MULTIVARIABLE KNOT POLYNOMIALS FROM BRAIDED HOPF ALGEBRAS WITH AUTOMORPHISMS

STAVROS GAROUFALIDIS AND RINAT KASHAEV

Abstract. We construct knot invariants from solutions to the Yang–Baxter equation associated to appropriately generalized left/right Yetter–Drinfel'd modules over a braided Hopf algebra with an automorphism. When applied to Nichols algebras, our method reproduces known knot polynomials and naturally produces multivariable polynomial invariants of knots. We discuss in detail Nichols algebras of rank 1 which recover the ADO and the colored Jones polynomials of a knot and two sequences of examples of rank 2 Nichols algebras, one of which starts with the product of two Alexander polynomials, and then conjecturally the Harper polynomial. The second sequence starts with the Links–Gould invariant (conjecturally), and then with a new 2-variable knot polynomial that detects chirality and mutation, and whose degree gives sharp bounds for the genus for a sample of 30 computed knots.

Contents

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Key words and phrases: Knots, Jones polynomial, ADO invariant, R-matrices, Yang–Baxter equation, Hopf algebras, braided Hopf algebras, tensor algebras, braided tensor algebras, quantum groups, Yetter–Drinfel'd modules, Nichols algebras, chirality, mutation, knot genus.

1. Introduction

Jones's discovery of his famous polynomial of knots had an enormous influence in knot theory and connected the subject of low dimensional topology and hyperbolic geometry to mathematical physics, giving rise to quantum topology [Jon87, Thu77, Wit89].

The Jones polynomial was originally defined by thinking of a knot as the closure of a braid, and by taking the (suitably normalized) trace of representations of the braid groups (with an arbitrary number of strands), which themselves were determined by a vector space V and an automorphism $R \in \operatorname{Aut}(V \otimes V)$ that satisfies the Yang–Baxter equation

$$R_1 R_2 R_1 = R_2 R_1 R_2 \in \operatorname{End}(V \otimes V \otimes V)\,, \tag{1}$$

where $R_1 = R \otimes I, R_2 = I \otimes R$.

It was soon realized that representations of simple Lie algebras and their deformations, known as quantum groups, were a natural source of solutions to the Yang–Baxter equations. This led to a plethora of polynomial invariants of knots; see for example Turaev [Tur88, RT90].

Another source of polynomial invariants (one for every complex root of unity) came from the work of Akutsu–Deguchi–Ohtsuki [ADO92]. It was conjectured by Habiro [Hab08, Conj.7.4] and later shown by Willets in [Wil22] that the collection of the colored Jones polynomials of a knot (colored by the irreducible representations of $\mathfrak{sl}_2$) determines and is determined by the collection of ADO invariants at roots of unity.

The definition of the above invariants requires an R-matrix together with a (ribbon) enhancement of it which, roughly speaking, is an endomorphism of V required to define the quantum trace, and hence the knot invariant. This comes from the Reshetikhin–Turaev functor which forms the basis of knot/link invariants in arbitrary 3-manifolds [RT90].

An R-matrix alone is in principle sufficient to define knot invariants. This was clarified by the second author by constructing invariants of knots from an R-matrix that satisfies some non-degeneracy conditions, called rigidity in [Kas21]. Rigid R-matrices indeed allow one to define state-sum invariants of planar projections of knots without any extra data. A description of how these invariants are defined is given in Section 2 below.

One can construct rigid R-matrices from any Hopf algebra with invertible antipode through Drinfel'd's quantum double construction which can be put into pure algebraic setting of multilinear algebra without finiteness assumptions of the Hopf algebra [Kas23].

In this paper, we propose a different approach of producing rigid R-matrices that does not use the quantum double of a Hopf algebra. The construction of these R-matrices, and the corresponding knot invariants, is schematically summarized in the following steps:

$$\left\{\begin{matrix}\text{Braided}\\ \text{Hopf algebras}\\ \text{with autos}\end{matrix}\right\} \to \left\{\begin{matrix}\text{Braided}\\ \text{left/right YD modules}\\ \text{with automorphisms}\end{matrix}\right\} \to \{R\text{-matrices}\} \to \{\text{Knot invariants}\} \quad (2)$$

The last arrow in (2) is the well-known Reshetikhin–Turaev functor reviewed in Section 2. The first and the second arrows are discussed in Sections 3.1 and 3.2 below. These sections are written in the maximum level of abstraction, using the language of category theory, for potential future applications to braided categories not coming from vector spaces.

