{69}$$

This follows by separating the integration variable of the inner edge of the 2–3 move in the integral $I^{\mathrm{pre}}_{\widetilde{\mathcal{T}},\widetilde{\theta}}(q)$ and applying the pentagon identity (50) to that variable. Since a meromorphic function is uniquely determined by its values on positive pre-angle structures and the map $m_3^2 : \mathcal{A}_{\widetilde{\mathcal{T}}} \to \mathcal{A}_{\mathcal{T}}$ is onto, it follows that $I^{\mathrm{pre}}_{\widetilde{\mathcal{T}}} = I^{\mathrm{pre}}_{\mathcal{T}} \circ m_3^2$. Now restrict to $\mathcal{B}_{\widetilde{\mathcal{T}}}$, use the above commutative diagram and the fact that m_3^2 is onto. It follows that $I^{\mathrm{bal}}_{\widetilde{\mathcal{T}}} = I^{\mathrm{bal}}_{\mathcal{T}} \circ m_3^2$. Using once again the commutative diagram and the fact that $\varpi_{\mathcal{T}}$ and $\varpi_{\widetilde{\mathcal{T}}}$ are onto, it follows that $I_{\widetilde{\mathcal{T}}} = I_{\mathcal{T}}$.

3.4. **The singularities of $I_{\mathcal{T}}(q)$.** The singularities of the integrand of $I_{\mathcal{T},e_\mu,e_\lambda}(q)$ are given by Lemma 2.2. To determine the singularities of the meromorphic function $I_{\mathcal{T}}$, perform one integral at a time and use the next lemma.

Lemma 3.8. Suppose $f(z)$ is a meromorphic function of z with singularities on $q^{-\mathbb{N}}$. Fix positive integers $a_1, \dots, a_p > 0$ and $b_1, \dots, b_n > 0$, let $s = (s_1, \dots, s_p)$, $t = (t_1, \dots, t_n)$ and consider the integral:

$$F(s,t) = \int_{\mathcal{C}} \prod_{i=1}^{p} f(s_i z^{a_i}) \prod_{j=1}^{n} f(t_j z^{-b_j}) \frac{dz}{z}$$

where $\mathcal{C}$ is a contour that separates $q^{-1-\mathbb{N}}$ from $q^{\mathbb{N}}$. Then, $F(s,t)$ is a meromorphic function of (s,t) with singularities at a subset of

$$\{(s,t) \mid s_i^{b_j} t_j^{a_i} \in q^{-a_i\mathbb{N} - b_j\mathbb{N}} \text{ for some } 1 \leq i \leq p, 1 \leq j \leq n\} \tag{70}$$

Proof. The singularities of the integrand is the set $\Sigma^-(s) \cup \Sigma^+(t)$ where

$$\Sigma^-(s) = \cup_{i=1}^{p} \{z^{a_i} \in s_i^{-1} q^{-\mathbb{N}}\}, \qquad \Sigma^+(t) = \cup_{j=1}^{n} \{z^{b_j} \in t_j q^{\mathbb{N}}\}.$$

As long as $\Sigma^-(s) \cup \Sigma^+(t)$ does not touch the contour $\mathcal{C}$, $F(s,t)$ is regular. It follows that if $F(s,t)$ is singular when pinching occurs, in other words we must have $z^{a_i} = s_i^{-1} q^{-k}$ and $z^{b_j} = t_j q^l$ for some i, j and some $k, l \in \mathbb{N}$. Thus $z^{a_i b_j} = (s_i^{-1} q^{-k})^{b_j} = (t_j q^l)^{a_i}$. The result follows. ∎

4. Examples and computations

4.1. **A non-1-efficient triangulation with two tetrahedra.** It is traditional in hyperbolic geometry to illustrate theorems concerning ideally triangulated manifolds with the case of the standard triangulation of the complement of the 4_1 knot. In our examples, we will deviate from this principle and begin by giving a detailed computation of the state integral for the case of a non-1-efficient ideal triangulation with two tetrahedra. This illustrates Propositions 3.1, 3.2, 3.3, 3.6 and Theorem 1.1, and also points out the inapplicability of Theorem 1.2.

Ideal triangulations can be efficiently described, constructed and manipulated by `SnapPy` and `Regina` [Bur, CDW], and we will follow their description below. In particular, ideal triangulations can be uniquely reconstructed by their isometry signature, and the latter is a string of letters and numbers. There are exactly 10 ideal triangulations with two tetrahedra of manifolds with one cusp, given in Table 3 of [GHHR16], 9 of them are 1-efficient, and one of them with isometry signature `cPcbbbdei` is not. Although we will not use it, this triangulation is not ordered. The underlying 3-manifold M is the union of a $T(2,4)$ torus link with the $T(1,3)$ (trefoil) torus knot. In `Regina` format, the tetrahedron gluings of the triangulation $\mathcal{T}$ of `cPcbbbdei` are given by:

tet	glued to	(012)	(013)	(023)	(123)
0		1(130)	1(023)	1(021)	1(132)
1		0(032)	0(201)	0(013)	0(132)

The edges of $\mathcal{T}$ are given by:

tet	edge	01	02	03	12	13	23
0		0	0	0	0	1	1
1		0	0	0	1	0	1

and the triangle faces of $\mathcal{T}$ are given by:

tet	face	012	013	023	123
0		0	1	2	3
1		2	0	1	3

The gluing equations, in `SnapPy` format and with the `Regina` ordering of the edges, are given by:

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 2 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \end{pmatrix}.$$

The $\overline{A}$, $\overline{B}$ and $\overline{C}$ matrices are given by:

$$\overline{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \overline{B} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}, \qquad \overline{C} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}.$$

The holonomy of the meridian and the longitude are given by:

$$\mu = -\beta_0 + \alpha_1, \qquad \lambda = 2\gamma_0 .$$

The ε variables are given by:

$$\varepsilon_0 = \alpha_0 + \beta_0 + 2\gamma_0 + \alpha_1 + 2\beta_1 + \gamma_1 - 2\pi, \qquad \varepsilon_1 = \alpha_0 + \beta_0 + \alpha_1 + \gamma_1 - 2\pi . \tag{71}$$

Using $\alpha_i + \beta_i + \gamma_i = \pi$ for $i = 0, 1$, we can solve the above equations in terms of the variables $\alpha_0, \varepsilon_0, \mu, \lambda$:

$$\begin{aligned} \alpha_0 &= \alpha_0 & \alpha_1 &= \pi - \alpha_0 - \lambda/2 + \mu \\ \beta_0 &= \pi - \alpha_0 - \lambda/2 & \beta_1 &= \varepsilon_0 - \lambda/2 \\ \gamma_0 &= \lambda/2 & \gamma_1 &= \alpha_0 - \varepsilon_0 + \lambda - \mu . \end{aligned} \tag{72}$$

This illustrates Proposition 3.2.

