ON THE MELVIN-MORTON-ROZANSKY CONJECTURE

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ABSTRACT. We prove a conjecture stated by Melvin and Morton (and elucidated further by Rozansky) saying that the Alexander-Conway polynomial of a knot can be read from some of the coefficients of the Jones polynomials of cables of that knot (i.e., coefficients of the "colored" Jones polynomial). We first reduce the problem to the level of weight systems using a general principle, which may be of some independent interest, and which sometimes allows to deduce equality of Vassiliev invariants from the equality of their weight systems. We then prove the conjecture combinatorially on the level of weight systems. Finally, we prove a generalization of the Melvin-Morton-Rozansky (MMR) conjecture to knot invariants coming from arbitrary semi-simple Lie algebras. As side benefits we discuss a relation between the Conway polynomial and immanants and a curious formula for the weight system of the colored Jones polynomial.

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1. Introduction

1.1. The conjecture. In this paper, we will mostly be concerned with proving and explaining some of the motivation for the following conjecture, due to Melvin and Morton [MM, Mo]:

Conjecture 1. *Let $\hat{J}_{sl(2),\lambda}(K) \in \mathbf{Q}(q)$ be the "framing independent colored Jones polynomial" of the knot K, i.e., the framing independent Reshetikhin-Turaev invariant*[1] *[RT] of K colored by the $(d = \lambda + 1)$-dimensional representation of $sl(2)$. Let $\hbar$ be a formal parameter, let $q = e^{\hbar}$, and let $[d]$ denote the "quantum integer d":*

$$[d] = \frac{q^{d/2} - q^{-d/2}}{q^{1/2} - q^{-1/2}} = \frac{e^{d\hbar/2} - e^{-d\hbar/2}}{e^{\hbar/2} - e^{-\hbar/2}}.$$

[1] I.e., $\hat{J}$ is obtained from the framing-dependent J either by multiplication by $q^{-C\cdot writhe}$ where C is the quadratic Casimir number of V_λ, or by evaluating J on K with its zero framing. We take the metric on $sl(2)$ to be the trace in the 2-dimensional representation.

Then, expanding $\hat{J}/[d]$ in powers of d and $\hbar$,

$$\frac{\hat{J}_{sl(2),\lambda}(K)(e^{\hbar})}{[d]} = \sum_{j,m\geq 0} a_{jm}(K) d^j \hbar^m,$$

we have:

(1) *"Above diagonal" coefficients vanish: $a_{jm}(K) = 0$ if $j > m$.*
(2) *"On diagonal" coefficients give the inverse of the Alexander-Conway polynomial:*

$$MM(K)(\hbar) \cdot A(K)(e^{\hbar}) = 1, \tag{1}$$

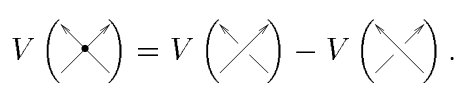
where $A(q)$ is the Alexander-Conway polynomial (in its "Conway" normalization, as in example 2.8) and MM is defined by

$$MM(K)(\hbar) = \sum_{m=0}^{\infty} a_{mm}(K)\hbar^m.$$

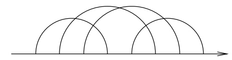
Notice that the colored Jones polynomial of a knot can be read from the Jones polynomials of cables of that knot (see, e.g. [MS]), and thus the above conjecture implies that the Alexander polynomial can be computed from the Jones polynomial and cabling operations.

Melvin and Morton arrived at (the rather unexpected) conjecture 1 after noticing it in some special cases, and by noticing that the two sides of (1) seem to behave in the same way when acted on by the 'Adams operations' of [B-N2]. In his visit to Cambridge in November 1993, we informed L. Rozansky of the conjecture, and he was able [Ro1] to find a path integral "proof" of it, which easily leads to a generalization to other Lie algebras, as shown in section 5. At the end of this introduction we will briefly review the main ideas of Rozansky's work on the MMR conjecture.

1.2. Preliminaries. Before we can sketch our proof of the MMR conjecture, let us recall some facts about Vassiliev invariants and chord diagrams, which are the main tools used in the proof. We follow the notation of [B-N2]; see also [Va1, Va2, BL, Ko1]. A *Vassiliev invariant* of type m is a knot invariant V which vanishes whenever it is evaluated on a knot with more than m double points, where the definition of V is extended to knots with double points via the formula



The algebra $\mathcal{V}$ of all Vassiliev invariants (with values in some fixed ring) is filtered, with the type m subspace $\mathcal{F}_m\mathcal{V}$ containing all type m Vassiliev invariants. The associated graded space of $\mathcal{V}$ is isomorphic to the space $\mathcal{W}$ of all *weight systems.* A degree m weight system is a homogeneous linear functional of degree m on the graded vector space $\mathcal{A}^r$ of *chord diagrams* like in figure 1 divided by the $4T$ and *framing independence* relations explained in figures 2 and 3.

Figure 1. A chord diagram: 

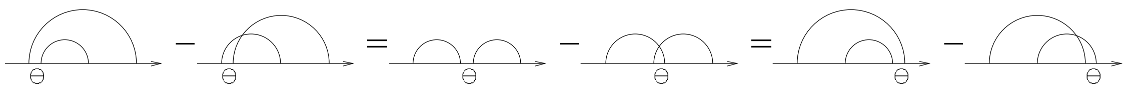
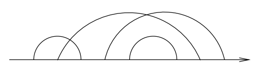


Figure 2. To get the $4T$ relations, add an arbitrary number of chords in arbitrary positions (only avoiding the short intervals marked by a 'no-entry' sign $\ominus$) to all six diagrams in exactly the same way.

Figure 3. The framing independence relation: any diagram containing a chord whose endpoints are not separated by the endpoints of other chords is equal to 0.

 $= 0$

$\mathcal{A}^r$ is graded by the number of chords in a chord diagram. It is a commutative and co-commutative Hopf algebra with multiplication defined by juxtaposition, and with co-multiplication Δ defined as the sum of all possible ways of 'splitting' a diagram. The co-algebra structure of $\mathcal{A}^r$ defines an algebra structure on $\mathcal{W}$. $\mathcal{A}$ is defined in the same way as $\mathcal{A}^r$, only without imposing the framing independence relation.

There are natural maps $W_m : \mathcal{F}_m\mathcal{V} \to \mathcal{G}_m\mathcal{W} = \mathcal{G}_m\mathcal{A}^{r\star}$, where $\mathcal{G}_m\,\mathrm{obj}$ denotes the degree m piece of a graded object obj. For a type m Vassiliev invariant V it is natural to think of $W_m(V)$ as "the m'th derivative of V". The maps W_m are compatible with the products of the spaces involved. Similar definitions can be made for framed knots, and the image of the corresponding map W_m will be $\mathcal{G}_m\mathcal{A}^\star$.

1.3. Plan of the proof. It is well known [Gu, B-N1, B-N2, BL, Lin] that the coefficients of both the Conway and the Jones polynomials are Vassiliev invariants. Normally, Vassiliev invariants are not determined by their weight systems. However, in section 2 we explain (following Kassel [Kas] and Le and Murakami [LM]) that when an invariant comes (in an appropriate sense) from a Lie algebra, it is in fact determined by its weight system. As this is the case for all the invariants appearing in conjecture 1 (or rather, in the version of it that we actually prove, theorem 1), it is enough to prove conjecture 1 (that is, theorem 1) on the level of weight systems.

To do this, we analyze the weight systems of the Conway polynomial and of the invariant *MM*. In section 3 we analyze the weight system W_C of the Conway polynomial. We find a simple characterization (theorem 2) of it, and then we use this characterization to show that $W_C(D)$ is the determinant of the intersection matrix $\mathrm{IM}(D)$ (definition 3.4) of the chord diagram D. In section 4 we go through a rather complicated analysis of the weight system of *MM*, finding that it is given by the permanent of the intersection matrix. We then conclude the proof of the conjecture by showing that, in the sense of weight systems,

$$\log\det \mathrm{IM} + \log \operatorname{per} \mathrm{IM} = 0, \tag{2}$$

and thus the two weight systems are inverses of each other. Equation (2) is proven in the ends of sections 3 and 4, where the logarithm of the two weight systems involved are given in terms of explicit formulas.

In section 5 we use similar techniques to generalize conjecture 1 to arbitrary semi-simple Lie algebras. In section 6.1 we discuss a curious relation between immanants and the algebra generated by the coefficients of the Conway polynomial, in section 6.2 we sketch how the techniques of section 4 can be used to get a formula for the weight system of the colored Jones polynomial, and in section 6.3 we conjecture a generalization of conjecture 1 beyond the realm of Lie algebras.

As noted before, we actually prove a variation of conjecture 1 in which the normalizations are somewhat 'better' from the point of view of sections 2 and 5:

Theorem 1. *Expanding $\hat{J}/d$ in powers of $\lambda = d-1$ and $\hbar$,*

$$\frac{\hat{J}_{sl(2),\lambda}(K)(e^\hbar)}{d} = \sum_{j,m\geq 0} b_{jm}(K)\lambda^j\hbar^m, \tag{3}$$

we have:

1. *"Above diagonal" coefficients vanish: $b_{jm}(K) = 0$ if $j > m$.*
2. *Up to a constant, "on diagonal" coefficients give the inverse of the Alexander-Conway polynomial:*

$$JJ(K)(\hbar) \cdot \frac{\hbar}{e^{\hbar/2} - e^{-\hbar/2}} A(K)(e^\hbar) = 1, \tag{4}$$

where JJ is defined by

$$JJ(K)(\hbar) = \sum_{m=0}^{\infty} b_{mm}(K)\hbar^m.$$

Claim 1.1. *Conjecture 1 and theorem 1 are equivalent.*

Proof. Let b'_{jm} be the coefficients of the expansion of $\hat{J}/d$ in powers of d and $\hbar$. It is clear that theorem 1 restated with b'_{jm} replacing b_{jm} is equivalent to the original theorem 1. We have:

$$\sum a_{jm} d^j \hbar^m = \frac{\hat{J}}{[d]} = \frac{d}{[d]} \cdot \frac{\hat{J}}{d} = \frac{e^{\hbar/2} - e^{-\hbar/2}}{\hbar} \cdot \frac{d\hbar}{e^{d\hbar/2} - e^{-d\hbar/2}} \cdot \sum b'_{jm} d^j \hbar^m \tag{5}$$

The first factor in the right hand side of (5) is a power series in $\hbar$ alone in which the coefficient of $\hbar^0$ is 1, and thus it (or its inverse) cannot take below- or on-diagonal terms to go above the diagonal, and it does not change the coefficients on the diagonal.

The second factor lives entirely *on* the diagonal and thus the first part of conjecture 1 is equivalent to the first part of theorem 1.

Restricted to the diagonal, (5) becomes

$$\sum a_{mm} d^m \hbar^m = \frac{d\hbar}{e^{d\hbar/2} - e^{-d\hbar/2}} \cdot \sum b'_{mm} d^m \hbar^m.$$

At $d = 1$, we get

$$MM = \frac{\hbar}{e^{\hbar/2} - e^{-\hbar/2}} \cdot JJ,$$

and it is clear that (1) and (4) are equivalent. $\square$

1.4. Rozansky's work. Rozansky arrives at the MMR conjecture using the path integral interpretation of the Jones polynomial given in Witten's seminal paper [Wi]. Needless to say, path integrals have not yet been mathematically defined, but they can be used as a rich source of motivation. In our case they do in fact lead to the correct conjecture, though our proof of the conjecture is not a translation of the path integral argument to rigorous math, and we don't know how to translate the path integral argument into rigorous math. For the convenience of the reader we outline Rozansky's argument below. The reader may find our account somewhat more readable than Rozansky's [Ro1], as we have isolated the parts relevant to conjecture 1 from his (much broader) paper, and skipped some of the details. We heartily recommend consulting with [Ro1] (as well as [Ro2, Ro3]) for the missing details and for many other related results.