The knot invariants defined by the steps in (2) require as input a braided Hopf algebra with automorphisms. A concrete source of braided Hopf algebras with a rich group of automorphisms are the Nichols algebras discussed in detail in Section 5. Roughly speaking, a Nichols algebra is the quotient of a naturally graded tensor algebra of a braided vector space by a suitable grading preserving maximal Hopf ideal. Any choice of an automorphism gives rise to appropriately generalized left and right Yetter–Drinfel'd module structures over the Nichols algebra. It turns out that the latter admits a natural quotient space in the case of the left generalized Yetter–Drinfel'd module and a natural submodule in the case of the right generalized Yetter–Drinfel'd module. In a sense, the choice of the braided Hopf algebra automorphism seems to correspond to a choice of a quantum double representation in the traditional approach.

In Sections 6 and 7 we study these generalized Yetter–Drinfel'd modules of Nichols algebras of diagonal type in the cases when the input braided vector space is of dimension one and two, which gives rise respectively to one and two-variable polynomial invariants of knots.

We end this introduction with some further comments.

1. An important feature of our construction is a braided Hopf algebra with an automorphism. The nontriviality of the automorphism is an essential part for constructing nontrivial knot invariants. In a sense the group of automorphisms replaces the representation theory.
2. Our approach unifies previous constructions of knot invariants (notably the colored Jones and the ADO polynomials, the Links-Gould and the Harper polynomials) coming from super/quantum groups, but also leads to a systematic construction of multivariable polynomial invariants of knots beyond the quantum groups.
3. A feature of the knot polynomials that we construct is that they depend on variables coming both from the braiding and the automorphism of the braided Hopf algebra. It is likely that some of our polynomial invariants of knots coincide with conjectured knot invariants that are discussed in the physics literature; see for instance the work of Gukov et al [GHN+21].
4. We expect that some of these invariants come from finite type invariants of knots [BN95], though we have not investigated this at the moment.
5. We expect that our polynomial invariants give lower bounds for the Seifert genus of a knot (as is known for the classical Alexander polynomial, but also in some other cases already, see [NvdV, KT]).
6. Regarding q-holonomic aspects, we expect our invariants to satisfy the analogous q-holonomic properties (that is, linear q-difference equations), as those that come from quantum groups (such as the colored Jones polynomials associated to a simple Lie algebra and

parametrized by weights of irreducible representations [GL05]) or those defined at roots of unity (such as the ADO invariant [BDGG]).

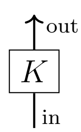
7. Finally, regarding asymptotic aspects, we expect that our invariants satisfy versions of the Volume Conjecture, analogous to those of the colored Jones polynomials [Kas97, MM01] and the ADO invariants [Mur08].

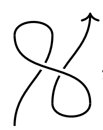
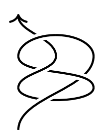
2. From R-matrices to knot invariants

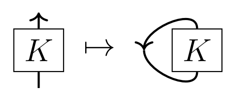
In this section we briefly describe the Reshetikhin–Turaev functor which allows to construct knot invariants from R-matrices. These invariants are defined by state-sums [RT90], using a variation of the construction from the second author's paper [Kas21]. There are three ingredients involved in this construction, namely suitable knot diagrams, rigid R-matrices, and the corresponding state-sums.

2.1. Knot diagrams and rigid R-matrices. We use a diagrammatic notation which is very important for the construction of knot invariants and has a long and successful history in knot theory [RT90, Tur94]. Basically, knots are represented by generic planar projections composed of local pieces which correspond to structural morphisms of a braided vector space, while the compatibility conditions ensure invariance under changes of planar projections. The notation leads naturally to the concept of a braided monoidal category, not necessarily in an abelian category, that vastly generalizes the notion of a braided vector space [TV17].