If x_i are the variables of the i-th edge for $i = 0, 1$ and θ is a pre-angle structure, then

$$
\begin{aligned}
B(\mathcal{T},x,\theta)(q) &= c(q)^2 G_q\left((-q)^{\frac{\alpha_0}{\pi}}\frac{x_0x_0}{x_0x_1}\right) G_q\left((-q)^{\frac{\beta_0}{\pi}}\frac{x_0x_1}{x_0x_0}\right) G_q\left((-q)^{\frac{\gamma_0}{\pi}}\frac{x_0x_1}{x_0x_1}\right)\\
&\qquad G_q\left((-q)^{\frac{\alpha_1}{\pi}}\frac{x_0x_1}{x_0x_0}\right) G_q\left((-q)^{\frac{\beta_1}{\pi}}\frac{x_0x_1}{x_0x_1}\right) G_q\left((-q)^{\frac{\gamma_1}{\pi}}\frac{x_0x_0}{x_0x_1}\right)\\
&= c(q)^2 G_q\left((-q)^{\frac{\alpha_0}{\pi}}\frac{x_0}{x_1}\right) G_q\left((-q)^{\frac{\beta_0}{\pi}}\frac{x_1}{x_0}\right) G_q\left((-q)^{\frac{\gamma_0}{\pi}}\right)\\
&\qquad G_q\left((-q)^{\frac{\alpha_1}{\pi}}\frac{x_1}{x_0}\right) G_q\left((-q)^{\frac{\beta_1}{\pi}}\right) G_q\left((-q)^{\frac{\gamma_1}{\pi}}\frac{x_0}{x_1}\right)\,.
\end{aligned}
\tag{73}
$$

When θ is positive, the absolutely convergent state-integral is given by:

$$
\begin{aligned}
I^{\mathrm{pre}}_{\mathcal{T},\theta}(q) &= \frac{c(q)^2}{(2\pi \mathsf{i})^2}\int_{\mathbb{T}^2} G_q\left((-q)^{\frac{\alpha_0}{\pi}}\frac{x_0}{x_1}\right) G_q\left((-q)^{\frac{\beta_0}{\pi}}\frac{x_1}{x_0}\right) G_q\left((-q)^{\frac{\gamma_0}{\pi}}\right)\\
&\qquad G_q\left((-q)^{\frac{\alpha_1}{\pi}}\frac{x_1}{x_0}\right) G_q\left((-q)^{\frac{\beta_1}{\pi}}\right) G_q\left((-q)^{\frac{\gamma_1}{\pi}}\frac{x_0}{x_1}\right)\frac{dx_0dx_1}{x_0x_1}\,.
\end{aligned}
\tag{74}
$$

After rescaling $x_0 \to x_0/x_1$, the integral is free of the x_1-variable and is given by:

$$
\begin{aligned}
I^{\mathrm{pre}}_{\mathcal{T},\theta}(q) &= \frac{c(q)^2}{2\pi \mathsf{i}} G_q\left((-q)^{\frac{\gamma_0}{\pi}}\right) G_q\left((-q)^{\frac{\beta_1}{\pi}}\right)\\
&\quad\times\int_{\mathbb{T}} G_q\left((-q)^{\frac{\alpha_0}{\pi}}x_0\right) G_q\left((-q)^{\frac{\beta_0}{\pi}}\frac{1}{x_0}\right) G_q\left((-q)^{\frac{\alpha_1}{\pi}}\frac{1}{x_0}\right) G_q\left((-q)^{\frac{\gamma_1}{\pi}}x_0\right)\frac{dx_0}{x_0}\,.
\end{aligned}
\tag{75}
$$

Using equation (73), the above integral becomes:

$$
\begin{aligned}
I^{\mathrm{pre}}_{\mathcal{T},\theta}(q) &= \frac{c(q)^2}{2\pi \mathsf{i}} G_q\left((-q)^{\frac{\lambda}{2\pi}}\right) G_q\left((-q)^{\frac{\varepsilon_0-\lambda/2}{\pi}}\right)\int_{\mathbb{T}} G_q\left((-q)^{\frac{\alpha_0}{\pi}}x_0\right)\\
&\quad G_q\left((-q)^{1+\frac{-\alpha_0-\lambda/2}{\pi}}\frac{1}{x_0}\right) G_q\left((-q)^{1+\frac{-\alpha_0-\lambda/2+\mu}{\pi}}\frac{1}{x_0}\right) G_q\left((-q)^{\frac{\alpha_0-\varepsilon_0+\lambda-\mu}{\pi}}x_0\right)\frac{dx_0}{x_0}\,.
\end{aligned}
\tag{76}
$$

Applying the change of variables

$$
(-q)^{\frac{\alpha_0}{\pi}}x_0 = y_0\,,
\tag{77}
$$

the above integral becomes:

$$
\begin{aligned}
I^{\mathrm{pre}}_{\mathcal{T},\theta}(q) &= \frac{c(q)^2}{2\pi \mathsf{i}} G_q\left((-q)^{\frac{\lambda}{2\pi}}\right) G_q\left((-q)^{\frac{\varepsilon_0-\lambda/2}{\pi}}\right)\\
&\quad\times\int_{C_\alpha} G_q(y_0)\, G_q\left((-q)^{1-\frac{\lambda/2}{\pi}}\frac{1}{y_0}\right) G_q\left((-q)^{1+\frac{-\lambda/2+\mu}{\pi}}\frac{1}{y_0}\right) G_q\left((-q)^{\frac{-\varepsilon_0+\lambda-\mu}{\pi}}y_0\right)\frac{dy_0}{y_0}
\end{aligned}
\tag{78}
$$

where C_α is the torus $|y_0| = |q^{\frac{\alpha_0}{\pi}}|$ and the integrand depends on $\varepsilon_0, \mu, \lambda$ and y_0. Notice that when θ is positive, Equation (73) implies that $\varepsilon_0 = \gamma_0 + \beta_1 > 0$ and $\varepsilon_1 = -\gamma_0 - \beta_1 < 0$.

Moreover, we can set $\varepsilon_0 = 0$ and obtain the uniformly convergent balanced integral

$$
\begin{aligned}
(79) \quad I^{\mathrm{bal}}_{\mathcal{T},\theta}(q) = \frac{c(q)^2}{2\pi\mathsf{i}} G_q\left((-q)^{\frac{\lambda}{2\pi}}\right) G_q\left((-q)^{\frac{-\lambda/2}{\pi}}\right) \\
\times \int_{C_\alpha} G_q(y_0)\, G_q\left((-q)^{1-\frac{\lambda/2}{\pi}}\frac{1}{y_0}\right) G_q\left((-q)^{1+\frac{-\lambda/2+\mu}{\pi}}\frac{1}{y_0}\right) G_q\left((-q)^{\frac{\lambda-\mu}{\pi}} y_0\right)\frac{dy_0}{y_0}
\end{aligned}
$$

which only depends on $(e_\mu, e_\lambda) = ((-q)^{\mu/\pi}, (-q)^{\lambda/(2\pi)})$. To simplify notation further, let $(s,t) = (e_\mu, e_\lambda)$. Then, we get:

$$
\begin{aligned}
(80) \quad I_{\mathcal{T},s,t}(q) = \frac{c(q)^2}{2\pi\mathsf{i}} G_q(t)\, G_q\left(t^{-1}\right) \\
\times \int_{C} G_q(y_0)\, G_q\left((-q)t^{-1}y_0^{-1}\right) G_q\left((-q)st^{-1}y_0^{-1}\right) G_q\left(s^{-1}t^2 y_0\right)\frac{dy_0}{y_0}
\end{aligned}
$$

where C is the torus $|y_0| = |q|^{\delta_0}$ for small $\delta_0 > 0$. This is the meromorphic function of Theorem 1.1. Next, we compute its singularities, starting from the singularities of the integrand. Using part (a) of Lemma 2.3, we see that the singularities of the integrand are given by

$$y_0 \in \Sigma^-(s,t) \cup \Sigma^+(s,t)\,,$$

where

$$\Sigma^-(s,t) = q^{-\mathbb{N}} \cup st^{-2}q^{-\mathbb{N}}, \qquad \Sigma^+(s,t) = (-q)t^{-1}q^{\mathbb{N}} \cup (-q)st^{-1}q^{\mathbb{N}}\,.$$