Let us recall Witten's interpretation of the Jones polynomial. For a framed, oriented knot K in S^3, a choice V_λ of an irreducible $SU(2)$ representation of heighest weight λ and an integer k, Witten introduces the following definition:

$$Z(K, V_\lambda; k) = \int_{\mathcal{A}} \mathcal{D}A\, e^{2\pi i k CS(A)}\, \mathcal{O}_{K,V_\lambda}(A)$$

where the (ill defined) path integral is over the space $\mathcal{A}$ of all $SU(2)$ connections on the trivial $SU(2)$ bundle over S^3, $CS : \mathcal{A} \to \mathbf{R}/\mathbf{Z}$ is the Chern-Simons action

$$CS(A) = \frac{1}{8\pi^2} \int_{S^3} tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A),$$

and $\mathcal{O}_{K,V_\lambda} : \mathcal{A} \to \mathbf{R}$ is the trace in the representation V_λ of the holonomy of the connection A along the knot K.

Using non-rigorous quantum field theory reasoning, Witten computed $Z(K, V_\lambda; k)$ and found that

$$Z(K, V_\lambda; k) = \sqrt{\frac{2}{k+2}} \sin\left(\frac{\pi}{k+2}\right) J_{sl(2),V_\lambda}(K) \left(\exp \frac{2\pi i}{k+2}\right),$$

where $J_{sl(2),V_\lambda}$ is the framing dependent colored Jones polynomial.

Now take a rational number $0 < a \ll 1$ (so that ka is a weight for many large integers k). Following Rozansky [Ro1], the path integral $Z(K, V_{ka}; k)$ (for such k) can be split into an integral over connections on a tubular neighborhood $\mathrm{Tub}(K)$ of the knot K and over connections on the complement $S^3 \setminus \mathrm{Tub}(K)$ with certain boundary conditions on the boundary $T^2 = \partial\,\mathrm{Tub}(K)$, followed by an integral over these boundary conditions. With the appropriate boundary conditions of [EMSS], the integral over the connections on $\mathrm{Tub}(K)$ can be restricted to an integral over flat connections, and on those it is proportional to $\delta(I_1 - e^{2\pi i a})$ *independently* of k, where I_1 is the holonomy along a meridian of K in $\partial\,\mathrm{Tub}(K)$ and $e^{2\pi i a}$ is considered in $SU(2)$ in the usual way. Therefore

$$Z(K, V_{ka}; k) = \int_{\mathcal{A}[S^3\setminus\mathrm{Tub}(K)]_a} \mathcal{D}A\, e^{2\pi i k CS'(A)} \tag{6}$$

where the integral is over the connections on $S^3 \setminus \mathrm{Tub}(K)$ with holonomy $e^{2\pi i a}$ along any meridian of K. Here CS' is a modified Chern-Simons action dictated by the boundary conditions.

Rozansky now applies stationary phase approximation to calculate the large k limit of $Z(K, V_{ka}; k)$. The critical points of CS' are the flat $SU(2)$ connections on the knot complement with holonomy $e^{2\pi i a}$ around a meridian. Modulo gauge equivalence, the moduli space of such connections consists of only one connection A_a, for sufficiently small values of a.

By the stationary phase approximation, the leading order term of the path integral is proportional to

$$\frac{1}{\sqrt{8}\pi}\left(\frac{4\pi^2}{k}\right)^{\frac{1}{2}\left(h^0(A_a)-h^1(A_a)\right)} \sqrt{\tau_{RS}(A_a)}\cdot e^{2\pi i k CS'(A_a)}$$

where $h^j(A_a)$ is the dimension of the j'th cohomology of $S^3\setminus\mathrm{Tub}(K)$ with coefficients twisted by A_a, and $\tau_{RS}(A_a)$ is the $SU(2)$ Ray-Singer torsion of $S^3\setminus\mathrm{Tub}(K)$ twisted by A_a. Furthermore one can check that $h^1(A_a) = 0$, $h^0(A_a) = 1$, and $CS'(A_a) = 0$. The Ray-Singer torsion splits into three factors, one for each algebra component of $SU(2)$. The torsion in the Cartan direction is 1, and in the remaining two directions the torsions are equal, and each contributes the square root of the $U(1) \subset SU(2)$ torsion using the representation of $\pi_1(S^3 \setminus \mathrm{Tub}(K))$ sending the meridian to $e^{2\pi i a} \in U(1)$. Summarizing, we get

$$\sqrt{\frac{2}{k+2}}\sin\left(\frac{\pi}{k+2}\right) J_{sl(2),V_{ka}}(K)\left(\exp\frac{2\pi i}{k+2}\right) \underset{k\to\infty}{\sim} \frac{1}{\sqrt{2k}}\tau_{RS}(S^3\setminus\mathrm{Tub}(K), e^{2\pi i a}).$$

Cheeger [Ch] and Muller [Mü] proved that the Ray-Singer torsion is equal to the Reidemeister torsion, which by Milnor [Mi] and Turaev [Tu] was shown to be proportional to the *inverse* of the Alexander polynomial $A(K)$ of K, evaluated at $e^{2\pi i a}$.

With the correct constant of proportionality $(2\sin\pi a)$ in place and ignoring factors that converge to 1 as $k\to\infty$, we get

$$\frac{\pi}{k}J_{sl(2),V_{ka}}(K)\left(\exp\frac{2\pi i}{k}\right)\underset{k\to\infty}{\longrightarrow}\frac{\sin\pi a}{A(K)(e^{2\pi i a})}.$$

See [Ro1, (2.8) and following paragraph] for an explanation why the J computed here is 'in zero framing'. Thus $J=\hat{J}$ and

$$\pi a\sum_{j,m\geq 0}b_{jm}(K)(2\pi i)^m a^j k^{j-m}\underset{k\to\infty}{\longrightarrow}\frac{\sin\pi a}{A(K)(e^{2\pi i a})}.$$

This proves (on the level of rigor of path integrals) that $b_{jm}=0$ if $j-m>0$, and, taking $a=\hbar/2\pi i$ and disregarding all strictly positive powers of k, it also proves theorem 1 (on the same level of rigor).

1.5. Acknowledgement. We wish to thank N. Bergeron, P. Diaconis, R. C. Kirby, H. R. Morton, L. Rozansky, S. Sawin, and C. H. Taubes for their many useful comments. Especially we wish to thank D. Kazhdan for his critical reading and P. M. Melvin for suggesting exercise 3.9 and the use of permanents.

2. A reduction to weight systems

Let us start with some generalities that (sometimes) allow us to deduce equality of invariants from the equality of their weight systems. In this section, we mostly interpret and adapt to our needs the deep results of Kassel [Kas] and Le and Murakami [LM], who followed Kohno [Koh] and Drinfel'd [Dr1, Dr2].

2.1. Canonical Vassiliev invariants. A fundamental (and not too surprising) result in the theory of Vassiliev invariants is that every degree m weight system comes from a type m Vassiliev invariant, and that the resulting Vassiliev invariant is well-defined up to Vassiliev invariants of lower types (see e.g. [Ko1] and [B-N2]); in other words, the sequence

$$0\longrightarrow\mathcal{F}_{m-1}\mathcal{V}\longrightarrow\mathcal{F}_m\mathcal{V}\longrightarrow\mathcal{G}_m\mathcal{A}^{r\star}\longrightarrow 0, \tag{7}$$

is exact. The standard way of proving this fact is to construct a splitting $V_m:\mathcal{G}_mA^{r\star}\to\mathcal{F}_m\mathcal{V}$ for each m. These splittings can be assembled together in a unique way to form a *universal Vassiliev invariant* Z with values in the graded completion of $\mathcal{A}^r$, satisfying

$$V_m(W)=W\circ Z \tag{8}$$

for each degree m weight system W. Indeed, usually Z is first constructed, and only then the splittings V_m are defined from it via (8).

A-priori, there appears to be no knot theoretic reason to expect that there would be a preferred choice for the splittings V_m, or, equivalently, for Z. However, rather surprisingly, it seems that such a preferred choice for Z does exist. Indeed, for reasons discovered by Drinfel'd [Dr1, Dr2] and elucidated further by Kassel [Kas] and Le and Murakami [LM], many of the known constructions [B-N3, Ca, Kas, Ko1, LM] of a universal Vassiliev invariant give the same (hard to compute but rather well behaved) answer.[2] Let us call this preferred universal Vassiliev invariant $\mathbf{Z}^{\mathbf{K}}$.

Definition 2.1. A *canonical type m Vassiliev invariant V* is a type m Vassiliev invariant lying in the image of the splitting of (7) defined by $\mathbf{Z}^{\mathbf{K}}$. In a simpler language, let $\mathbf{Z}^{\mathbf{K}}_m$ be the projection of $\mathbf{Z}^{\mathbf{K}}$ into $\mathcal{G}_m\mathcal{A}^r$. V is a canonical type m Vassiliev invariant iff

$$V = W_m(V) \circ \mathbf{Z}^{\mathbf{K}}_m.$$

Definition 2.2. Let $\hbar$ be a formal parameter. A *Vassiliev power series* is an element

$$V \in \sum_{m=0}^{\infty} (\mathcal{F}_m\mathcal{V})\hbar^m.$$

That is to say, it is a power series $V = V_0 + V_1\hbar + \dots$ in which the coefficient V_m of $\hbar^m$ is a Vassiliev invariant of type m. The *weight system* $W(V)$ of V will be the sum of the weight systems of the coefficients of V (which makes sense in the graded completion $\bar{\mathcal{W}}$ of $\mathcal{W}$):

$$W(V) = \sum_{m=0}^{\infty} W_m(V_m) \in \bar{\mathcal{W}}.$$

Definition 2.3. A Vassiliev power series $V = \sum V_m\hbar^m$ is called *canonical* if each of its coefficients V_m is canonical. Equivalently, if $\hbar^{\deg}$ is the operator that multiplies every degree m diagram by $\hbar^m$ and $\mathbf{Z}^{\mathbf{K}}_\hbar \stackrel{\text{def}}{=} \hbar^{\deg} \circ \mathbf{Z}^{\mathbf{K}}$, then V is canonical iff

$$V = W(V) \circ \mathbf{Z}^{\mathbf{K}}_\hbar.$$

Obviously, two canonical Vassiliev power series (or canonical Vassiliev invariants) are equal iff their weight systems are equal. Sometimes, as is the case in this paper, it is easier to verify equality of weight systems and then use it to deduce the equality of the corresponding canonical invariants rather than proving the equality of the invariants directly.

2.2. Examples of canonical Vassiliev power series. In this section we will establish, through a sequence of examples, that the invariants appearing in theorem 1 are canonical.

[2][B-N2, Pi2] differ only by a normalization, and the incomplete perturbative Chern-Simons constructions [AS1, AS2, B-N1, Ko2] are conjectured to also give the same answer.