Following [Kas21], we now explain concretely the knot diagrams used. An (oriented) long knot diagram K is an oriented knot diagram in $\mathbb{R}^2$ with two open ends called "in" and "out":

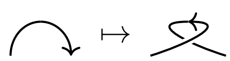
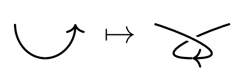
$K =$ 

Examples: $K =$ , $K =$ .

A long knot diagram can be closed to a planar projection of a knot: 

The vertical direction plays a preferred role for long knot diagrams.

The normalization $\dot{K}$ of K is the diagram obtained from K by the replacements of local extrema oriented from left to right

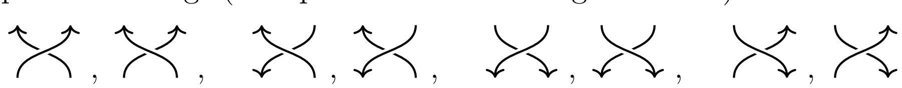
 and  (3)

at all posible locations of K. We say that K is normal if $K = \dot{K}$.

The building blocks of normal diagrams are given by four types of segments

 (4)

and eight types of crossings (four positive and four negative ones)

 (5)

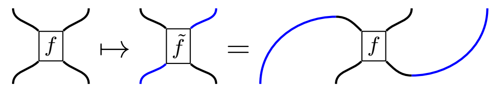
We next define R-matrices and their rigid version. An R-matrix over a vector space V is an automorphism $r \in \mathrm{Aut}(V \otimes V)$ of $V \otimes V$ that satisfies the quantum Yang–Baxter relation

$$r_1 r_2 r_1 = r_2 r_1 r_2, \qquad r_1 := r \otimes \mathrm{id}_V, \; r_2 := \mathrm{id}_V \otimes r\,. \tag{6}$$

Let V^* denote the dual vector space and $\langle \cdot, \cdot \rangle : V^* \otimes V \to \mathbb{F}$ denote the natural evaluation map. Assume that V is a finite-dimensional, and fix a basis B of V and the corresponding dual basis $\{b^*\}_{b \in B}$ of V^*.

Given $f \in \mathrm{End}(V \otimes V)$, we define its partial transpose $\tilde{f} \colon V^* \otimes V \to V \otimes V^*$ by

$$\tilde{f}(a^* \otimes b) = \sum_{c,d \in B} \langle a^* \otimes c^*, f(b \otimes d) \rangle c \otimes d^*, \tag{7}$$



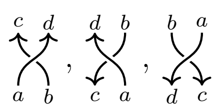
We call an R-matrix r rigid if $\widetilde{r^{\pm 1}}$ are invertible.

2.2. **State-sum invariants of knots.** We now have all the ingredients to define the state-sum invariants of normal knot diagrams. Fix a rigid R-matrix r over a finite dimensional vector space V, equipped with a basis B. For a normal long knot diagram K, let E_K and C_K denote its sets of edges and crossings, respectively.

A state s of K is a map $s \colon E_K \to B$ that assigns an element of B to each edge of K. The weight $w_s(K)$ of the state s of K is the product of local weights

$$w_s(K) = \prod_{c \in C_K} w_s(c)\,, \tag{8}$$

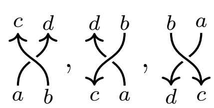
where the local weights are defined by

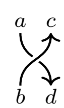


$$\overset{w_s}{\longmapsto} \langle c^* \otimes d^*, r(a \otimes b) \rangle,$$


$$\overset{w_s}{\longmapsto} \left\langle a \otimes c^*, (\widetilde{r^{-1}})^{-1}(b \otimes d^*) \right\rangle \tag{9}$$

for positive crossings and likewise for negative crossings



$$\overset{w_s}{\longmapsto} \langle c^* \otimes d^*, r^{-1}(a \otimes b) \rangle,$$


$$\overset{w_s}{\longmapsto} \left\langle a \otimes c^*, (\widetilde{r})^{-1}(b \otimes d^*) \right\rangle. \tag{10}$$

These arrangements of the R-matrices at the crossings are the same as in [Kas21, Eqns.(16)-(19)].

The main theorem of this construction is the topological invariance of the state-sum; see [Res89, RT90, Tur94] and in the form stated below, [Kas21, Thm.1].