The contour of integration has to separate $\Sigma^-(s,t)$ from $\Sigma^+(s,t)$. The integral can only be singular when pinching occurs, that is, for (s,t) such that $\Sigma^-(s,t)$ intersects $\Sigma^+(s,t)$. This happens precisely when

$$(81) \qquad t \in -q^{\mathbb{Z}\setminus\{0\}} \qquad \text{or} \qquad s^{-1}t \in -q^{\mathbb{Z}\setminus\{0\}}\,.$$

Using the notation of the q-rays (1), the above set is given by

$$-q\Sigma_{0,1} \cup -q^{-1}\Sigma_{0,-1} \cup -q\Sigma_{-1,1} \cup -q^{-1}\Sigma_{1,-1}\,.$$

Thus, the above integral is a meromorphic function which is regular on the complement of the set $\Sigma_{\mathcal{T}}$. Note that some of the points of the set (81) might be regular points of the integral, this happens for instance when the residue at simple poles vanishes. Note also that the set (81) is disjoint from the real torus $|s| = |t| = 1$, so the integral can be expanded into Laurent series convergent on the real torus $|s| = |t| = 1$. However, the prefactor $G_q(t)G_q(t^{-1})$ has singularities on the set $t \in q^{\mathbb{Z}}$ (and those are actual, i.e., not removable), which prevent the meromorphic function $I_{\mathcal{T},s,t}(q)$ from being expanded into Laurent series on the torus $|s| = |t| = 1$. In conclusion, the meromorphic function $I_{\mathcal{T},s,t}(q)$ is regular in the complement of the shifted q-rays

$$(82) \qquad \Sigma_{0,1} \cup \Sigma_{0,-1} \cup -q\Sigma_{0,1} \cup -q^{-1}\Sigma_{0,-1} - q\Sigma_{-1,1} \cup -q^{-1}\Sigma_{1,-1}\,.$$

4.2. **The 4_1 knot.** Next, we discuss the example of the 4_1 knot, giving the first and last steps of the above computations, and asking the reader to fill in the intermediate steps.

The gluing equations, in `SnapPy` format and with the `Regina` ordering of the edges, are given by:

$$\begin{pmatrix} 2 & 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 & 2 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 1 & -1 & -3 \end{pmatrix} .$$

The above matrix determines the following information. The $\overline{A}$, $\overline{B}$ and $\overline{C}$ matrices are given by:

$$\overline{A} = \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}, \qquad \overline{B} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \overline{C} = \begin{pmatrix} 0 & 0 \\ 2 & 2 \end{pmatrix} .$$

The holonomy of the meridian and the longitude are given by:

$$\mu = \alpha_0 - \gamma_1, \qquad \lambda = \alpha_0 + \beta_0 + \gamma_0 + \alpha_1 - \beta_1 - 3\gamma_1 .$$

The ε variables are given by:

$$\varepsilon_0 = 2\alpha_0 + \beta_0 + 2\alpha_1 + \beta_1 - 2\pi, \qquad \varepsilon_1 = \beta_0 + 2\gamma_0 + \beta_1 + 2\gamma_1 - 2\pi .$$

Using $\alpha_i + \beta_i + \gamma_i = \pi$ for $i = 0, 1$, we can solve the above equations in terms of the variables $\alpha_0, \varepsilon_0, \mu, \lambda$:

$$\begin{aligned} \alpha_0 &= \alpha_0 & \alpha_1 &= \alpha_0 + \lambda/2 - \mu \\ \beta_0 &= \pi - 2\alpha_0 + \varepsilon_0 - \lambda/2 & \beta_1 &= \pi - 2\alpha_0 - \lambda/2 + 2\mu \\ \gamma_0 &= \alpha_0 - \varepsilon_0 + \lambda/2 & \gamma_1 &= \alpha_0 - \mu . \end{aligned} \tag{83}$$

Let θ be a positive semi-angle structure. Rescaling $x_0 \to x_0/x_1$, applying the change of variables

$$(-q)^{-\frac{\alpha_0}{\pi}} x_0 = y_0 , \tag{84}$$

letting $(s, t) = (e_\mu, e_\lambda)$, and following the steps of the previous example we obtain that the state-integral is given by

$$\begin{aligned} I^{\mathrm{pre}}_{\mathcal{T}_{4_1},\theta}(q) = \frac{c(q)^2}{2\pi \mathsf{i}} \int_{C_\alpha} \frac{dy_0}{y_0} & G_q\left(y_0^{-1}\right) G_q\left(s^{-1}y_0^{-1}\right) G_q\left((-q)^{-\varepsilon_0} t y_0^{-1}\right) G_q\left(s^{-1} t y_0^{-1}\right) \\ & \times G_q\left(-q(-q)^{\varepsilon_0} t^{-1} y_0^2\right) G_q\left(-q s^2 t^{-1} y_0^2\right) \end{aligned} \tag{85}$$

where the contour of integration C_α (determined by (84)) is given by $|y_0| = |q^{-\frac{\alpha_0}{\pi}}|$. Moreover, we can set $\varepsilon_0 = 0$, and obtain the meromorphic function of the 4_1 knot:

$$\begin{aligned} I_{\mathcal{T}_{4_1},s,t}(q) = \frac{c(q)^2}{2\pi \mathsf{i}} \int_{C} \frac{dy_0}{y_0} & G_q\left(y_0^{-1}\right) G_q\left(s^{-1}y_0^{-1}\right) G_q\left(t y_0^{-1}\right) G_q\left(s^{-1} t y_0^{-1}\right) \\ & \times G_q\left(-q t^{-1} y_0^2\right) G_q\left(-q s^2 t^{-1} y_0^2\right) \end{aligned} \tag{86}$$

where C is the torus $|y_0| = 1^+$.

4.3. **The sister of the 4_1 knot.** Next, we present the invariant for the sister `m003` of the 4_1 knot. The gluing equations, in `SnapPy` format and with the `Regina` ordering of the edges, are given by:

$$\begin{pmatrix} 2 & 0 & 1 & 2 & 0 & 1 \\ 0 & 2 & 1 & 0 & 2 & 1 \\ 0 & -2 & 0 & 2 & 0 & 0 \\ 0 & -1 & 0 & 2 & -1 & 0 \end{pmatrix} .$$

Using the above matrix, we can compute the $\overline{A}$, $\overline{B}$, $\overline{C}$ matrices, the holonomy of the meridian and longitude, the ε variables, and express all variables in terms of the variables $\alpha_0, \varepsilon_0, \mu, \lambda$:

$$\begin{aligned} \alpha_0 &= \alpha_0 & \alpha_1 &= \alpha_0 - \varepsilon_0 + \lambda \\ \beta_0 &= \alpha_0 - \varepsilon_0 + \lambda - \mu/2 & \beta_1 &= \alpha_0 - \varepsilon_0 + \mu/2 \\ \gamma_0 &= \pi - 2\alpha_0 + \varepsilon_0 - \lambda + \mu/2 & \gamma_1 &= \pi - 2\alpha_0 + 2\varepsilon_0 - \lambda - \mu/2 \,. \end{aligned} \tag{87}$$