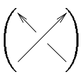
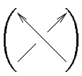
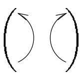
Example 2.4. The type 0 invariant 1, whose value on all knots (having no double points) is 1, is both a canonical type 0 Vassiliev invariant and a canonical Vassiliev power series. Its weight system ϵ is defined by

$$\epsilon(D) = \begin{cases} 1 & \text{if } \deg D = 0 \text{ (namely, if } D = \underline{\qquad\quad}\!\!\rightarrow \text{ is the empty diagram),} \\ 0 & \text{otherwise.} \end{cases}$$

Kassel [Kas, theorem 8.3, chapter XX] and Le and Murakami [LM, theorem 10], using the techniques of Kohno [Koh] and Drinfel'd [Dr1, Dr2], have shown that the Reshetikhin-Turaev [RT] invariant associated with a semi-simple Lie algebra $\mathfrak{g}$ and a representation V (and a metric t on $\mathfrak{g}$) is a canonical Vassiliev power series when evaluated at $q = e^{\hbar}$ and expanded in powers of $\hbar$.[3] (Both the framed version $J_{\mathfrak{g},V}$ and unframed version $\hat{J}_{\mathfrak{g},V}$ are canonical; for the framed version, $\mathcal{A}$ has to replace $\mathcal{A}^r$ in the definitions of this section. For the unframed version (at least when V is irreducible), simply notice that it can always be obtained from the framed version by multiplying the Lie algebra by an Abelian Lie algebra). We will use this crucial result twice, in example 2.5 and in example 2.6.

Example 2.5. By [Kas, LM], the invariant $\hat{J}_{sl(2),\lambda}$ of conjecture 1 is a canonical Vassiliev power series, and hence the invariants b_{jm} of theorem 1 are canonical of type m, and JJ is a canonical Vassiliev power series. The invariants a_{jm} and MM are not canonical as $[d]$ depends on $\hbar$.

Example 2.6. The HOMFLY polynomial, defined by the relations

$e^{N\hbar/2} H \Big($  $\Big) - e^{-N\hbar/2} H \Big($  $\Big) = (e^{\hbar/2} - e^{-\hbar/2}) H \Big($  $\Big),$

$$H\,(c\text{-component unlink}) = \left(\frac{e^{N\hbar/2} - e^{-N\hbar/2}}{e^{\hbar/2} - e^{-\hbar/2}} \right)^c,$$

is a canonical Vassiliev power series, as it is the Reshetikhin-Turaev invariant associated with the Lie algebra $sl(N)$ in its defining representation.

Example 2.7. Divide the HOMFLY polynomial by N and take the limit $N \to 0$. The limit exists because the limit

$$\lim_{N \to 0} \frac{e^{N\hbar/2} - e^{-N\hbar/2}}{N} = \hbar$$

[3]Thus they gave an affirmative answer to problem 4.9 of [B-N2].

exists. The result is a canonical Vassiliev power series $\tilde{C}$ satisfying

(9)
$$\tilde{C}\left(\nwarrow\!\!\!\!\!\nearrow\right) - \tilde{C}\left(\nwarrow\!\!\!\!\!\nearrow\right) = (e^{\hbar/2} - e^{-\hbar/2})\tilde{C}\left(\uparrow\,\uparrow\right),$$
$$\tilde{C}\,(c\text{-component unlink}) = \begin{cases} \frac{\hbar}{e^{\hbar/2}-e^{-\hbar/2}} & \text{if } c = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Recall that the Conway polynomial C [Co, Kau] (considered as a polynomial in $\hbar$) is defined by the relations:

(10)
$$C\left(\nwarrow\!\!\!\!\bullet\!\!\!\!\nearrow\right) \stackrel{\text{def}}{=} C\left(\nwarrow\!\!\!\!\!\nearrow\right) - C\left(\nwarrow\!\!\!\!\!\nearrow\right) = \hbar C\left(\uparrow\,\uparrow\right),$$
$$C\,(c\text{-component unlink}) = \begin{cases} 1 & \text{if } c = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Comparing (9) and (10), we see that the Conway polynomial itself is not a canonical Vassiliev power series, but its renormalized reparametrized version

$$\tilde{C}(\hbar) = \frac{\hbar}{e^{\hbar/2} - e^{-\hbar/2}} C(e^{\hbar/2} - e^{-\hbar/2})$$

is a canonical Vassiliev power series.

Example 2.8. The Alexander polynomial, defined by $A(z) = C(z^{1/2} - z^{-1/2})$ is *not* a canonical Vassiliev power series, but it becomes canonical when multiplied by $\frac{\hbar}{e^{\hbar/2}-e^{-\hbar/2}}$ and evaluated at $z = e^{\hbar}$ (as this product is $\tilde{C}$).

2.3. Products. The product (in the natural sense) of two Vassiliev power series is a Vassiliev power series, and the weight system of such a product is the product of the weight systems of the factors.

Proposition 2.9. *The product of any two canonical Vassiliev power series is a canonical Vassiliev power series.*

Proof. It can be shown that the universal Vassiliev invariant $\mathbf{Z^K}$ is 'group-like'; it satisfies $\Delta\mathbf{Z^K}(K) = \mathbf{Z^K}(K) \otimes \mathbf{Z^K}(K)$ for any knot K. This property is an immediate consequence of the Kontsevich integral formula for $\mathbf{Z^K}$ described in [Ko1, B-N2][4].

[4] A similar but different statement is [LM, theorem 4].

Now, if $V_{1,2}$ are canonical, then

$$
\begin{aligned}
(W(V_1V_2)\circ \mathbf{Z}^{\mathbf{K}}_\hbar)(K) &= (W(V_1)W(V_2))(\mathbf{Z}^{\mathbf{K}}_\hbar(K)) && \text{[B-N2, exercise 3.10]}\\
&= (W(V_1)\otimes W(V_2))(\Delta \mathbf{Z}^{\mathbf{K}}_\hbar(K)) && \mathcal{A}^r \text{ is a Hopf algebra}\\
&= (W(V_1)\otimes W(V_2))(\mathbf{Z}^{\mathbf{K}}_\hbar(K)\otimes \mathbf{Z}^{\mathbf{K}}_\hbar(K)) && \mathbf{Z}^{\mathbf{K}} \text{ is group-like}\\
&= (W(V_1)\circ \mathbf{Z}^{\mathbf{K}}_\hbar)(K)(W(V_2)\circ \mathbf{Z}^{\mathbf{K}}_\hbar)(K)\\
&= V_1(K)V_2(K),
\end{aligned}
$$

and thus $V_1 \cdot V_2$ is also canonical. □

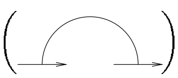
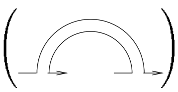
It follows from examples 2.4, 2.5, and 2.8 and from proposition 2.9 that both sides of equation (4) are canonical Vassiliev power series, and thus it is enough to prove (4) (as well as the vanishing of b_{jm} for $j > m$) on the level of weight systems. That is, we need to show that

$$W_{JJ} \cdot W_C = \epsilon,$$

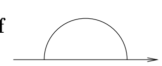
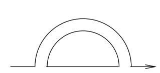
where W_{JJ} is the weight system of JJ, W_C is the weight system of $\tilde{C}$ (which is equal to the weight system of C), and ϵ is as in example 2.4.

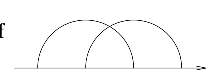
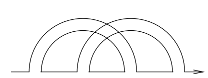
3. The Conway polynomial

3.1. The Conway weight system. The defining relations (10) of C, become the following relations on the level of W_C:

W_C () $= W_C$ () and $W_C\,(c \text{ cycles}) = \begin{cases} 1 & \text{if } c = 0 \\ 0 & \text{otherwise.} \end{cases}$

In other words, to compute W_C of a given chord diagram D, "thicken" all chords in D into bands, and count the number of cycles in the resulting diagram; if it is greater than 0, $W_C(D)$ is 0, and otherwise it is 1. For example,

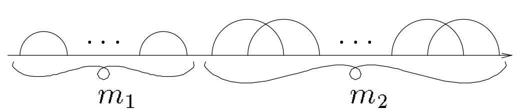
$\theta \stackrel{\text{def}}{=}$  $\longrightarrow$  $\longrightarrow$ 1 cycle $\longrightarrow 0$,

$X \stackrel{\text{def}}{=}$  $\longrightarrow$  $\longrightarrow$ 0 cycles $\longrightarrow 1$.

These two examples can be combined as in the following definition:

Definition 3.1. An (m_1, m_2)-*caravan* or simply a *caravan* is the chord diagram $\theta^{m_1}X^{m_2}$ made of m_1 single-hump-camels and m_2 double-hump-camels, as in figure 4. It is a chord diagram of degree $m = m_1 + 2m_2$.

Figure 4. An (m_1, m_2)-caravan:



Proposition 3.2.

$$W_C\,(\textit{an } (m_1, m_2)\textit{-caravan}) = \begin{cases} 1 & \textit{if } m_1 = 0 \\ 0 & \textit{otherwise.} \end{cases} \qquad \square$$

3.2. The $2T$ relation. It is clear that W_C is invariant under the "$2T$" or "slide" relations shown in figure 5. Indeed, after thickening the chords l and r, it is clear that it is possible to 'slide' l over r as in figure 6 without changing the topology of the resulting diagram.

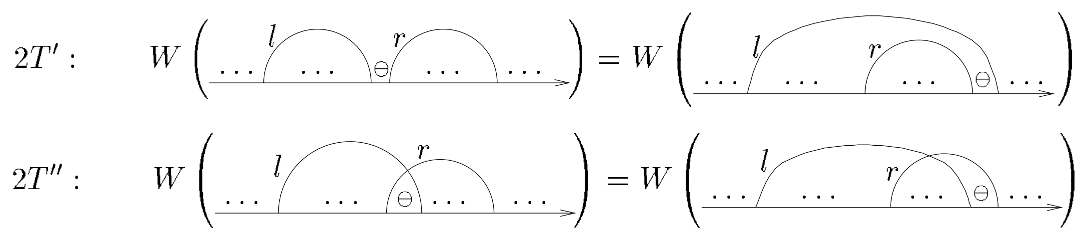


Figure 5. The $2T$ relations. In these figures, ellipsis denote possible other chords, while a 'no-entry' sign ($\ominus$) means that no chords can end in the corresponding interval. For definiteness, we drew the 'far' end of the chord l left of the chord r, but it can be anywhere else in the diagram.

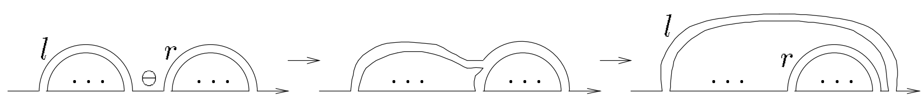


Figure 6. Deriving the relation $2T'$ by sliding l over r.

Let $\mathcal{G}_m\mathcal{D}$ be the set of all chord diagrams of degree m. The following theorem[5] is a characterization of the Conway weight system:

Theorem 2. *If a map $W : \mathcal{G}_m\mathcal{D} \to \mathbf{Z}$ satisfies the $2T$ relations and the same 'initial condition' as in proposition 3.2, then it is the Conway weight system W_C.*

Proof. It is enough to show that modulo $2T$ relations, every chord diagram D is equivalent to a caravan. If D has a pair of intersecting chords r_1 and r_2, thicken both of them and slide all other chords out and to the left as in figure 7. The result is that a double-hump-camel (an X diagram) is factored out. Use induction to simplify the rest. If D has no pairs of intersecting chords, than it must have a 'small' chord r, a chord whose endpoints are not separated by the endpoints of any other chords. Thicken r, and slide all other chords over it and to the left. The result is that a

[5] P. M. Melvin commented that this is simply the classification theorem for surfaces presented as 'a box with handles'.

single-hump-camel (a θ diagram) is factored out. Again, use induction to simplify the rest. $\square$

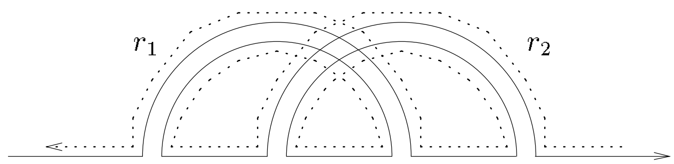


Figure 7. Factoring out a double-hump-camel. Slide all other chords out following the path marked by a dotted line.