Theorem 2.1. *Let a normal long knot diagram K have an equal number of negative and positive crossings. Then, the linear map*

$$J_r(K) \colon V \to V, \quad J_r(K)a = \sum_{s \in B^{E_K},\ s_{in} = a} w_s(K) s_{out} \tag{11}$$

is independent of the basis of V and it is thus an End(V)-valued invariant of oriented knots.

When the vector space V is equipped with a basis B (as in all of our examples below), then the oriented knot invariant above is a matrix-valued invariant.

Note that this construction can be extended to the context of arbitrary monoidal categories with duality.

The above invariant should not be confused with the universal invariants of knots taking values in quotients of completed Hopf algebras that come from quantum groups, considered by Lawrence [Law90], Ohtsuki [Oht93] and Habiro [Hab08].

3. From braided Hopf algebras with automorphisms to R-matrices

3.1. **From Hopf f-objects to left/right Yetter–Drinfel'd f-objects.** In this section we discuss the left arrow of (2). We deliberately phrase our results in the language of braided monoidal (not-necessarily abelian) categories to allow versatility of future applications. A detailed discussion of the concepts of a monoidal category, braided monoidal category, rigid monoidal category, category of functors, algebra and coalgebra objects in monoidal categories, modules over algebra objects and comodules over coalgebra objects and their morphisms can be found in the book by Turaev–Virelizier [TV17, Sec.1.6].

All monoidal categories that we consider are assumed to be strict. In writing compositions of morphisms $g : X \to Y$ and $f : Y \to Z$ in a category, we will suppress the composition symbol, so that we write fg instead of $f \circ g$. Moreover, in the case of monoidal categories, we assume the preference of the composition over the monoidal product, so that, for example, $fg \otimes h$ will mean $(fg) \otimes h$.

When a functor $F : \mathcal{D} \to \mathcal{C}$ is considered as an object of the functorial category $\mathcal{C}^{\mathcal{D}}$, it will be called *functorial object* or just *f-object* for brevity.

Let $\mathcal{C}$ be a braided monoidal category. Denote by $\mathcal{C}^{\mathbb{Z}}$ the braided monoidal category of functors $F : \mathbb{Z} \to \mathcal{C}$ where the additive group of integers $\mathbb{Z}$ is viewed as a category with one object $*$ whose automorphism group is $\mathbb{Z}$. We denote by $\tau : \otimes \to \otimes^{\mathrm{op}}$ the braiding of $\mathcal{C}^{\mathbb{Z}}$ which assigns to any pair of f-objects F and G a functorial morphism $\tau_{F,G} : F \otimes G \to G \otimes F$ that at the unique object $*$ of $\mathbb{Z}$ evaluates to the morphism of $\mathcal{C}$

$$(\tau_{\mathcal{C}})_{F(*),G(*)} : F(*) \otimes G(*) \to G(*) \otimes F(*)$$

where $\tau_{\mathcal{C}}$ is the braiding in $\mathcal{C}$.

Remark 3.1. Given the fact that the group $\mathbb{Z}$ is freely generated by one element 1, an object G of the functor category $\mathcal{C}^{\mathbb{Z}}$ is uniquely determined by the pair (A, ϕ) where A is the object of $\mathcal{C}$ obtained as the image by G of the unique object $*$ of $\mathbb{Z}$, and $\phi : A \to A$ is the automorphism of A obtained as the image by G of the generating element 1 of $\mathbb{Z}$. With this interpretation, a morphism from (A, ϕ) to (B, ψ) is a morphism $f : A \to B$ in $\mathcal{C}$ such that $\psi f = f\phi$. The monoidal product of two pairs $(A, \phi) \otimes (B, \psi)$ is given by the pair $(A \otimes B, \phi \otimes \psi)$.