Let θ be a positive semi-angle structure. Rescaling $x_0 \to x_0/x_1$, applying the change of variables

$$(-q)^{\frac{\alpha_0}{\pi}} x_0 = y_0 \,, \tag{88}$$

letting $(s,t) = (e_\mu, e_\lambda)$, and following the steps of the previous example we obtain that the state-integral is given by

$$\begin{aligned} I^{\mathrm{pre}}_{\mathcal{T}_{m003},\theta}(q) = \frac{c(q)^2}{2\pi \mathsf{i}} \int_{C_\alpha} \frac{dy_0}{y_0} & G_q\left((-q)^{1+2\varepsilon_0} s^{-1/2} t^{-2} y_0^{-2}\right) G_q\left((-q)^{1+\varepsilon_0} s^{1/2} t^{-2} y_0^{-2}\right) G_q\left(y_0\right) \\ & \times G_q\left((-q)^{-\varepsilon_0} s^{1/2} y_0\right) G_q\left((-q)^{-\varepsilon_0} t^2 y_0\right) G_q\left((-q)^{-\varepsilon_0} s^{-1/2} t^2 y_0\right) \,. \end{aligned} \tag{89}$$

where the contour of integration C_α (determined by (88)) is given by $|y_0| = |(-q)^{\frac{\alpha_0}{\pi}}|$. Moreover, we can set $\varepsilon_0 = 0$, and obtain the meromorphic function of the sister of the 4_1 knot:

$$\begin{aligned} I_{\mathcal{T}_{m003},s,t}(q) = \frac{c(q)^2}{2\pi \mathsf{i}} \int_{C} \frac{dy_0}{y_0} & G_q\left(-q s^{-1/2} t^{-2} y_0^{-2}\right) G_q\left(-q s^{1/2} t^{-2} y_0^{-2}\right) G_q\left(y_0\right) G_q\left(s^{1/2} y_0\right) \\ & \times G_q\left(t^2 y_0\right) G_q\left(s^{-1/2} t^2 y_0\right) \,. \end{aligned} \tag{90}$$

where C is the torus $|y_0| = 1^+$.

4.4. **The unknot.** Next, we compute the invariant for the unknot, and find a surprise: we can compute the integral exactly. The unknot has three triangulations with two tetrahedra given in Table 3 of [GHHR16]. One of them has isometry signature `cMcabbgds`. The gluing equations, in `SnapPy` format and with the `Regina` ordering of the edges, are given by:

$$\begin{pmatrix} 1 & 2 & 2 & 2 & 2 & 2 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \\ 1 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} .$$

Following the steps of the previous examples, we have:

$$\varepsilon_0 = \alpha_0 + 2\beta_0 + 2\gamma_0 + 2\alpha_1 + 2\beta_1 + 2\gamma_1 - 2\pi, \qquad \varepsilon_1 = \alpha_0 - 2\pi \tag{91}$$

which simplifies to $\varepsilon_0 = -\alpha_0 + 2\pi$ and in particular is bigger than π when θ is positive. We can express all variables in terms of the variables $\alpha_0, \varepsilon_0, \mu, \lambda$:

$$\begin{aligned} \alpha_0 &= 2\pi - \varepsilon_0 & \alpha_1 &= -\pi + \varepsilon_0 + \lambda + \mu/2 \\ \beta_0 &= \beta_0 & \beta_1 &= 2\pi - \varepsilon_0 - \lambda \\ \gamma_0 &= -\pi - \beta_0 + \varepsilon_0 & \gamma_1 &= -\mu/2\,. \end{aligned} \tag{92}$$

After rescaling $x_0 \to x_0/x_1$, applying the change of variables

$$(-q)^{\frac{-\beta_0}{\pi}} x_0 = y_0\,, \tag{93}$$

letting $(s,t) = (e_\mu, e_\lambda)$, and following the steps of the previous example we obtain that the state-integral is given by

$$\begin{aligned} I^{\mathrm{pre}}_{\mathcal{T}_{\mathrm{unknot}},\theta}(q) = \frac{c(q)^2}{2\pi\mathsf{i}} & G_q\left((-q)^{2-\varepsilon_0}\right) G_q\left(s^{-1/2}\right) G_q\left((-q)^{2-\varepsilon_0}t^{-2}\right) G_q\left((-q)^{-1+\varepsilon_0}s^{1/2}t^2\right) \\ &\times \int_{C_\alpha} \frac{dy_0}{y_0} G_q\left(-q^{-1}y_0\right) G_q\left(y_0^{-1}\right)\,. \end{aligned} \tag{94}$$

where C_α is the torus $|y_0| = |(-q)^{-\frac{\beta_0}{\pi}}|$. On the other hand, Equations (7) and (8) give that

$$G_q\left(-q^{-1}y_0\right) G_q\left(y_0^{-1}\right) = \frac{1}{(1+y_0)(1+q^{-1}y_0)}\,,$$

so that

$$\begin{aligned} I^{\mathrm{bal}}_{\mathcal{T}_{\mathrm{unknot}},\theta}(q) = \frac{c(q)^2}{2\pi\mathsf{i}} & G_q\left((-q)^{2-\varepsilon_0}\right) G_q\left(s^{-1/2}\right) G_q\left((-q)^{2-\varepsilon_0}t^{-2}\right) G_q\left((-q)^{-1+\varepsilon_0}s^{1/2}t^2\right) \\ &\times \int_{C_\alpha} \frac{dy_0}{y_0} \frac{1}{(1+y_0)(1+q^{-1}y_0)}\,. \end{aligned} \tag{95}$$

When θ is positive, the contour C_α encircles both singularities -1 and $-q$ of the integrand, and a residue calculation reveals that the integral is zero. It follows that

$$I_{\mathcal{T}_{\mathrm{unknot}},s,t}(q) = 0\,. \tag{96}$$

4.5. **The trefoil.** Next, we compute the invariant for the trefoil. The gluing equations, in `SnapPy` format and with the `Regina` ordering of the edges, are given by:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 2 & 2 & 2 & 1 & 2 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & -4 & 4 & -1 & 0 \end{pmatrix}.$$

The state-integral is given by

$$(97) \qquad I^{\mathrm{pre}}_{\mathcal{T}_{3_1},\theta}(q) = \frac{c(q)^2}{2\pi\mathsf{i}} G_q\left((-q)^{1+\varepsilon_0/2}s^2t^{-1}\right) G_q\left((-q)^{1+\varepsilon_0/2}s^{-2}t\right) \times \int_{C_\alpha} \frac{dy_0}{y_0} G_q\left(y_0^{-1}\right) G_q\left(s^{-1}y_0^{-1}\right) G_q\left((-q)^{-\varepsilon_0/2}s^3t^{-1}y_0\right) G_q\left((-q)^{-\varepsilon_0/2}s^{-2}ty_0\right)$$

where the contour of integration C_α is given by $|y_0| = |(-q)^{\frac{-\alpha_1}{\pi}}|$. Moreover, we can set $\varepsilon_0 = 0$ and obtain the meromorphic function given by:

$$(98) \qquad I_{\mathcal{T}_{3_1},s,t}(q) = \frac{c(q)^2}{2\pi\mathsf{i}} G_q\left(-qs^2t^{-1}\right) G_q\left(-qs^{-2}t\right) \times \int_{C} \frac{dy_0}{y_0} G_q\left(y_0^{-1}\right) G_q\left(s^{-1}y_0^{-1}\right) G_q\left(s^3t^{-1}y_0\right) G_q\left(s^{-2}ty_0\right)$$

where C is the torus $|y_0| = 1^+$.