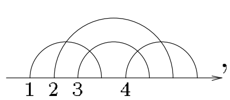
Exercise 3.3. Show that the space of maps $W : \mathcal{G}_m\mathcal{D} \to \mathbf{Z}$ satisfying the $2T$ relations is spanned by the coefficients of various powers of N in $D \mapsto W_{gl(N),V_N}(D)$, where $W_{gl(N),V_N}(D)$ is the weight assigned to D using the Lie algebra $gl(N)$ in its defining representation V_N as in section 4.1 below. Show that such a map that also satisfies the framing independence relations has to be proportional to W_C.

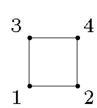
3.3. The intersection graph and the intersection matrix. In this section, we will use theorem 2 to find a determinant formula for W_C.

Definition 3.4. (See also [CDL1, CDL2, CDL3]) Let D be a degree m chord diagram. The *labeled intersection graph* $\mathrm{LIG}(D)$ of D will be the graph whose vertices are the chords of D, numbered from 1 to m by the order in which they appear along the 'base line' of D from left to right, and in which two vertices are connected by an edge iff the corresponding two chords in D intersect. The *intersection matrix* $\mathrm{IM}(D)$ of D is the anti-symmetric variant of the $m \times m$ adjacency matrix of $\mathrm{LIG}(D)$ defined by

$$\mathrm{IM}(D)_{ij} = \begin{cases} \operatorname{sign}(i-j) & \text{if chords } i \text{ and } j \text{ of } D \text{ intersect (where chords} \\ & \text{of } D \text{ are numbered from left to right),} \\ 0 & \text{otherwise.} \end{cases}$$

Example 3.5.

$D =$ 

, $\mathrm{LIG}(D) =$ 

, $\mathrm{IM}(D) = \begin{pmatrix} 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$.

Example 3.6. The labeled intersection graph of an (m_1, m_2)-caravan is the disconnected union of m_1 single vertices and m_2 graphs like •——•. Its intersection matrix is block diagonal, with the blocks on the diagonal being m_1 copies of the 1×1 zero matrix and m_2 copies of the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Exercise 3.7. Show that if the labeled intersection graph of a chord diagram is connected, then the diagram is determined by its intersection matrix. Deduce that in general the intersection matrix determines the class of the diagram modulo $4T$ relations.

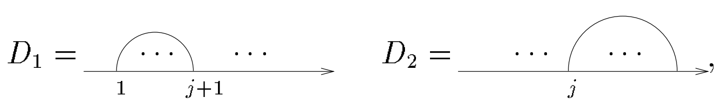
Hint 3.8. Start from a connected labeled intersection graph of a chord diagram, remove one vertex so that the resulting graph is still connected (this is possible!), use induction, and show that there is a unique way to re-install the missing chord.

In the light of the above exercise, it is not surprising that one can find a formula for the weight system of the Conway polynomial in terms of the intersection matrix, as found in the theorem below. A mild generalization of this theorem is in section 6.1. Even though the exercise suggests it should be possible, we have not been able to find nice formulae for other weight systems (beyond those of section 6.1) in terms of the intersection matrix.

Theorem 3. *For any chord diagram D,*

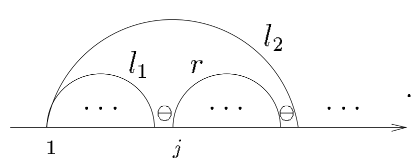
$$W_C(D) = \det \mathrm{IM}(D)$$

Proof. Let $W : \mathcal{G}_m\mathcal{D} \to \mathbf{Z}$ be defined by $W(D) = \det \mathrm{IM}(D)$. By theorem 2, it is enough to prove that W satisfies the $2T$ relations and the initial conditions of proposition 3.2. The latter fact is trivial; simply compute the determinant of the block diagonal matrix in example 3.6. Let us now prove that W satisfies the $2T$ relations. First, notice that W is 'independent of the basepoint of D'. That is, if the diagram D_2 is obtained from the diagram D_1 by moving the left-most vertex of D_1 to the right end,



then $W(D_1) = W(D_2)$. Indeed, except for the labeling the intersection graphs of D_1 and D_2 are the same, and so $\mathrm{IM}(D_2)$ is obtained from $\mathrm{IM}(D_1)$ by reversing all the signs in the first row of $\mathrm{IM}(D_1)$, re-installing it as row number j for some j, and then doing exactly the same to the first column of $\mathrm{IM}(D_1)$. The effect of the row operations is to multiply $\det \mathrm{IM}(D_1)$ by some sign, and then the column operations multiply by the same sign once again. The end result is that $\det \mathrm{IM}(D_1) = \det \mathrm{IM}(D_2)$, as required.

By repeating the above process a few times, we may assume that the chord l in the $2T'$ relation is chord number 1, and so we need to prove that $W(D_1) = W(D_2)$ where D_1 (D_2) is the diagram obtained by ignoring l_2 (l_1) in the figure



In this figure, it is clear that any other chord can intersect either none of the chords l_1, l_1, and r, or exactly two of them. Using this and some case-checking, it is clear that $\mathrm{IM}(D_2)$ is obtained from $\mathrm{IM}(D_1)$ by adding its jth rows to its first row, and then doing the same column operation. Therefore $\det \mathrm{IM}(D_1) = \det \mathrm{IM}(D_2)$, as required. The same argument also proves the $2T''$ relation. $\square$

In the following two exercises, we outline two alternative proofs of theorem 3:

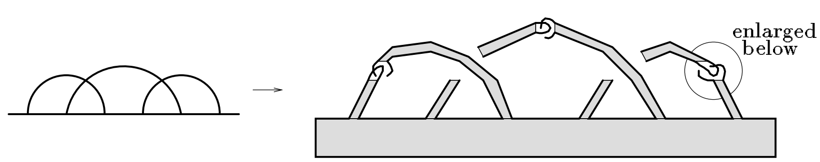
Exercise 3.9. (Melvin) Let F be the surface obtained by thickening a chord diagram D (that is, thicken all chords *and* the base line), and let ∂F be its boundary. $W_C(D) = 1$ if $H_0(\partial F) = \mathbf{Z}$, and otherwise, $W_C(D) = 0$. Now consider the following long exact sequence:

$$\begin{tikzcd}
H_1(F) \arrow[r, "p_\star"] & H_1(F, \partial F) \arrow[r, "\delta"] \arrow[d, "\gamma\ \binom{\text{Poincaré}}{\text{duality}}"] & H_0(\partial F) \arrow[r, "i_\star"] & H_0(F) = \mathbf{Z} \arrow[r] & 0 \\
& H^1(F) & & &
\end{tikzcd}$$

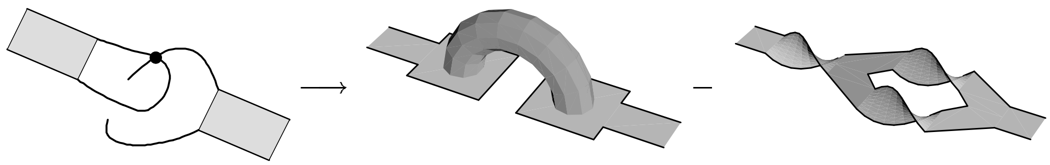
We are interested in knowing when $H_0(\partial F) = \mathbf{Z}$, which is when $p_\star$ is an epimorphism, which is when $\gamma \circ p_\star$ is an epimorphism. Show that in the basis suggested by the chords of D, $\gamma \circ p_\star$ is given by the matrix $\mathrm{IM}(D)$, and use this to deduce theorem 3 at least up to signs.

Exercise 3.10. Deduce theorem 3 from the fact (see e.g. [Kau, chapter 7]) that the Alexander polynomial of a knot K is given by $\det(z^{-1}\theta - z\theta^T)$, where θ is Seifert pairing matrix for some Seifert surface for K, and θ^T is its transpose.

Hint 3.11. First, take the 'pre-Seifert surface' of a specific singular embedding of a chord diagram as in:



Then resolve all the double points to overcrossings and undercrossings, while extending the 'pre-Seifert surface' to a Seifert surface as in:



It is now easy to compute the $2m \times 2m$ Seifert pairing matrices of the resulting surfaces in terms of the $m \times m$ intersection matrix of the original chord diagram and the over/under choices at the double points.

3.4. The logarithm of the Conway weight system. Expanding $\det \mathrm{IM}(D)$ as a sum over permutations, we only need to consider those permutations of chords(D) which map any chord to a different chord intersecting it. Such permutations can be considered as 'walks' on $\mathrm{LIG}(D)$. Let us introduce the relevant terminology:

Definition 3.12. A *Hamilton cycle* in $\mathrm{LIG}(D)$ is a directed cycle H of length > 1 in $\mathrm{LIG}(D)$ containing no repeated vertices. For example, the graph in example 3.5 has two Hamilton cycles of length 4, four of length 2, and none of any other length. The *descent* $d(H)$ of a Hamilton cycle H is the number of label-decreases along the cycle. For example, the cycle $1 \to 2 \to 4 \xrightarrow{\star} 3 \xrightarrow{\star} 1$ in example 3.5 has descent 2, corresponding to the two stared label-decreases. A *cycle decompositions* $H = \uplus H_\alpha$ is a cover of the vertex set of $\mathrm{LIG}(D)$ by a collection of unordered disjoint Hamilton cycle, and the *descent* $d(H)$ of H is defined by $d(H) = \sum d(H_\alpha)$.

Expanding $\det \mathrm{IM}(D)$, and taking account of signs, we find that

$$W_C(D) = \sum_{H = \biguplus_\alpha H_\alpha} (-1)^{\sigma_H} (-1)^{d(H)}, \tag{11}$$

where σ_H is the permutation of the vertices of $\mathrm{LIG}(D)$ underlying H. Notice that if H contains a cycle of odd length, then $(-1)^{d(H)}$ is odd under reversing the orientation of that cycle, while $(-1)^{\sigma_H}$ does not change under that operation. Therefore, summation can be restricted to cycle decompositions containing no odd cycles. For such cycle decompositions, $(-1)^{\sigma_H} = (-1)^{|H|}$, where $|H|$ is the number of cycles in H, and thus

$$W_C(D) = \sum_{H = \biguplus_\alpha H_\alpha} (-1)^{|H|} (-1)^{d(H)}. \tag{12}$$

Recall (see e.g. [B-N2]) that the algebra structure on weight systems is defined by

$$(W_1 \cdot W_2)(D) = \sum_{\substack{\text{splittings} \\ D = D_1 \dot\cup D_2}} W_1(D_1) \cdot W_2(D_2). \tag{13}$$

Using the power series expansion of the exponential function, we find that

$$(\exp W)(D) = \sum_{\substack{\text{unordered splittings} \\ D = \dot\cup D_\alpha}} \prod_\alpha W(D_\alpha),$$

and if W depends on D only through $\mathrm{LIG}(D)$, we find

$$(\exp W)(D) = \sum_{\substack{\text{unordered splittings} \\ \mathrm{LIG}(D) = \dot\cup G_\alpha}} \prod_\alpha W(G_\alpha),$$

using the obvious definition for a splitting of a labeled graph.