Definition 3.2. A *Hopf f-object* is an f-object $H : \mathbb{Z} \to \mathcal{C}$ together with functorial morphisms (natural transformations)

$$\nabla : H \otimes H \to H, \quad \eta : \mathbb{I} \to H, \quad \Delta : H \to H \otimes H, \quad \epsilon : H \to \mathbb{I}, \quad S : H \to H \tag{12}$$

such that (H, ∇, η) is an algebra f-object, (H, Δ, ϵ) is a coalgebra f-object and

$$(\nabla \otimes \nabla)(\mathrm{id}_H \otimes \tau_{H,H} \otimes \mathrm{id}_H)(\Delta \otimes \Delta) = \Delta\nabla, \tag{13}$$

$$\nabla(S \otimes \mathrm{id}_H)\Delta = \eta\epsilon = \nabla(\mathrm{id}_H \otimes S)\Delta. \tag{14}$$

We will always assume that S is an invertible (functorial) morphism. As in the theory of Hopf algebras, the functorial morphisms ∇, η, Δ, ϵ and S are respectively called product, unit, coproduct, counit and antipode.

Definition 3.3. Let $H\colon \mathbb{Z} \to \mathcal{C}$ be a Hopf f-object. A *left Yetter–Drinfel'd f-object* over H is a triple (Y, λ_L, δ_L) where $Y\colon \mathbb{Z} \to \mathcal{C}$ is an f-object of $\mathcal{C}^{\mathbb{Z}}$, and $\lambda_L\colon H\otimes Y \to Y$, $\delta_L\colon Y \to H\otimes Y$ are morphisms of $\mathcal{C}^{\mathbb{Z}}$ such that (Y, λ_L) is a left H-module f-object, (Y, δ_L) is a left H-comodule f-object, and

$$\begin{aligned}(\nabla \otimes \mathrm{id}_Y)(\mathrm{id}_H \otimes\tau_{Y,H})(\delta_L\lambda_L \otimes \phi_H)(\mathrm{id}_H \otimes\tau_{H,Y})(\Delta \otimes \mathrm{id}_Y)\\ = (\nabla \otimes \lambda_L)(\mathrm{id}_H \otimes\tau_{H,H} \otimes \mathrm{id}_Y)(\Delta \otimes \delta_L)\end{aligned} \tag{15}$$

where $\phi_H\colon H \to H$ is the functorial isomorphism that at the unique object $*$ of $\mathbb{Z}$ evaluates as

$$(\phi_H)_* = H(1)\colon H(*) \to H(*).$$

Taking into account the self-dual nature of Hopf objects, it is useful to have the dual version of Definition 3.3 which reads as follows.

Definition 3.4. A *right Yetter–Drinfel'd f-object* over a Hopf f-object $H\colon \mathbb{Z} \to \mathcal{C}$ is a triple (Y, λ_R, δ_R) where $Y\colon \mathbb{Z} \to \mathcal{C}$ is an f-object of $\mathcal{C}^{\mathbb{Z}}$, and $\lambda_R\colon Y \otimes H \to Y$, $\delta_R\colon Y \to Y \otimes H$ are functorial morphisms of $\mathcal{C}^{\mathbb{Z}}$ such that (Y, λ_R) is a right H-module f-object, (Y, δ_R) is a right H-comodule f-object, and

$$\begin{aligned}(\mathrm{id}_Y \otimes\nabla)(\tau_{H,Y} \otimes \mathrm{id}_H)(\phi_H \otimes \delta_R\lambda_R)(\tau_{Y,H} \otimes \mathrm{id}_H)(\mathrm{id}_Y \otimes\Delta)\\ = (\lambda_R \otimes \nabla)(\mathrm{id}_Y \otimes\tau_{H,H} \otimes \mathrm{id}_H)(\delta_R \otimes \Delta)\,.\end{aligned} \tag{16}$$

We will return and give further clarifications to these definitions later in Subsection 4.2 after introducing the graphical notation of string diagrams.

For a Hopf f-object $H\colon \mathbb{Z} \to \mathcal{C}$, we denote by $\Delta^{(2)}$ and $\nabla^{(2)}$ the twice iterated coproduct and product, respectively, defined by

$$\begin{aligned}\nabla^{(2)} : H \otimes H \otimes H \to H, \qquad & \nabla^{(2)} = \nabla(\nabla \otimes \mathrm{id}_H)\\ \Delta^{(2)} : H \to H \otimes H \otimes H, \qquad & \Delta^{(2)} = (\Delta \otimes \mathrm{id}_H)\Delta\,.\end{aligned} \tag{17}$$

The following theorem provides constructions of left/right Yetter–Drinfel'd f-objects over a Hopf f-object $H\colon \mathbb{Z} \to \mathcal{C}$.