4.6. **The 5_2 knot.** Next, we present the invariant of the 5_2 knot. The gluing equations, in `SnapPy` format (we use the homological longitude and the `Regina` ordering of the edges), are given by:

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 2 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 2 & 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & -3 & 1 & 0 & -2 & 0 & 0 & 1 \end{pmatrix}.$$

After rescaling $x_i \to x_i/x_2$ and a change of variables $x_0 = (-q)^{\alpha_0}y_0$ and $x_1 = (-q)^{-\alpha_1/2}s^{1/2}t^{1/2}y_1$, the state-integral is given by

$$(99) \quad I^{\mathrm{pre}}_{\mathcal{T}_{5_2},\theta}(q) = \frac{c(q)^3}{(2\pi\mathsf{i})^2} \int_{C_\alpha} \frac{dy_0dy_1}{y_0y_1} G_q\left(y_0^{-1}\right) G_q\left(sy_0^{-1}\right) G_q\left((-q)^{-\varepsilon_1}s^2y_0^{-1}\right) \times G_q\left(-qs^{-2}t^{-1}t^{-1}y_0y_1^{-2}\right) G_q\left((-q)^{1+\varepsilon_0/4}y_0y_1^{-1}\right) G_q\left((-q)^{1+3\varepsilon_0/4+\varepsilon_1}s^{-3}t^{-1}y_0y_1^{-1}\right) \times G_q\left((-q)^{-\varepsilon_0/4}y_1\right) G_q\left((-q)^{-3\varepsilon_0/4}sty_1\right) G_q\left(sty_1^2\right)$$

where C_α is the torus given by $|y_0| = |(-q)^{\frac{-\alpha_0}{\pi}}|$ and $|y_1s^{1/2}t^{1/2}| = |(-q)^{\frac{\alpha_1}{2\pi}}|$. Moreover, we can set $\varepsilon_0 = \varepsilon_1 = 0$ and obtain the meromorphic function given by:

$$(100) \qquad I_{\mathcal{T}_{5_2},s,t}(q) = \frac{c(q)^3}{(2\pi\mathsf{i})^2} \int_{C} \frac{dy_0dy_1}{y_0y_1} G_q\left(y_0^{-1}\right) G_q\left(sy_0^{-1}\right) G_q\left(s^2y_0^{-1}\right) G_q\left(-qs^{-2}t^{-1}t^{-1}y_0y_1^{-2}\right) \times G_q\left(-qy_0y_1^{-1}\right) G_q\left(-qs^{-3}t^{-1}y_0y_1^{-1}\right) G_q\left(y_1\right) G_q\left(sty_1\right) G_q\left(sty_1^2\right)$$

where C is the torus given by $|y_0| = 1^+$ and $|y_1s^{1/2}t^{1/2}| = 1^-$.

4.7. **The 6_1 knot.** Finally, we present the invariant of the 6_1 knot. The gluing equations, in `SnapPy` format (we use the homological longitude and the `Regina` ordering of the edges), are given by:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 2 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 2 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & -2 \end{pmatrix}.$$

After rescaling $x_i \to x_i/x_3$ and a change of variables

$$x_0 = (-q)^{\alpha_2} y_0, \qquad x_1 = (-q)^{\alpha_0} y_1, \qquad x_2 = (-q)^{-\alpha_1/2} t^{1/2} y_2 \tag{101}$$

the state-integral is given by

$$\begin{aligned} I^{\mathrm{pre}}_{\mathcal{T}_{6_1},\theta}(q) = \frac{c(q)^4}{(2\pi \mathsf{i})^3} \int_{C_\alpha} & \frac{dy_0 dy_1 dy_2}{y_0 y_1 y_2} G_q\left(y_0^{-1}\right) G_q\left(y_1^{-1}\right) G_q\left(s y_1^{-1}\right) G_q\left((-q)^{\varepsilon_2} s^{-2} y_0^{-1} y_1\right) \\ & \times G_q\left(-q s^{-1} t^{-1} y_1 y_2^{-2}\right) G_q\left((-q)^{1/2-\varepsilon_0/4+\varepsilon_1/4} t^{-1} y_2^{-1}\right) G_q\left((-q)^{3/2+\varepsilon_0/4+3\varepsilon_1/4} y_0 y_2^{-1}\right) \\ & \times G_q\left((-q)^{3/2+3\varepsilon_0/4+\varepsilon_1/4} t^{-1} y_0 y_1 y_2^{-1}\right) G_q\left((-q)^{-1/2-3\varepsilon_0/4-\varepsilon_1/4} t y_1^{-1} y_2\right) \\ & \times G_q\left((-q)^{1/2+\varepsilon_0/4-\varepsilon_1/4-\varepsilon_2} s^2 t y_0 y_1^{-1} y_2\right) G_q\left((-q)^{-1/2-\varepsilon_0/4-3\varepsilon_1/4} y_0^{-1} y_1 y_2\right) G_q\left(-q t y_2^2\right) \end{aligned} \tag{102}$$

where C_α is the torus given by $|y_0| = |(-q)^{\frac{-\alpha_2}{\pi}}|$, $|y_1| = |(-q)^{\frac{-\alpha_0}{\pi}}|$ and $|y_2 t^{1/2}| = |(-q)^{\frac{\alpha_1}{2\pi}}|$. Moreover, we can set $\varepsilon_0 = \varepsilon_1 = \varepsilon_2 = 0$ and obtain the meromorphic function given by:

$$\begin{aligned} I_{\mathcal{T}_{6_1},s,t}(q) = \frac{c(q)^4}{(2\pi \mathsf{i})^3} \int_{C_\alpha} & \frac{dy_0 dy_1 dy_2}{y_0 y_1 y_2} G_q\left(y_0^{-1}\right) G_q\left(y_1^{-1}\right) G_q\left(s y_1^{-1}\right) G_q\left(s^{-2} y_0^{-1} y_1\right) \\ & \times G_q\left(-q s^{-1} t^{-1} y_1 y_2^{-2}\right) G_q\left((-q)^{1/2} t^{-1} y_2^{-1}\right) G_q\left((-q)^{3/2} y_0 y_2^{-1}\right) \\ & \times G_q\left((-q)^{3/2} t^{-1} y_0 y_1 y_2^{-1}\right) G_q\left((-q)^{-1/2} t y_1^{-1} y_2\right) \\ & \times G_q\left((-q)^{1/2} s^2 t y_0 y_1^{-1} y_2\right) G_q\left((-q)^{-1/2} y_0^{-1} y_1 y_2\right) G_q\left(-q t y_2^2\right) \end{aligned} \tag{103}$$

where C is the torus given by $|y_0| = 1^+$, $|y_1| = 1^+$ and $|y_2 t^{1/2}| = 1^-$.

Acknowledgements. The authors wish to thank R. Siejakowski for a careful reading of the manuscript and for pointing out typographical errors and A. Mellit for enlightening discussions. S.G. was supported in part by grant DMS-14-06419 of the US National Science Foundation. R.K. was supported in part by the Swiss National Science Foundation and the center of excellence grant "Center for quantum geometry of Moduli Spaces" of the Danish National Research Foundation. The paper was presented by S.G. in a conference at the Max-Planck Institute in Bonn (Modular Forms are Everywhere) in honor of D. Zagier's 65th birthday. S.G. wishes to thank the organizers, and particularly K. Bringmann and D. Zagier for their invitation and hospitality.

Appendix A. A quantum dilogarithm over the LCA group $\mathbb{T} \times \mathbb{Z}$

The function (19) with q real has been introduced and studied in the functional analytic context of Hilbert spaces and quantum $E(2)$ group by Woronowicz in [Wor92]. In this section, we derive some of its operator properties by using the theory of quantum dilogarithms over Pontryagin self-dual LCA groups developed in [AKa, Kasb].

Throughout the section, for a Hilbert space H, we write $A\colon H \to H$, if A is a not necessarily bounded linear operator in H whose domain is dense in H. The Hermitian conjugate of A will be denoted A^*. Below, we will use freely the standard Dirac's bra-ket notation.