Proposition 3.13.

$$(\log W_C)(D) = -\sum_H (-1)^{d(H)},$$

where the sum extends over all Hamilton cycles H covering all the vertices of $\mathrm{LIG}(D)$ *(i.e., all cycle decompositions into a single cycle).*

Proof. Simply exponentiate both sides of this equation and use the discussion in the proceeding paragraph to recover (12). □

4. Understanding W_{JJ}

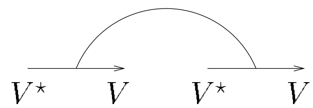
The purpose of this section is to understand W_{JJ}, the weight system underlying the invariant JJ. The invariant JJ, as defined in the statement of theorem 1, has to do with the Lie algebra $sl(2)$. So let us start by recalling the relation between Lie algebras and weight systems.

4.1. Lie algebras and weight systems. Let $\mathfrak{g}$ be a Lie algebra over some ground field $\mathbf{F}$, let t be a metric (ad-invariant symmetric non-degenerate quadratic form) on $\mathfrak{g}$, and let V be a representation of $\mathfrak{g}$. Given this information, one can construct a weight system [B-N1, B-N2]. Let us recall how this is done.

Choose some basis $\{\mathfrak{g}_a\}_{a=1}^{\dim \mathfrak{g}}$ of $\mathfrak{g}$. Let (t_{ab}) be the matrix corresponding to the metric t in the basis $\{\mathfrak{g}_a\}$; that is, $t_{ab} = t(\mathfrak{g}_a, \mathfrak{g}_b)$. Let the matrix (t^{ab}) be the inverse of the matrix (t_{ab}), and let $B \in (V^\star \otimes V) \otimes (V^\star \otimes V) = \mathrm{End}(V \otimes V)$ be given by

$$B = \sum_{a,b=1}^{\dim \mathfrak{g}} t^{ab} \mathfrak{g}_a \otimes \mathfrak{g}_b.$$

We will represent B symbolically by the diagram

(14) $B \longleftrightarrow$ 

With this notation for B, one can view a chord diagram of degree m as a recipe for how to contract m copies of B and get a tensor $\mathcal{T}(D) \in \operatorname{End} V$. This is best explained by an example; see figure 8.

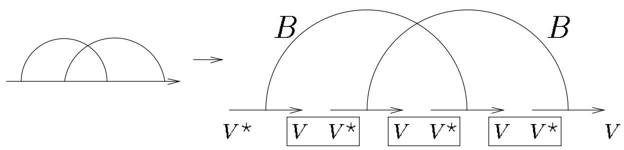
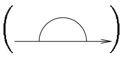


Figure 8. The construction of $\mathcal{T}(D)$. The B components are as in (14), and pairs of spaces surrounded by a box should be contracted. The two un-boxed spaces are $V^\star$ and V, and thus the result is a tensor in $V^\star \otimes V = \operatorname{End} V$.

One can show (see [B-N1, B-N2]) that the resulting tensor $\mathcal{T}(D)$ is independent of the choice of the basis of $\mathfrak{g}$ (indeed, already B is independent of that choice), is an intertwinner, and that the map $D \mapsto \operatorname{tr}\mathcal{T}(D)$ satisfies the $4T$ relation, and hence it descends to a map $W_{\mathfrak{g},V} : \mathcal{A} \to \mathbf{F}$ (the metric t is usually suppressed from the notation). If V is an irreducible representation and C is its quadratic Casimir number (the ratio $W_{\mathfrak{g},V}$ () $/W_{\mathfrak{g},V}$ ()), one can define

$$\hat{W}_{\mathfrak{g},V} = W_{\mathfrak{g}\oplus u(1),\hat{V}},$$

where $\hat{V} = V \otimes \sqrt{-C}$ and $\sqrt{-C}$ denotes the 1-dimensional representation of the 1-dimensional Lie algebra $u(1)$, in which the unit norm generator acts by multiplication by $\sqrt{-C}$. Notice that the representations V and $\hat{V}$ are in the same vector space, and that $\hat{W}_{\mathfrak{g},V}(D)$ can be computed using the same procedure as in figure 8, only everywhere replacing B by $\hat{B}$, where $\hat{B} = B - C \cdot I$.

Recall from section 2.2 that $J_{\mathfrak{g},V}(q)$ is the (framing dependent) Reshetikhin-Turaev knot invariant associated with the algebra $\mathfrak{g}$ and the representation V (and the metric t), and that (when V is irreducible) $\hat{J}_{\mathfrak{g},V}(q) = q^{-C\cdot writhe} \cdot J_{\mathfrak{g},V}(q)$ is its framing independent version. Consider both invariants as Vassiliev power series in the formal parameter $\hbar$ by substituting $q = e^\hbar$.

Proposition 4.1. *The weight system (in the sense of definition 2.2) of $J_{\mathfrak{g},V}$ is $W_{\mathfrak{g},V}$ and (when V is irreducible) the weight system of $\hat{J}_{\mathfrak{g},V}$ is $\hat{W}_{\mathfrak{g},V}$.*

Proof. The framing dependent part is in [Pi1]; it follows easily from the relation $R - R^{-1} = \hbar B + o(\hbar)$ satisfied by the quantum Yang-Baxter matrix R. The framing independent part follows from the fact [B-N2, exercise 6.33] that the weight system corresponding to a direct sum of Lie algebras (and tensor products of representations) is the product of the weight systems of the algebras (and representations) involved, and from a direct (and very simple) analysis of the weight system of $\exp(-\hbar C \cdot \mathit{writhe})$ and of the weight system $W_{u(1),\sqrt{-C}}$. $\square$

Let us now switch from general consideration to the particular case of $\mathfrak{g} = sl(2)$ and $V = V_\lambda$.

4.2. Understanding $\hat{B}$. In one of the standard models[6] of the representation V_λ, it is spanned by vectors $v_0, \dots, v_\lambda$, satisfying

$$hv_k = (\lambda - 2k)v_k,$$

$$yv_k = (k+1)v_{k+1}, \qquad \text{and} \qquad xv_k = (\lambda - k + 1)v_{k-1},$$

where

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{15}$$

is the standard basis of $sl(2)$. Using the standard scalar product on $sl(2)$ ($\langle M_1, M_2\rangle = \operatorname{tr}(M_1 M_2)$), we have $\frac{1}{2}\langle h, h\rangle = \langle x, y\rangle = \langle y, x\rangle = 1$, with all other scalar products between h, x, and y vanishing.

Therefore,

$$\hat{B} = y \otimes x + x \otimes y + \frac{1}{2} h \otimes h - C \cdot I,$$

where C, the quadratic Casimir number of V_λ, is given by $C = \lambda(\lambda+2)/2$ (see e.g. [Hu, exercise 4 in section 23]).

By an explicit computation, we find that

$$\hat{B}(v_k \otimes v_{k'}) = (k+1)(\lambda - k' + 1) v_{k+1} \otimes v_{k'-1} \tag{16}$$
$$+ (\lambda - k + 1)(k'+1) v_{k-1} \otimes v_{k'+1} + \frac{1}{2}\left((\lambda - 2k)(\lambda - 2k') - \lambda(\lambda+2)\right) v_k \otimes v_{k'}$$

$$= \lambda(B^+ + B^- + I)(v_k \otimes v_{k'}) + (\text{terms of degree 0 in } \lambda)\,, \tag{17}$$

where

$$B^+ = \sum_{\epsilon = 0,1} (-1)^\epsilon B^+_\epsilon; \qquad B^+_\epsilon(v_k \otimes v_{k'}) = -(k+1) v_{k+\epsilon} \otimes v_{k'-\epsilon},$$

[6] Here and later in this paper, we follow the notation of [Hu] for Lie algebras and their representations.

and

$$B^- = \sum_{\epsilon=0,1} (-1)^\epsilon B^-_\epsilon; \qquad B^-_\epsilon(v_k \otimes v_{k'}) = -(k'+1)v_{k-\epsilon} \otimes v_{k'+\epsilon}.$$

Proof of part 1 of theorem 1. Recall that $\hat{B} = \hat{B}(\lambda)$ depends on λ. We wish to study this λ dependence. The different $\hat{B}(\lambda)$'s lie in different spaces, but this is not a serious problem: Let $\hat{V}_\infty$ be the vector space spanned by infinitely many basis vectors $\{v_k\}_{k=0}^\infty$, and extend $\hat{B}(\lambda)$ for all λ to be elements of $\mathrm{End}(\hat{V}_\infty \otimes \hat{V}_\infty)$ using the explicit formula (16). For a chord diagram D, $\mathcal{T}(D) \in \mathrm{End}(\hat{V}_\infty)$ can be constructed as before as in figure 8, and when restricted to $\hat{V}_\lambda$, the new definition generalizes the old one.

Now that the different $\hat{B}(\lambda)$'s can be compared, equation (17) shows that $\hat{B}(\lambda)$ is at most linear in λ and thus $\mathcal{T}(D)$ is at most of degree m in λ. Taking the trace of an intertwinner (back again in $\hat{V}_\lambda$!) multiplies by $\lambda+1$, the dimension of $\hat{V}_\lambda$, and that factor is canceled by the denominator in (3). Finally, by the general considerations of section 2, the result on the level of knot invariants follows from the level of weight systems to the level of knot invariants. $\square$

4.3. Understanding W_{JJ}. Clearly, in computing $W_{JJ}(D)$ for some degree m chord diagram D, it is enough to consider $B^+ + B^- + I$, the coefficient of λ in $\hat{B}$. So let $T(D)$ be the operator constructed as in figure 8, only with $B^+ + B^- + I$ replacing B. As $\mathcal{T}(D)$ is an intertwinner, $T(D) = W_{JJ}(D)I$. Similarly, let $T'(D)$ be the same, only with $B^+ + B^-$ replacing B, and let $W'_{JJ}(D)$ satisfy $T'(D) = W'_{JJ}(D)I$. It is easy to verify that $W_{JJ} = W'_{JJ} \cdot W_1$, where the product is taken using the coproduct on $\mathcal{A}$ (the space spanned by chord diagrams), and $W_1 \in \mathcal{A}^\star$ satisfies $W_1(D) = 1$ for *any* chord diagram D.

Let D be a degree m chord diagram, and let $(C_\gamma)_{\gamma=1}^m$ be the chords of D, numbered from left to right as in definition 3.4. We are interested in computing $T(D)v_{k(1)}$, or, almost equivalently, $T'(D)v_{k(1)}$, for some non-negative integer $k(1)$. Looking again at figure 8 and at (17), we see that $T'(D)v_{k(1)}$ can be computed as follows:

- Sum over the 4^m possible ways of marking the chords $(C_\gamma)_{\gamma=1}^m$ of D by signs $s(\gamma) \in \{+,-\}$ and numbers $\epsilon(\gamma) \in \{0,1\}$, corresponding to the choice between $\{B_0^+, B_1^+, B_0^-, B_1^-\}$. Take the term marked by (s,ϵ) with a sign $\prod_\gamma (-1)^{\epsilon(\gamma)}$.
- For each fixed choice of (s,ϵ), add a term determined as follows: Set $k = k(1)$. 'Feed' the marked diagram $D^{(s,\epsilon)}$ with the vector v_k on the left, and push it right passing it through the vertices of D. Each vertex corresponds to some simple operation, dictated by the marking on the chord C_γ connected to it. The operation is to add or subtract $\epsilon(\gamma)$ to k, and to multiply by either 1 or $-(k+1)$, using the current value of k for the multiplication. The end result, as read at the right end of $D^{(s,\epsilon)}$, is proportional to the original $v_{k(1)}$; our term is the corresponding constant of proportionality.