Theorem 3.5. *For any Hopf f-object $H\colon \mathbb{Z} \to \mathcal{C}$,*
(a) the triple (H, ∇, δ_L) is a left Yetter–Drinfel'd f-object over H, where

$$\delta_L := (\nabla \otimes \mathrm{id}_H)(\mathrm{id}_H \otimes\tau_{H,H})(\mathrm{id}_{H\otimes H} \otimes S\phi_H)\Delta^{(2)}; \tag{18}$$

(b) the triple (H, λ_R, Δ) is a right Yetter–Drinfel'd f-object over H, where

$$\lambda_R := \nabla^{(2)}(S\phi_H \otimes \mathrm{id}_{H\otimes H})(\tau_{H,H} \otimes \mathrm{id}_H)(\mathrm{id}_H \otimes\Delta)\,. \tag{19}$$

3.2. **From left/right Yetter–Drinfel'd f-objects to R-matrices.** The next theorem constructs R-matrices from left/right Yetter–Drinfel'd f-objects corresponding to the second arrow in (2).

Theorem 3.6. *Let $H: \mathbb{Z} \to \mathcal{C}$ be a Hopf f-object and (Y, λ, δ) be a left, respectively a right, Yetter–Drinfel'd f-object over H. Then*

$$\rho_L = (\lambda \otimes \mathrm{id}_Y)(\mathrm{id}_H \otimes \tau_{Y,Y})(\delta \otimes \phi_Y), \tag{20}$$

respectively

$$\rho_R = (\phi_Y \otimes \lambda)(\tau_{Y,Y} \otimes \mathrm{id}_H)(\mathrm{id}_Y \otimes \delta), \tag{21}$$

is an R-matrix, that is a solution of the following braid group type Yang–Baxter relation in the automorphism group $\mathrm{Aut}(Y \otimes Y \otimes Y)$:

$$(\rho \otimes \mathrm{id}_Y)(\mathrm{id}_Y \otimes \rho)(\rho \otimes \mathrm{id}_Y) = (\mathrm{id}_Y \otimes \rho)(\rho \otimes \mathrm{id}_Y)(\mathrm{id}_Y \otimes \rho). \tag{22}$$

Moreover, this R-matrix is rigid if the f-object Y is rigid.

The proof of these theorems is given in the next section, using a diagrammatic calculus. A corollary of Theorem (3.6) gives an invariant of knots.

Theorem 3.7. *Fix a rigid left or a right Yetter–Drinfel'd f-object Y over a Hopf f-object H with corresponding R-matrix ρ. Then, there exists a knot invariant*

$$\{\textit{Knots in } S^3\} \to \mathrm{End}(Y), \qquad K \mapsto J_\rho(K)\,. \tag{23}$$

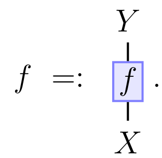
4. Proofs

4.1. **Diagrammatics of braided Hopf algebras with automorphisms.** The Hopf f-objects introduced in Section 3 are categorical versions of pairs (H, ϕ) where H is a braided Hopf algebra and ϕ is an automorphism of H. At around the same time of the Reshetikhin–Turaev construction of knot invariants via diagrammatics, there was a parallel intense activity in the theory of Hopf algebras motivated in part by the theory of quantum groups developed by Drinfel'd [Dri87] and Jimbo [Jim86]. There is a string diagrammatic calculus designed to prove tensor identities in Hopf algebras that avoids using explicit coordinate formulas for the tensors involved.

This string diagrammatic calculus extends to the case of braided Hopf algebras, introduced by Majid around 1990 [Maj94, Maj95], and used extensively by many authors including Radford, Kuperberg and Kauffman [Rad12, Kup91, KR95]. A survey of the various directions of braided Hopf algebras around 2000 is given by Takeuchi in [Tak00].