The group $\mathbb{T} \times \mathbb{Z}$ is a self-dual LCA group with the *gaussian exponential*

$$\langle \cdot \rangle\colon \mathbb{T} \times \mathbb{Z} \to \mathbb{T}, \quad \langle z, m\rangle = z^m, \quad \text{for all} \quad (z,m) \in \mathbb{T} \times \mathbb{Z}. \tag{104}$$

The *Fourier kernel* is fixed as the co-boundary of the gaussian exponential

$$\langle z, m; w, n\rangle := \frac{\langle zw, m+n\rangle}{\langle z, m\rangle \langle w, n\rangle} = z^n w^m. \tag{105}$$

We define a unitary *Fourier operator* $\mathsf{F}\colon L^2(\mathbb{T} \times \mathbb{Z}) \to L^2(\mathbb{T} \times \mathbb{Z})$ by the integral kernel

$$\langle z, m|\mathsf{F}|w, n\rangle = \langle z, m; w, n\rangle. \tag{106}$$

For any (measurable) function $f\colon \mathbb{T} \times \mathbb{Z} \to \mathbb{C}$, we associate three normal operators as follows. The multiplication operator by f:

$$f(\mathsf{q})\colon L^2(\mathbb{T} \times \mathbb{Z}) \to L^2(\mathbb{T} \times \mathbb{Z}), \quad \langle z, m|f(\mathsf{q}) = f(z,m)\langle z, m|, \tag{107}$$

its unitary conjugate by the Fourier operator

$$f(\mathsf{p}) := \mathsf{F} f(\mathsf{q}) \mathsf{F}^*, \tag{108}$$

and the unitary conjugate of the latter by the inverse of the (unitary) multiplication operator $\langle \mathsf{q}\rangle$ by the gaussian exponential (104):

$$f(\mathsf{p}+\mathsf{q}) := \langle \mathsf{q}\rangle^* f(\mathsf{p}) \langle \mathsf{q}\rangle. \tag{109}$$

We remark that all three operators $f(\mathsf{q})$, $f(\mathsf{p})$ and $f(\mathsf{p}+\mathsf{q})$ have spectrum given by the closure of the image of f.

Lemma A.1. The function

$$\eta\colon \mathbb{T} \times \mathbb{Z} \to \mathbb{C}_{\neq 0}, \quad (z,m) \mapsto -zq^{1-m}, \tag{110}$$

satisfies the following operator equations:

$$\eta(\mathsf{q})\eta(\mathsf{p}) = q^2\eta(\mathsf{p})\eta(\mathsf{q}), \quad \eta(\mathsf{q})^*\eta(\mathsf{p}) = \eta(\mathsf{p})\eta(\mathsf{q})^*, \quad \eta(\mathsf{p}+\mathsf{q}) = -\eta(\mathsf{p})\eta(\mathsf{q}). \tag{111}$$

Proof. In the Hilbert space $L^2(\mathbb{Z})$, define a self-adjoint operator h by

$$\langle m|\mathsf{h} = m\langle m|, \quad \text{for all} \quad m \in \mathbb{Z}, \tag{112}$$

and a unitary operator z by

$$\langle m|\mathsf{z} = \langle m-1|, \quad \text{for all} \quad m \in \mathbb{Z}. \tag{113}$$

These operators satisfy the commutation relation

$$[\mathsf{h},\mathsf{z}] := \mathsf{hz} - \mathsf{zh} = \mathsf{z} \tag{114}$$

which is verified as follows:

$$\langle m|[\mathsf{h},\mathsf{z}] = \langle m|\mathsf{hz} - \langle m|\mathsf{zh} = m\langle m-1| - (m-1)\langle m-1| = \langle m|\mathsf{z}. \tag{115}$$

Similarly, in the Hilbert space $L^2(\mathbb{T})$, we define a self-adjoint operator H by

$$\langle z|\mathsf{H} = z\frac{\partial}{\partial z}\langle z|, \quad \text{for all} \quad z \in \mathbb{T}, \tag{116}$$

and a unitary operator Z by

$$\langle z|\mathsf{Z} = z\langle z|, \quad \text{for all} \quad z \in \mathbb{T}. \tag{117}$$

It is easily verified that

$$[\mathsf{H},\mathsf{Z}] := \mathsf{HZ} - \mathsf{ZH} = \mathsf{Z}. \tag{118}$$

Moreover, if $\mathsf{J}\colon L^2(\mathbb{Z}) \to L^2(\mathbb{T})$ is the isomorphism defined by the integral kernel

$$\langle z|\mathsf{J}|m\rangle = z^m, \quad \text{for all} \quad (z,m) \in \mathbb{T}\times\mathbb{Z}, \tag{119}$$

then we have the equalities

$$\mathsf{Jh} = \mathsf{HJ}, \quad \mathsf{Jz} = \mathsf{ZJ}. \tag{120}$$

Identifying $\mathsf{H},\mathsf{Z},\mathsf{h},\mathsf{z}$ with their natural counterparts in $L^2(\mathbb{T}\times\mathbb{Z})$, we have

$$\mathsf{Fh} = \mathsf{HF}, \quad \mathsf{Fz} = \mathsf{ZF}, \quad \mathsf{FH} = -\mathsf{hF}, \quad \mathsf{FZ} = \mathsf{z}^{-1}\mathsf{F}, \tag{121}$$

and

$$\mathsf{h}\langle\mathsf{q}\rangle = \langle\mathsf{q}\rangle\mathsf{h}, \quad \mathsf{Z}\langle\mathsf{q}\rangle = \langle\mathsf{q}\rangle\mathsf{Z}, \quad \mathsf{H}\langle\mathsf{q}\rangle = \langle\mathsf{q}\rangle(\mathsf{H}+\mathsf{h}), \quad \mathsf{z}\langle\mathsf{q}\rangle = \langle\mathsf{q}\rangle\mathsf{z}\mathsf{Z}^{-1} \tag{122}$$

so that

$$f(\mathsf{q}) = f(\mathsf{Z},\mathsf{h}), \quad f(\mathsf{p}) = f(\mathsf{z}^{-1},\mathsf{H}), \quad f(\mathsf{p}+\mathsf{q}) = f(\mathsf{z}^{-1}\mathsf{Z},\mathsf{H}+\mathsf{h}) \tag{123}$$

for any $f\colon \mathbb{T}\times\mathbb{Z} \to \mathbb{C}$, where, in the right hand sides, the functions with operator arguments are understood in the spectral sense. In the case of $f = \eta$, we thus have

$$\eta(\mathsf{q}) = -\mathsf{Z}q^{1-\mathsf{h}}, \quad \eta(\mathsf{p}) = -\mathsf{z}^{-1}q^{1-\mathsf{H}}, \quad \eta(\mathsf{p}+\mathsf{q}) = -\mathsf{z}^{-1}\mathsf{Z}q^{1-\mathsf{H}-\mathsf{h}} \tag{124}$$

and the relations (111) are verified straightforwardly. □

Next, observe that the function

$$\mu\colon \mathbb{T}\times\mathbb{Z} \to \mathbb{C}, \quad (z,m) \mapsto \left(\eta(z,m); q^2\right)_\infty, \tag{125}$$

nowhere vanishes and satisfies the operator five term identity

$$\mu(\mathsf{p})\mu(\mathsf{q}) = \mu(\mathsf{q})\mu(\mathsf{p}+\mathsf{q})\mu(\mathsf{p}) \tag{126}$$

as a consequence of Lemma A.1 and the formal power series identity in non-commuting indeterminates

$$\left(\mathsf{v}; q^2\right)_\infty \left(\mathsf{u}; q^2\right)_\infty = \left(\mathsf{u}; q^2\right)_\infty \left(-\mathsf{vu}; q^2\right)_\infty \left(\mathsf{v}; q^2\right)_\infty, \quad \mathsf{uv} = q^2\mathsf{vu}, \tag{127}$$

which is equivalent to the q-binomial formula

$$\sum_{n=0}^{\infty} \frac{(a;q^2)_n}{(q^2;q^2)_n} z^n = \frac{(az;q^2)_\infty}{(z;q^2)_\infty}, \tag{128}$$

see [Kasa]. Indeed, by substituting u by $\eta(\mathsf{q})$ and v by $\eta(\mathsf{p})$, we convert (127) into (126). Lemma A.1 also implies that the function

$$\phi_q \colon \mathbb{T}\times\mathbb{Z} \to \mathbb{T}, \quad (z,m) \mapsto \mu(z,m)/\mu(1/z,m) = \mu(z,m)/\overline{\mu(z,m)}, \tag{129}$$

satisfies the unitarized version of identity (126):

$$\phi_q(\mathsf{p})\phi_q(\mathsf{q}) = \phi_q(\mathsf{q})\phi_q(\mathsf{p}+\mathsf{q})\phi_q(\mathsf{p}). \tag{130}$$

Moreover, ϕ_q satisfies an inversion relation, see below (131), which allows us to identify it as an example of a quantum dilogarithm over $\mathbb{T}\times\mathbb{Z}$.