To make the above algorithm more precise and write the result in a closed form, we need to make some definitions. First, number the vertices of D from left to right, beginning with 1 and ending with $2m$. Let i_γ^+ (i_γ^-) be the number of the left (right) end of the chord C_γ, and let the *domain* of C_γ be

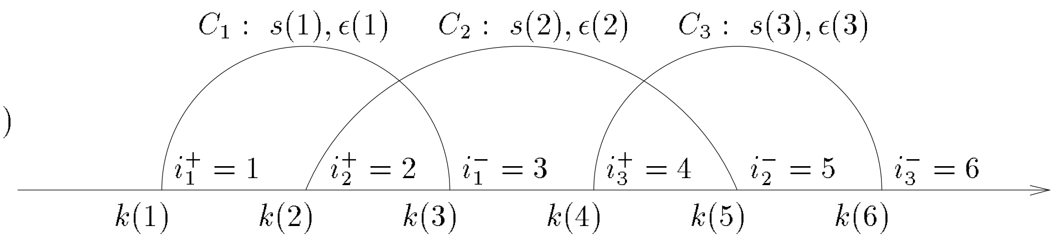
$$\operatorname{dom} C_\gamma = (i_\gamma^+, i_\gamma^-] = \{i \in \mathbf{N} : i_\gamma^+ \leq i < i_\gamma^-\}.$$

Let $k(i)$ be the value of k before passing the i'th vertex. It is easy to check that

$$k(i) = k(1) + \sum_{\{\gamma : i \in \operatorname{dom} C_\gamma\}} s(\gamma)\epsilon(\gamma).$$

Our notation is summarized by the following example:

(18)



Using this notation, the algorithm becomes the following formula:

$$W'_{JJ}(D) = (-1)^m \sum_{\substack{s \in \{+,-\}^m \\ \epsilon \in \{0,1\}^m}} \prod_{\gamma=1}^m (-1)^{\epsilon(\gamma)} \left(1 + k(i_\gamma^{s(\gamma)})\right)$$

Define the 'difference' operators $\delta/\delta\epsilon(\gamma)$ on polynomials P in the variables $\epsilon(\gamma)$, $\gamma = 1, \dots, m$ by

$$\frac{\delta P}{\delta \epsilon(\gamma)} = P|_{\epsilon(\gamma)=1} - P|_{\epsilon(\gamma)=0}. \tag{19}$$

With this definition,

$$W'_{JJ}(D) = (-1)^m \sum_{s \in \{+,-\}^m} \left(\prod_{\gamma=1}^m \frac{\delta}{\delta\epsilon(\gamma)} \right) \left(\prod_{\gamma=1}^m \left(1 + k(1) + \sum_{\{\delta : i_\gamma^{s(\gamma)} \in \operatorname{dom} C_\delta\}} s(\delta)\epsilon(\delta) \right) \right)$$

Notice that in the above formula we take the m'th partial difference (with respect to $\epsilon(1), \dots, \epsilon(m)$) of a polynomial of degree at most m in these variables. By an easy to prove partial difference analog of Taylor's theorem, the result is the coefficient of $\epsilon(1) \cdots \epsilon(m)$ in

$$(-1)^m \sum_{s \in \{+,-\}^m} \prod_{\gamma=1}^m \left(1 + k(1) + \sum_{\{\delta : i_\gamma^{s(\gamma)} \in \operatorname{dom} C_\delta\}} s(\delta)\epsilon(\delta) \right).$$

As only one $\epsilon(\delta)$ can be picked up from any factor in the product over $\gamma = 1, \dots, m$, this coefficient is the (properly signed) number of choices of an $\epsilon(\delta)$ for each of these γ's, or, in other words,

$$W'_{JJ}(D) = (-1)^m \sum_{s \in \{+,-\}^m} \sum_{\{\Delta \in S_m : \forall\gamma\ i_\gamma^{s(\gamma)} \in \operatorname{dom} C_{\Delta(\gamma)}\}} \prod_{\gamma=1}^{m} s(\Delta(\gamma)).$$

The condition in the summation over the permutation Δ can be made a little stronger. Notice that if for a given γ both $i_\gamma^+ \in \operatorname{dom} C_{\Delta(\gamma)}$ and $i_\gamma^- \in \operatorname{dom} C_{\Delta(\gamma)}$ (that is, both ends of the chord C_γ are within the domain of the chord $\Delta(\gamma)$), then the terms with $s(\gamma) = (+)$ cancel the terms with $s(\gamma) = (-)$ in the above sum, and thus summation can be restricted to the cases where this does not happen. In these cases, for each Δ there is a unique choice for the $s(\gamma)$'s for which $\forall\gamma\ i_\gamma^{s(\gamma)} \in \operatorname{dom} C_{\Delta(\gamma)}$. Denote this choice by $s(\Delta, \gamma)$ and get

$$W'_{JJ}(D) = \sum_{\{\Delta \in S_m : \forall\gamma\ C_\gamma \text{ intersects or equals } C_{\Delta(\gamma)}\}} \prod_{\gamma=1}^{m} (-s(\Delta, \gamma)).$$

Finally, if $\gamma = \Delta(\gamma)$, then necessarily $s(\gamma) = (+)$ and thus $s(\Delta, \gamma) = (+)$. This means that the possibility 'C_γ equals $C_{\Delta(\gamma)}$' can be removed from the above equation by multiplying it by W_1. Thus,

$$W_{JJ}(D) = \sum_{\{\Delta \in S_m : \forall\gamma\ C_\gamma \text{ intersects } C_{\Delta(\gamma)}\}} \prod_{\gamma=1}^{m} (-s(\Delta, \gamma)).$$

A moment's reflection shows that this formula proves the following proposition:

Proposition 4.2. *$W_{JJ}(D)$ is is the permanent* $\operatorname{per}\mathrm{IM}(D)$ *of the intersection matrix* $\mathrm{IM}(D)$ *of D. (Recall that the permanent of a matrix is defined as a sum over permutations in exactly the same way as the determinant, only without the signs).*

4.4. The logarithm of the *JJ* weight system.

Proposition 4.3.

$$(\log W_{JJ})(D) = \sum_H (-1)^{d(H)},$$

where the sum extends over all cycle decompositions of $\mathrm{LIG}(D)$ *into a single cycle.*

Proof. Expand $\operatorname{per}\mathrm{IM}(D)$ as a sum over permutations just as in (11), and get

$$W_{JJ}(D) = \sum_{H = \cup_\alpha H_\alpha} (-1)^{d(H)}.$$

Now take the logarithm as in proposition 3.13. □

Comparing this with proposition 3.13, we find that $\log W_C + \log W_{JJ} = 0$, concluding the proof of the Melvin-Morton-Rozansky conjecture.

5. The MMR conjecture for general semi-simple Lie algebras

Let $\hat{J} = \hat{J}_{\mathfrak{g},V_\lambda}(K) \in \mathbf{Q}(q)$ be the framing-independent Reshetikhin-Turaev invariant of the knot K for the semi-simple Lie algebra $\mathfrak{g}$ and the irreducible representation V_λ of $\mathfrak{g}$ of highest weight λ. (The metric on $\mathfrak{g}$ will be the Killing form $\langle\cdot,\cdot\rangle$). In this section we will prove an analog of theorem 1 (and thus of conjecture 1) for $\hat{J}$.

Choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, denote by Φ the set of all roots of $\mathfrak{g}$ in $\mathfrak{h}^\star$, and by Φ^+ the set of all positive roots. Let $\langle\cdot,\cdot\rangle$ also denote the scalar product on $\mathfrak{h}^\star$ induced by the Killing form.

The following theorem is suggested by the same reasoning as in section 1.4, only replacing $SU(2)$ by $\mathfrak{g}$. The main difference is that $\tau_{RS}(A_a)$ splits into a product of $\dim\mathfrak{g}$ Abelian torsions, rather than just 3. The torsions along the Cartan directions are still 1, while those along the negative roots pair with those along the positive roots to give a product of Alexander polynomials (appearing under the alias $\tilde{C}$, discussed in examples 2.7 and 2.8):

Theorem 4. *(Proven in sections 5.1–5.4). Regarding $\hat{J}(K)(e^\hbar)/\dim V_\lambda$ as a power series in $\hbar$ whose coefficients are polynomials in λ, we have:*

1. *The coefficient $\hat{J}_m$ of $\hbar^m$ is of degree at most m in λ.*
2. *If $JJ_\mathfrak{g}$ is the power series in $\hbar$ whose degree m coefficient is the homogeneous degree m piece of $\hat{J}_m$, then*

$$JJ_\mathfrak{g}(K)(\hbar)\cdot\prod_{\alpha\in\Phi^+}\tilde{C}(K)\left(\langle\lambda,\alpha\rangle\hbar\right) = 1. \tag{20}$$

(Since on a simple Lie algebra every invariant scalar product is a multiple of the Killing form and the left-hand-side of (20) is clearly multiplicative under taking the direct sum of Lie algebras, it follows that (20) still holds when $\langle\cdot,\cdot\rangle$ is replaced by an arbitrary invariant scalar product on $\mathfrak{g}$, in both the $\tilde{C}$ part of the equation and in the definition of $\hat{J}$.)

As in section 2, it is enough to prove theorem 4 on the level of weight systems. Furthermore, in the light of theorem 1, in order to prove (20) it is enough to prove that

$$W_{JJ,\mathfrak{g}} = \prod_{\alpha\in\Phi^+} W_{JJ}\circ\langle\lambda,\alpha\rangle^{\deg}, \tag{21}$$

where $\langle\lambda,\alpha\rangle^{\deg}$ is defined as in definition 2.3.

5.1. Lie-algebraic preliminaries. Let $\mathfrak{g} = \mathfrak{h}\oplus(\oplus_{\alpha\in\Phi}L_\alpha)$ be the root space decomposition of $\mathfrak{g}$. Recall (e.g. [Hu]) that $\mathfrak{h}$ is orthogonal to all the L_α's, that L_α is orthogonal to L_β unless $\alpha+\beta=0$ and that one can find $x_\alpha\in L_\alpha$, $y_\alpha\in L_{-\alpha}$, for all $\alpha\in\Phi$ so that

(22) Setting $h_\alpha = [x_\alpha, y_\alpha]$, the triple $\{x_\alpha, y_\alpha, h_\alpha\}$ spans a subalgebra of $\mathfrak{g}$ isomorphic to $sl(2)$ via the map $(x_\alpha, y_\alpha, h_\alpha) \mapsto (x, y, h)$, where $\{x, y, h\}$ are as in (15).

(23) $\langle x_\alpha, y_\alpha \rangle = 2/\langle \alpha, \alpha \rangle$.

(24) For any $\lambda \in \mathfrak{h}^\star$ and $\alpha \in \Phi \subset \mathfrak{h}^\star$, one has $\lambda(h_\alpha) = 2\langle \lambda, \alpha \rangle / \langle \alpha, \alpha \rangle$.