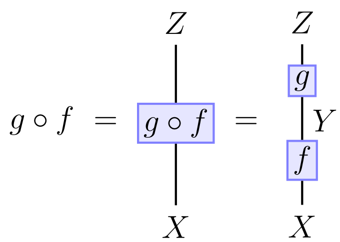
The string diagrammatics of the generators and relations of a Hopf algebra are given in [Maj95]. For a recent treatement, see [Kas23], namely, Eqns. (1.81)-(1.85) for the generators, Eqns. (1.86)-(1.91) for the relations and Eqns. (1.68)-(1.73) for the diagrammatic notation. For the convenience of the reader, we recall below the definitions of these morphisms, relations, and the string diagrammatic notation.

Let $\mathcal{C}$ be a category. To any morphism $f\colon X \to Y$ in $\mathcal{C}$, we associate a graphical picture



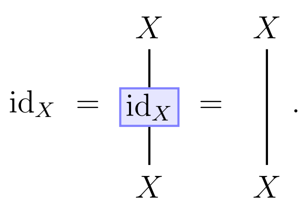
(24)

If $f\colon X \to Y$ and $g\colon Y \to Z$ are two composable morphisms, then their composition is described by the vertical concatenation of graphs



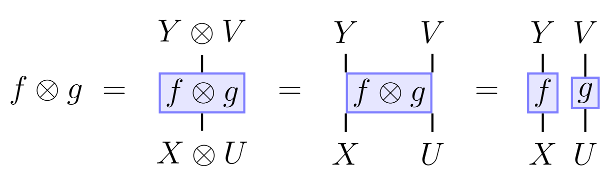
(25)

In particular, for the identity morphism id_X it is natural to use just a line



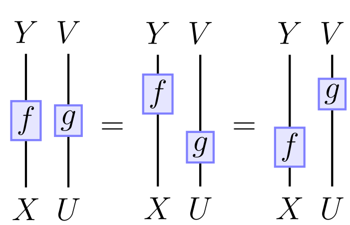
(26)

The string diagrams are especially useful in the case when $\mathcal{C}$ is a strict monoidal category, because the tensor (monoidal) product can be drawn by the horizontal juxtaposition. Namely, for two morphisms $f\colon X \to Y$ and $g\colon U \to V$, their tensor product $f \otimes g\colon X \otimes U \to Y \otimes V$ is drawn as follows:



(27)

For example, the graphical equalities



(28)

correspond to the well known relations in the tensor calculus

$$f \otimes g = (f \otimes \mathrm{id}_V)(\mathrm{id}_X \otimes g) = (\mathrm{id}_Y \otimes g)(f \otimes \mathrm{id}_U). \tag{29}$$

By taking into account the distinguished role of the identity object I, it is natural to associate to it the empty graph.

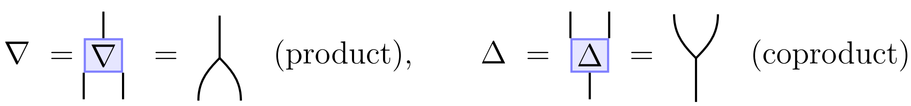
Let $\mathcal{C}$ be a symmetric monoidal category with tensor product $\otimes$, the opposite tensor product $\otimes^{\mathrm{op}}$, unit object I and symmetry $\sigma\colon \otimes \to \otimes^{\mathrm{op}}$. Recall that a Hopf object in $\mathcal{C}$ is an object H endowed with five structural morphisms

$$\nabla\colon H\otimes H\to H,\quad \eta\colon I\to H,\quad \Delta\colon H\to H\otimes H,\quad \epsilon\colon H\to I,\quad S\colon H\to H \tag{30}$$

called, respectively, product, unit, coproduct, counit and antipode, that satisfy the following relations or axioms