Lemma A.2. The function (129) satisfies the following inversion relation

$$\phi_q(z,m)\phi_q(1/z,-m) = z^m = \langle z,m\rangle, \quad \text{for all} \quad (z,m)\in\mathbb{T}\times\mathbb{Z}. \tag{131}$$

Proof. We have

$$\begin{aligned}\phi_q(z,m)\phi_q(1/z,-m) &= \frac{(-q^{1-m}z;q^2)_\infty\,(-q^{1+m}/z;q^2)_\infty}{(-q^{1-m}/z;q^2)_\infty\,(-q^{1+m}z;q^2)_\infty} \\ &= \frac{\theta_q(zq^{-m})}{\theta_q(zq^m)} = \frac{\theta_q(q^{-2m}zq^m)}{\theta_q(zq^m)} = q^{-m^2}\,(zq^m)^m = z^m\end{aligned} \tag{132}$$

where in the second equality we have used (26) and in the forth equality the functional equation (32). $\square$

Appendix B. The quantum dilogarithm and the Beta pentagon relation

In this section, we generalize the result of [AKa] to include the LCA groups which do not admit division by 2. Note that most of the equations of this section (for instance, (137), (140), (145), (149)) are valid as distributions.

Let A be a self-dual LCA group with a gaussian exponential $\langle\cdot\rangle\colon A\to\mathbb{T}$ and the Fourier kernel

$$\langle x;y\rangle = \frac{\langle x+y\rangle}{\langle x\rangle\langle y\rangle}, \quad \text{for all} \quad (x,y)\in A^2. \tag{133}$$

For a non-negative integer $n\in\mathbb{N}$, denote

$$[n] := \mathbb{Z}_{\geq 0}\cap\mathbb{Z}_{\leq n}, \quad \text{for all} \quad n\in\mathbb{Z}_{\geq 0}\,. \tag{134}$$

According to [AKa], a bounded function

$$f\colon [4]\times A\to\mathbb{C}, \quad (i,x)\mapsto f_i(x), \tag{135}$$

is called of *Faddeev type* if it satisfies the non-constant version of the operator pentagon relation

$$f_1(\mathsf{p})f_3(\mathsf{q}) = f_4(\mathsf{q})f_2(\mathsf{p}+\mathsf{q})f_0(\mathsf{p})\,. \tag{136}$$

The latter is equivalent to the functional integral identity

$$\tilde{f}_1(x)\tilde{f}_3(u)\langle x;u\rangle = \int_A \tilde{f}_4(u-z)\tilde{f}_2(z)\tilde{f}_0(x-z)\langle z\rangle\,\mathrm{d}z,\quad \text{for all}\quad (x,u)\in A^2, \tag{137}$$

where

$$\tilde{f}_i(x) := (\mathsf{F}^{-1}f_i)(x) = \int_A \langle x;-y\rangle f_i(y)\,\mathrm{d}y. \tag{138}$$

Defining

$$\hat{f}_i(x) := \tilde{f}_i(x)\langle x\rangle, \tag{139}$$

we rewrite (137) in the form

$$\hat{f}_1(x)\hat{f}_3(u) = \int_A \langle x-z;z-u\rangle \hat{f}_4(u-z)\hat{f}_2(z)\hat{f}_0(x-z)\,\mathrm{d}z,\quad \text{for all}\quad (x,u)\in A^2, \tag{140}$$

We call a subgroup $B\subset A$ *isotropic* if it satisfies the condition

$$\langle b;b'\rangle = 1,\quad \text{for all}\quad (b,b')\in B^2. \tag{141}$$

Lemma B.1. Let a bi-character $\chi\colon A^2\to\mathbb{T}$ be such that

$$\langle x;y\rangle = \chi(x,y)\chi(y,x),\quad \text{for all}\quad (x,y)\in A^2, \tag{142}$$

and $B\subset A$ an isotropic subgroup. Then, for any function $f_i(x)$ of Faddeev type, the function

$$g\colon [4]\times A^2\to\mathbb{C},\quad (i,x,y)\mapsto g_i(x,y) := \chi(x,y)\int_B \hat{f}_i(x+b)\langle b;y\rangle\,\mathrm{d}b \tag{143}$$

is automorphic, i.e., satisfies

$$g_i(x+b,y) = \chi(-y,b)g_i(x,y),\quad g_i(x,y+b) = \chi(x,b)g_i(x,y),\qquad \text{for all}\quad (i,b,x,y)\in[4]\times B\times A^2, \tag{144}$$

and satisfies the integral identity

$$g_1(x,y)g_3(u,v) = \int_{A/B} g_4(u-z,v+z-x)g_2(z,y+v)g_0(x-z,y+z-u)\,\mathrm{d}z \tag{145}$$

Proof. The automorphicity properties (144) are verified in a straightforward manner, while to derive the integral identity (145), we write

$$
\begin{aligned}
(146)\quad & \frac{g_1(x,y)g_3(u,v)}{\chi(x,y)\chi(u,v)} \\
&= \int_{A\times B^2} \langle x+b-z; z-u-c\rangle\langle b;y\rangle\langle c;v\rangle \hat{f}_4(u+c-z)\hat{f}_2(z)\hat{f}_0(x+b-z)\,\mathrm{d}(z,b,c) \\
&= \int_{A\times B^2} \langle x-z; z-u\rangle\langle b;y+z-u\rangle\langle c;v+z-x\rangle \hat{f}_4(u+c-z)\hat{f}_2(z)\hat{f}_0(x+b-z)\,\mathrm{d}(z,b,c) \\
&= \int_A \frac{\langle x-z;z-u\rangle g_4(u-z,v+z-x)\hat{f}_2(z)g_0(x-z,y+z-u)}{\chi(u-z,v+z-x)\chi(x-z,y+z-u)}\,\mathrm{d}z \\
&= \int_A \frac{g_4(u-z,v+z-x)\hat{f}_2(z)g_0(x-z,y+z-u)}{\chi(u-z,v)\chi(x-z,y)}\,\mathrm{d}z
\end{aligned}
$$

so that

$$
\begin{aligned}
(147)\quad & g_1(x,y)g_3(u,v) = \int_A \chi(z,y+v)g_4(u-z,v+z-x)\hat{f}_2(z)g_0(x-z,y+z-u)\,\mathrm{d}z \\
&= \int_{(A/B)\times B} \chi(z+b,y+v)g_4(u-z-b,v+z+b-x)\hat{f}_2(z+b)g_0(x-z-b,y+z+b-u)\,\mathrm{d}(z,b) \\
&= \int_{(A/B)\times B} \chi(z+b,y+v)\chi(v+y,b)g_4(u-z,v+z-x)\hat{f}_2(z+b)g_0(x-z,y+z-u)\,\mathrm{d}(z,b) \\
&= \int_{A/B} g_4(u-z,v+z-x)g_2(z,y+v)g_0(x-z,y+z-u)\,\mathrm{d}z
\end{aligned}
$$