An additional property worth recalling is

(25) For any $\alpha, \beta \in \Phi$, $[L_\alpha, L_\beta] \subset L_{\alpha+\beta}$.

Choose a total ordering $<$ of Φ^+ for which $\alpha, \beta < \alpha + \beta$ for any $\alpha, \beta \in \Phi^+$. (For example, you can order the roots by the lengths of their projections on some generic vector in the fundamental Weyl chamber). Let $v_0 \in V_\lambda$ be a highest weight vector; that is, a vector satisfying $hv_0 = \lambda(h)v_0$ for all $h \in \mathfrak{h}$ and $x_\alpha v_0 = 0$ for all $\alpha \in \Phi^+$. Let $\mathbf{Z}_+\Phi^+ = \{\sum_{\alpha \in \Phi^+} k_\alpha \bar{\alpha} : \forall \alpha\, k_\alpha \in \mathbf{Z}_+\}$ be the semi-group of formal linear combinations of symbols $\bar{\alpha}$, one for each $\alpha \in \Phi^+$, with non-negative integer coefficients. Define a map $\{\cdot\} : \mathbf{Z}_+\Phi^+ \to \mathfrak{h}^\star$ by $\{\sum k_\alpha \bar{\alpha}\} = \sum k_\alpha \alpha$. Order $\mathbf{Z}_+\Phi^+$ lexicographically, that is, declare that $\sum k_\alpha \bar{\alpha} < \sum k'_\alpha \bar{\alpha}$ iff for some β, $k_\beta < k'_\beta$ and $k_\alpha = k'_\alpha$ for all $\alpha < \beta$. For any $k \in \mathbf{Z}_+\Phi^+$, set

$$v_k = \left(\prod_{\alpha \in \Phi^+} \frac{y_\alpha^{k_\alpha}}{k_\alpha!} \right) v_0, \tag{26}$$

where the k_α's are the coefficients of k and the product is taken using a decreasing order for the y_α's, so that, for example, if $\alpha > \beta$, then

$$v_k = \left(\cdots \frac{y_\alpha^{k_\alpha}}{k_\alpha!} \cdots \frac{y_\beta^{k_\beta}}{k_\beta!} \cdots \right) v_0. \tag{27}$$

The action of $\mathfrak{g}$ on V_λ is given by the following

Lemma 5.1. *With the notation as above we have that*

$$\begin{aligned}
hv_k &= (\lambda - \{k\})(h) v_k, && (28)\\
y_\alpha v_k &= (k_\alpha + 1) v_{k+\bar{\alpha}} + \sum_{\substack{j \in \mathbf{Z}_+\Phi^+ \\ j > k+\bar{\alpha}}} c_1(\alpha, k, j) v_j && (29)\\
x_\alpha v_k &= \frac{2}{\langle \alpha, \alpha \rangle} \langle \lambda, \alpha \rangle v_{k-\bar{\alpha}} + \sum_{\substack{j \in \mathbf{Z}_+\Phi^+ \\ j > k-\bar{\alpha}}} c_2(\lambda, \alpha, k, j) v_j + O(1), && (30)
\end{aligned}$$

where c_1 does not depend on λ, c_2 is linear in λ, and here and in the next few paragraphs $O(1)$ means terms independent of λ.

The importance of the precise form of the 'remainder terms' in the above lemma will be better understood after reading the proof of lemma 5.2. We therefore postpone the proof of lemma 5.1 to section 5.4.

5.2. Understanding $\hat{B}$. As in section 4, the key to understanding $W_{JJ,\mathfrak{g}}$ is to first understand $\hat{B} \in \mathrm{End}(\hat{V}_\lambda \otimes \hat{V}_\lambda)$, where $\hat{V}_\lambda = V_\lambda \otimes \sqrt{-C}$ and $\sqrt{-C}$ denotes the 1-dimensional representation of the 1-dimensional Lie algebra $u(1)$, in which the unit norm generator acts by multiplication by $\sqrt{-C}$, and C is the quadratic Casimir number of V_λ.

Let $\{h_i\}_{i=1}^r$ be an arbitrary $\langle\cdot,\cdot\rangle$-orthonormal basis of $\mathfrak{h}$. Using (23), we find that

$$\hat{B} = \sum_{\alpha\in\Phi^+} \frac{\langle\alpha,\alpha\rangle}{2}(x_\alpha\otimes y_\alpha + y_\alpha\otimes x_\alpha) + \sum_{i=1}^r h_i\otimes h_i - C\cdot I.$$

Since the quadratic Casimir number C of the representation V_λ is $\langle\lambda+2\rho,\lambda\rangle$, where $\rho = 1/2\sum_{\alpha\in\Phi^+}\alpha$ is half the sum of the positive roots [Hu, exercise 4 in section 23], we also have that

$$\begin{aligned}
&\left(\sum_{i=1}^r h_i\otimes h_i - C\right) v_k\otimes v_{k'} \\
&\quad= \Big(((\lambda-\{k\})\otimes(\lambda-\{k'\}))\Big(\sum h_i\otimes h_i\Big) - C\Big) v_k\otimes v_{k'} && \text{by lemma 5.1}\\
&\quad= (\langle\lambda-\{k\},\lambda-\{k'\}\rangle - \langle\lambda,\lambda+2\rho\rangle)\, v_k\otimes v_{k'} && \text{by Pythagoras' theorem}\\
&\quad= -\langle\lambda, 2\rho+\{k\}+\{k'\}\rangle v_k\otimes v_{k'} + O(1) \\
&\quad= -\sum_{\alpha\in\Phi^+}\langle\lambda,\alpha\rangle(1+k_\alpha+k'_\alpha)v_k\otimes v_{k'} + O(1) && \text{expanding } \rho,\ \{k\},\ \{k'\}.
\end{aligned}$$

Using the above formula and lemma 5.1 we get that

$$\hat{B} = \sum_{\alpha\in\Phi^+}\langle\lambda,\alpha\rangle(B_\alpha^+ + B_\alpha^- + I) + B_{rest} + O(1) \tag{31}$$

where

$$\begin{aligned}
B_\alpha^+(v_k\otimes v_{k'}) &= -(k_\alpha+1)\sum_{\epsilon=0,1}(-1)^\epsilon v_{k+\epsilon\bar{\alpha}}\otimes v_{k'-\epsilon\bar{\alpha}}\\
B_\alpha^-(v_k\otimes v_{k'}) &= -(k'_\alpha+1)\sum_{\epsilon=0,1}(-1)^\epsilon v_{k-\epsilon\bar{\alpha}}\otimes v_{k'+\epsilon\bar{\alpha}}\\
B_{rest}(v_k\otimes v_{k'}) &= \sum_{\substack{j,j'\in\mathbf{Z}_+\Phi^+\\ j+j'>k+k'}} c_3(\lambda,k,k',j,j')v_j\otimes v_{j'},
\end{aligned}$$

and where c_3 (which is a simple combination of $c_{1,2}$) is linear in λ.

Since $\hat{B}$ is at most linear in λ we conclude the first part of theorem 4 as in section 4.2.

5.3. Understanding $W_{JJ,\mathfrak{g}}$. Reading section 4.3 once again and looking at figure 8, we see that $W_{JJ,\mathfrak{g}}(D)$ is a certain summation over all the possible ways of labeling the chords of D by I, B_α^+, B_α^-, or B_{rest}.

Lemma 5.2. *In the summation making $W_{JJ,\mathfrak{g}}(D)$, terms containing a chord labeled by B_{rest} can be ignored.*

Proof. This statement is best proven by an example. Let $k(i)$ be the value of k before passing the i's vertex of D, as in (18) (but notice that now $k(i)$ is in $\mathbf{Z}_+\Phi^+$ rather than in $\mathbf{Z}_+$). Similarly, let $k(7)$ be the value of k after passing the sixth vertex (assuming, for the sake of this example, that D is the diagram in (18)). As $\mathcal{T}(D)$ is an intertwinner, it has to be a multiple of the identity and thus $k(7) = k(1)$. On the other hand, by (31) (and remembering that in as much as $W_{JJ,\mathfrak{g}}$ is concerned, we need not care about the $O(1)$ term), we find that

$$\begin{aligned} k(1) + k(3) &\geq k(2) + k(4), \\ k(2) + k(5) &\geq k(3) + k(6), \\ k(4) + k(6) &\geq k(5) + k(7). \end{aligned}$$

Adding these inequalities, we get $k(1) \geq k(7)$, and this inequality becomes strict if any of the previous ones is strict. As we know that $k(1) \geq k(7)$ cannot be strict, we learn that none of the previous ones is, and thus we can ignore B_{rest} (as it would correspond to a strict inequality). $\square$

Therefore, in computing $W_{JJ,\mathfrak{g}}(D)$, it is enough to consider

$$\sum_{\alpha\in\Phi^+} \langle\lambda, \alpha\rangle (B_\alpha^+ + B_\alpha^- + I). \tag{32}$$

Nicely enough, the different summands in (32) are 'decoupled'. For each α, $B_\alpha^\pm$ cares only about the α components of the $k(i)$'s, and changes only these components. This amounts to saying that $W_{JJ,\mathfrak{g}}$ is the product of the weight systems corresponding to the different summands. Comparing the definition of $B_\alpha^\pm$ with the definition of $B^\pm$ in section 4, we find that we've proven (21) and hence we've proven theorem 4. $\square$

5.4. Proof of lemma 5.1. (28) is just the well known statement that the y_α's act as 'lowering operators'. To prove (29), let us compute $y_\alpha \prod_\beta (y_\beta^{k_\beta}/k_\beta!)$ (using the same convention as in (27) for the ordering of products). To bring this expression to the form of (26), we need to commute y_α to its place, next to the term $y_\alpha^{k_\alpha}/k_\alpha!$. This done, the result is

$$\cdots y_\alpha \frac{y_\alpha^{k_\alpha}}{k_\alpha!} \cdots = \cdots (k_\alpha + 1) \frac{y_\alpha^{k_\alpha+1}}{(k_\alpha+1)!} \cdots,$$

explaining the first term in (29). However, en route to its place, we needed to commute y_α with various y_β's for which $\beta > \alpha$. By (25) and the choice of the order

$<$, such commutators are proportional to y_γ's with even bigger γ's, explaining the remainder term in (29). To be fair, the resulting y_γ's also need to be taken to their respective places, at the cost of some more commutators proportional to even bigger y_δ's, but that doesn't disturb (29). A complete argument can be given using the PBW theorem for the subalgebra of $\mathfrak{g}$ generated by $\{y_\beta : \beta > \alpha\}$, but we don't feel this is necessary.

The proof of (30) is a little harder, but goes along similar lines. Consider an expression like $x_\alpha \prod_\beta (y_\beta^{k_\beta}/k_\beta!)$. Commuting x_α all the way to the right, we get a product that kills the heighest weight vector v_0. Along the way, we pick up three kinds of commutators:

(1) First, we pick some $[x_\alpha, y_\beta]$'s, with $\beta > \alpha$. By (25), if $\beta > \alpha$, $[x_\alpha, y_\beta]$ is proportional to some y_γ, resulting in terms which are products of y's, and thus they fall into the third summand of (30), $O(1)$.