□

Now, we point out that the integral identity (145) is an equivalent form of the automorphic Beta pentagon identity. Namely, if we define

$$
(148)\qquad \phi_i(x,y) := g_i(-y,x+y) \quad\Leftrightarrow\quad g_i(x,y) = \phi_i(x+y,-x),
$$

then

$$
(149)\quad \phi_1(x,y)\phi_3(u,v) = \int_{A/B} \phi_4(u+y,v-z)\phi_2(x+y+u+v-z,z)\phi_0(x+v,y-z)\,\mathrm{d}z\,.
$$

In the case of the quantum dilogarithm ϕ_q constructed in Appendix A, we have

$$
(150)\qquad A = \mathbb{T}\times\mathbb{Z}, \quad \langle z,m\rangle = z^m, \quad \langle z,m;w,n\rangle = z^n w^m
$$

so that

$$
\begin{aligned}
\hat{\phi}_q(z,m) &= z^m \int_{\mathbb{T}\times\mathbb{Z}} \phi_q(t,k)\langle z,m;1/t,-k\rangle\,\mathrm{d}(t,k) = z^m\int_{\mathbb{T}\times\mathbb{Z}} \phi_q(t,k)z^{-k}t^{-m}\,\mathrm{d}(t,k) \\
&= z^m\int_{\mathbb{T}\times\mathbb{Z}} \psi(1/t,k)z^{-k}t^{-m}\,\mathrm{d}(t,k) = z^m\int_{\mathbb{T}} \psi^0(1/t,1/z)t^{-m}\frac{\mathrm{d}t}{2\pi i t}\,.
\end{aligned}
$$

We choose

$$B = \mathbb{Z} \subset \mathbb{T} \times \mathbb{Z} \tag{151}$$

and

$$\chi((z,m),(w,n)) = w^m, \tag{152}$$

so that the automorphic factors are trivial:

$$\chi((z,m),(1,n)) = 1, \quad \text{for all} \quad (z,m,n) \in \mathbb{T} \times \mathbb{Z}^2. \tag{153}$$

Thus,

$$\begin{aligned} g((z,m),(w,n)) &= \chi((z,m),(w,n)) \sum_{k\in\mathbb{Z}} \hat{\phi}_q(z,m+k)\langle 1,k;w,n\rangle \\ &= w^m \sum_{k\in\mathbb{Z}} \hat{\phi}_q(z,m+k) w^k = \sum_{k\in\mathbb{Z}} \hat{\phi}_q(z,k) w^k \\ &= \int_{\mathbb{T}} \sum_{k\in\mathbb{Z}} (zw/t)^k \psi^0(1/t,1/z) \frac{\mathrm{d}t}{2\pi i t} \\ &= \int_{\mathbb{T}} \delta_{\mathbb{T}}(zw/t) \psi^0(1/t,1/z) \frac{\mathrm{d}t}{2\pi i t} = \psi^0(1/(zw),1/z) \end{aligned}$$

and

$$\phi(((z,m),(w,n)) = g((1/w,-n),(zw,m+n)) = \psi^0(1/z,w). \tag{154}$$

Taking into account the symmetry of the Beta pentagon identity under the negation of all arguments, we conclude that $\psi^0(1/z,w)$ and $\psi^0(z,1/w)$ both satisfy the Beta pentagon identity.

References

[Ahl78] Lars Ahlfors, *Complex analysis*, third ed., McGraw-Hill Book Co., New York, 1978, An introduction to the theory of analytic functions of one complex variable, International Series in Pure and Applied Mathematics.

[AKa] Jørgen Ellegaard Andersen and Rinat Kashaev, *Complex Quantum Chern-Simons*, arXiv:1409.1208, Preprint 2014.

[AKb] ———, *A new formulation of the Teichmüller TQFT*, arXiv:1305.4291, Preprint 2013.

[AK14] ———, *A TQFT from Quantum Teichmüller theory*, Comm. Math. Phys. **330** (2014), no. 3, 887–934.

[Bur] Benjamin Burton, *Regina, software for 3-manifold topology and normal surface theory*, http://regina.sourceforge.net (30/01/2015).

[CDW] Marc Culler, Nathan Dunfield, and Jeffrey Weeks, *SnapPy, a computer program for studying the topology of 3-manifolds*, Available at http://snappy.computop.org (30/01/2015).

[DG13] Tudor Dimofte and Stavros Garoufalidis, *The quantum content of the gluing equations*, Geom. Topol. **17** (2013), no. 3, 1253–1315.

[DGG13] Tudor Dimofte, Davide Gaiotto, and Sergei Gukov, *3-manifolds and 3d indices*, Adv. Theor. Math. Phys. **17** (2013), no. 5, 975–1076.

[DGG14] ———, *Gauge theories labelled by three-manifolds*, Comm. Math. Phys. **325** (2014), no. 2, 367–419.

[Gar16] Stavros Garoufalidis, *The 3D index of an ideal triangulation and angle structures*, Ramanujan J. **40** (2016), no. 3, 573–604.

[GHHR16] Stavros Garoufalidis, Craig Hodgson, Neil Hoffman, and Hyam Rubinstein, *The 3D-index and normal surfaces*, Illinois J. Math. **60** (2016), no. 1, 289–352.

[GHRS15] Stavros Garoufalidis, Craig Hodgson, Hyam Rubinstein, and Henry Segerman, *1-efficient triangulations and the index of a cusped hyperbolic 3-manifold*, Geom. Topol. **19** (2015), no. 5, 2619–2689.

[GK17] Stavros Garoufalidis and Rinat Kashaev, *From state integrals to q-series*, Math. Res. Lett. **24** (2017), no. 3, 781–801.

[Kasa] Rinat Kashaev, *The q-binomial formula and the Rogers dilogarithm identity*, arXiv:math/0407078, Preprint 2004.

[Kasb] ______, *The Yang-Baxter relation and gauge invariance*, arXiv:1510.03043, Preprint 2015.

[Kas14] ______, *Beta pentagon relations*, Theoret. and Math. Phys. **181** (2014), no. 1, 1194–1205, Russian version appears in Teoret. Mat. Fiz. **1**81, (2014), no. 1, 73–85.

[KLV16] Rinat Kashaev, Feng Luo, and Grigory Vartanov, *A TQFT of Turaev-Viro type on shaped triangulations*, Ann. Henri Poincaré **17** (2016), no. 5, 1109–1143.

[LT08] Feng Luo and Stephan Tillmann, *Angle structures and normal surfaces*, Trans. Amer. Math. Soc. **360** (2008), no. 6, 2849–2866.

[NZ85] Walter Neumann and Don Zagier, *Volumes of hyperbolic three-manifolds*, Topology **24** (1985), no. 3, 307–332.

[Thu77] William Thurston, *The geometry and topology of 3-manifolds*, Universitext, Springer-Verlag, Berlin, 1977, http://msri.org/publications/books/gt3m.

[Wor92] Stanisław Woronowicz, *Operator equalities related to the quantum $E(2)$ group*, Comm. Math. Phys. **144** (1992), no. 2, 417–428.

[Zag07] Don Zagier, *The dilogarithm function*, Frontiers in number theory, physics, and geometry. II, Springer, Berlin, 2007, pp. 3–65.

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160, USA
http://www.math.gatech.edu/~stavros
Email address: stavros@math.gatech.edu

Section de Mathématiques, Université de Genève, 2-4 rue du Lièvre, Case Postale 64, 1211 Genève 4, Switzerland
http://www.unige.ch/math/folks/kashaev
Email address: Rinat.Kashaev@unige.ch