(2) We then pick the term containing $[x_\alpha, y_\alpha^{k_\alpha}]$, which, using (22), gives

$$\prod_{\beta>\alpha} \frac{y_\beta^{k_\beta}}{k_\beta!} \cdot \frac{1}{k_\alpha!} \left(\sum_{i=1}^{k_\alpha} y_\alpha^{i-1} h_\alpha y_\alpha^{k_\alpha - i} \right) \cdot \prod_{\beta<\alpha} \frac{y_\beta^{k_\beta}}{k_\beta!}.$$

By (28), applied to v_0 this is $\lambda(h_\alpha) v_{k-\bar{\alpha}} + O(1)$, and by (24), this is

$$\frac{2}{\langle \alpha, \alpha \rangle} \langle \lambda, \alpha \rangle v_{k-\bar{\alpha}} + O(1),$$

explaining the first term in (30).

(3) Finally, we get terms containing $[x_\alpha, y_\beta]$'s, with $\beta < \alpha$. By (25), if $\beta < \alpha$, $[x_\alpha, y_\beta]$ is proportional to some x_γ with $\gamma < \alpha$. Such x_γ are pushed to the right recursively using the same procedure we've used so far, at the cost of (at most) terms independent of λ and terms linear in λ, as in case (2) above, but with $v_{k-\bar{\gamma}}$ (or $v_{k-\bar{\delta}}$ for even smaller δ) replacing $v_{k-\bar{\alpha}}$. Such term fall into the middle term of (30). $\square$

6. Odds and ends

6.1. Immanants and the Conway polynomial. Theorem 3 and proposition 4.2 show (in particular) that both the map $D \mapsto \det \mathrm{IM}(D)$ and the map $D \mapsto \operatorname{per} \mathrm{IM}(D)$ are weight systems. It is tempting to look for common generalizations of these two weight systems. In this section, which may be of some independent interest, we sketch just such a generalization. The basic idea is that just where the character of the alternating representation of the symmetric group S_m is used in the definition of det and the character of the trivial representation is used in the definition of per, one can put the character of an arbitrary representation of S_m:

Definition 6.1. Let $[\sigma]$ denote the conjugacy class of a permutation σ. Let ZS_m be the free $\mathbf{Z}$-module generated by the conjugacy classes of S_m. Let $ZS_\star$ be the graded

$\mathbf{Z}$-module whose degree m piece is ZS_m. The natural embedding $\iota : S_m \times S_n \to S_{m+n}$ makes $ZS_\star$ an algebra by setting $[\sigma][\tau] = [\iota(\sigma, \tau)]$. Identifying $ZS_\star$ with its dual by declaring each individual conjugacy class $[\sigma]$ to be of unit norm, the product on $ZS_\star$ becomes a co-product on $ZS_\star^\star = ZS_\star$.

Exercise 6.2. Verify that with the above product and co-product $ZS_\star$ becomes a graded commutative and co-commutative Hopf algebra, and that the primitive elements of $ZS_\star$ are exactly the classes of cyclic permutations (and thus $ZS_\star$ has exactly one generator in each degree).

Definition 6.3. (Compare with [Lit]) Let M be an $m \times m$ matrix. The *universal immanant* $\operatorname{imm} M$ of M is defined by

$$\operatorname{imm} M = \sum_{\sigma \in S_m} [\sigma] \prod_{i=1}^{m} M_{i\sigma i} \in ZS_m.$$

(Exactly the same as the definitions of $\det M$, only with $[\sigma]$ replacing $(-1)^\sigma$).

Composing the universal immanant with characters of arbitrary representations of S_m, one gets specific complex valued "immanants". Taking the representation to be the alternating representation, one gets $\det M$. Taking it to be the trivial representation, one gets $\operatorname{per} M$. Much is known about many other immanants; see e.g. [GJ, St1, St2].

In our context, we will be interested in the universal immanant of the intersection matrix of a chord diagram. By abuse of notation, we will write $\operatorname{imm} D$ for $\operatorname{imm} \mathrm{IM}(D)$.

Theorem 5. (1) *The map* imm : $\{$*chord diagrams*$\} \to ZS_\star$ *descends to a well defined map* $\operatorname{imm} : \mathcal{A}^r \to ZS_\star$.
(2) *The thus defined* $\operatorname{imm} : \mathcal{A}^r \to ZS_\star$ *is a morphism of Hopf algebras.*
(3) *The image of the adjoint map* $\operatorname{imm}^\star : ZS_\star^\star = ZS_\star \to \mathcal{A}^{r\star} = \mathcal{W}$ *is the subalgebra of* $\mathcal{W}$ *generated by the weight systems of the coefficients of the Conway polynomial.*

Proof. (sketch) Let L_m be the degree m piece of $\log W_C$, and let $C_m \in S_m$ be a cyclic permutation. Re-interpreted in our new language, proposition 3.13 is simply the statement $\operatorname{imm}^\star[C_m] = -L_m$ and equation (13) becomes the multiplicativity of $\operatorname{imm}^\star$. It follows that the image of $\operatorname{imm}^\star$ is equal to the subalgebra of the algebra of functionals on chord diagrams generated by the L_m's. As L_m is known to be a weight system and the product of two weight systems is again a weight system, it follows that the image of $\operatorname{imm}^\star$ is in $\mathcal{W}$ and thus imm descends to $\mathcal{A}^r$. Finally notice that the algebra generated by the L_m's is equal to to algebra generated by the weight systems of the coefficients of the Conway polynomial. $\square$

It is easy to check (or deduce from theorem 5) that $\mathrm{imm}^\star[\sigma]=0$ if σ has a cycle of an odd length. By evaluating $\mathrm{imm}^\star[\sigma]$ on chord diagrams whose intersection graph is a union of polygons of an even number of sides, one can see that $\mathrm{imm}^\star$ restricted to permutations with no cycles of odd length is injective.

Exercise 6.4. Check that if $\mathrm{IM}(D)$ is replaced by $\mathrm{IM}(D)+\lambda I$ for any non-zero constant λ and $\mathcal{A}^r$ and $\mathcal{W}$ are replaced by $\mathcal{A}$ and $\mathcal{A}^\star$ in the statement of theorem 5, the theorem remains valid, with the unique element of $\mathcal{G}_1\mathcal{A}^\star$ adjoined to the generators of the image of $\mathrm{imm}^\star$.

6.2. A curious formula for the weight system of the colored Jones polynomial. (A sketch). The key to the understanding of W_{JJ} in section 4.3 was to rewrite (16) in a nicer form, equation (17). There is an even nicer form, however, that also includes the terms independent of λ: (suppressing '$\otimes$' symbols)

$$\begin{aligned}\hat{B}(v_kv_{k'}) =&\lambda\Big((k+1)(\underbrace{v_{k+1}v_{k'-1}-v_kv_{k'}}_{\text{part 1}})-(k'+1)(\underbrace{v_kv_{k'}-v_{k-1}v_{k'+1}}_{\text{part 2}})+v_kv_{k'}\Big)\\ &+(k-k')(\underbrace{v_{k+1}v_{k'-1}-v_{k-1}v_{k'+1}}_{\text{part 3}})+v_{k+1}v_{k'-1}+v_{k-1}v_{k'+1}\\ &-kk'(\underbrace{v_{k+1}v_{k'-1}-2v_kv_{k'}+v_{k-1}v_{k'+1}}_{\text{part 4}}).\end{aligned}$$

Following roughly the same steps as in section 4.3, parts 1 and 2 of the above equation become 'derivatives' like in (19). Part 3 also becomes a derivative, but with an additional factor of 2 as in it '$\Delta k=2$'. Part 4 becomes a 'second derivative', and all other parts remain '0th order'. These 'differentiations' mean that we want to look at the coefficients of certain monomials in the ϵ's of section 4.3, and when all the dust settles we remain with the following (completely self-contained) formula:

Theorem 6. *Let D be a chord diagram of degree m, and let $i^\pm_\gamma$ and $\operatorname{dom} C_\gamma$ be as in section 4.3. Let $\epsilon(\gamma)$ be commuting indeterminates, and let*

$$k(i)=\sum_{\{\gamma:i\in\operatorname{dom}C_\gamma\}}\epsilon(\gamma).$$

Then $W_{\hat{J}}(D)$ (the weight of D in the weight system of the framing-independent Reshetikhin-Turaev invariant of $sl(2)$ in the $(\lambda+1)$st dimensional representation) is the term independent of all the $\epsilon(\gamma)$'s in

$$(\lambda+1)\prod_{\gamma=1}^m\left((\lambda+2)\left(1+\frac{k(i^+_\gamma)-k(i^-_\gamma)}{\epsilon(\gamma)}\right)-2\frac{k(i^+_\gamma)k(i^-_\gamma)}{\epsilon(\gamma)^2}\right).\quad\square$$

Exercise 6.5. Deduce the equality $W_{JJ}(D)=\operatorname{per}\mathrm{IM}(D)$ from the above theorem.

Arguing similarly but starting from the 'framed' $B = x \otimes y + y \otimes x + h \otimes h/2$, one finds that the weight of D in the weight system of the framing-*dependent* Reshetikhin-Turaev invariant of $sl(2)$ in the $(\lambda+1)$st dimensional representation is the term independent of all the $\epsilon(\gamma)$'s in

$$(\lambda+1)\prod_{\gamma=1}^{m}\left((\lambda+2)\left(1+\frac{\lambda}{2}+\frac{k(i_\gamma^+)-k(i_\gamma^-)}{\epsilon(\gamma)}\right)-2\frac{k(i_\gamma^+)k(i_\gamma^-)}{\epsilon(\gamma)^2}\right).$$

Remark 6.6. Experimentally (on a computer) we found that the above formulas appear to be (by far) the best method for computing the corresponding weight systems. But, in some sense, we do not understand them very well:

(1) Our only proof that the above formulas satisfy the $4T$ relation is by tracing them back to $sl(2)$. It would be interesting to find a direct proof.
(2) We do not know how to generalize these formulas to other Lie algebras.
(3) We do not know how to view these formulas in the context of Rozansky's work. More specifically, it should be possible to push exercise 6.5 a little further and get formulas for the 'sub-diagonal' invariants $JJ_n = \sum_m b_{m-n,m}\hbar^m$ (for small n), and it should be possible to expand (6) in powers of $1/k$ using Feynman diagrams. The $1/k^n$ term in (6) should equal JJ_n. In this paper we dealt with the case $n=0$ but we don't know how to deal with higher values of n.

6.3. A further generalization. If, as conjectured in [B-N2], all weight systems come from Lie algebras, then there should be a way of stating and proving theorem 4 without any reference to Lie algebras. We do not have a precise analog of the statement; without a Lie algebra, it is not clear what λ is and in which space it should be. However, on the level of group representations, $\psi^n V_\lambda = V_{n\lambda} +$ (representations of a smaller highest weight), and thus the Adams operations ψ^n, which have a generalization to arbitrary weight systems [B-N2], can play a role similar to 'scaling λ'. We thus arrive at the following conjecture:

Conjecture 2. *Let W be an arbitrary weight system, let n be an integer, and let $\hat{W}^n = \widehat{\psi^{n\star}W}$ be the deframed version of $W \circ \psi^n$, where ψ^n is the nth Adams operation on chord diagrams. Then*

(1) *For any fixed chord diagram D of degree m, $\hat{W}^n(D)$ is a polynomial in n of degree at most m.*
(2) *Let $\hat{W}^{n,m}(D)$ be the degree m piece of $\hat{W}^n(D)$. Then the weight system $\hat{W}^{n,m}$ is in the algebra generated by the coefficients of the Conway polynomial.*

A similar statement should hold on the level of knot invariants, using the '0-framing' of a knot for the Adams operations.

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