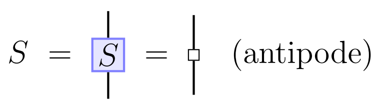
$$\text{associativity}:\ \nabla(\nabla\otimes \mathrm{id}_H)=\nabla(\mathrm{id}_H\otimes\nabla) \tag{31a}$$
$$\text{coassociativity}:\ (\Delta\otimes \mathrm{id}_H)\Delta=(\mathrm{id}_H\otimes\Delta)\Delta \tag{31b}$$
$$\text{unitality}:\ \nabla(\eta\otimes \mathrm{id}_H)=\mathrm{id}_H=\nabla(\mathrm{id}_H\otimes\eta) \tag{31c}$$
$$\text{counitality}:\ (\epsilon\otimes \mathrm{id}_H)\Delta=\mathrm{id}_H=(\mathrm{id}_H\otimes\epsilon)\Delta \tag{31d}$$
$$\text{invertibility}:\ \nabla(\mathrm{id}_H\otimes S)\Delta=\eta\epsilon=\nabla(S\otimes \mathrm{id}_H)\Delta \tag{31e}$$
$$\text{compatibility}:\ (\nabla\otimes\nabla)(\mathrm{id}_H\otimes\sigma_{H,H}\otimes \mathrm{id}_H)(\Delta\otimes\Delta)=\Delta\nabla. \tag{31f}$$

Let us introduce the following graphical notation for the structural maps of H (all lines correspond to the object H)



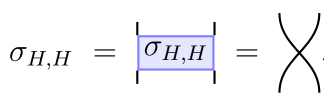
(32)

(33)



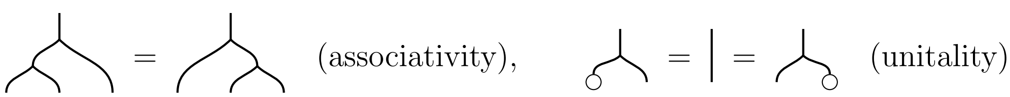
(34)

We complete this with the graphical notation for the symmetry

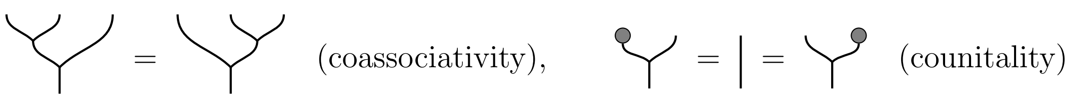


(35)

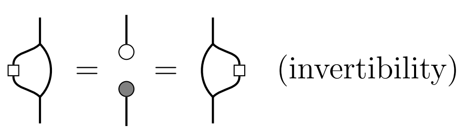
The relations or axioms of a Hopf object take the following graphical form:



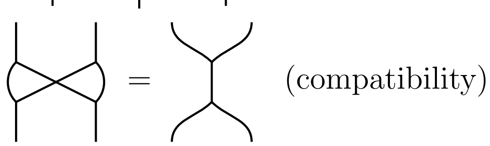
(36)



(37)



(38)



(39)

Our first refinement is the notion of a braided Hopf algebra in a braided monoidal category. It generalizes the notion of a Hopf object (defined in the context of a symmetric monoidal category) by replacing the symmetry $\sigma_{H,H}$ in the compatibility axiom (31f), (39) by the braiding $\tau = \tau_{H,H} : H \otimes H \to H \otimes H$ that satisfies the Yang–Baxter equation

$$(\tau \otimes \mathrm{id}_H)(\mathrm{id}_H \otimes \tau)(\tau \otimes \mathrm{id}_H) = (\mathrm{id}_H \otimes \tau)(\tau \otimes \mathrm{id}_H)(\mathrm{id}_H \otimes \tau). \tag{40}$$

In other words, a braided Hopf object (in a braided monoidal category) is defined by the same set of structural maps (30) that satisfy relations (31a)–(31e), while in the compatibility relation (31f), (39) the symmetry σ is replaced by the braiding τ

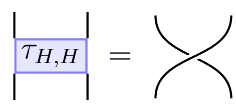
$$\text{compatibility}: \ (\nabla \otimes \nabla)(\mathrm{id}_H \otimes \tau_{H,H} \otimes \mathrm{id}_H)(\Delta \otimes \Delta) = \Delta\nabla. \tag{41}$$

One can show that in any braided Hopf algebra, the antipode satisfies the relations

$$S\nabla = \nabla\tau_{H,H}(S \otimes S), \qquad \Delta S = (S \otimes S)\tau_{H,H}\Delta \tag{42}$$

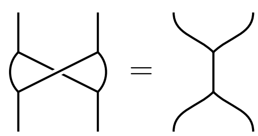
which can be proven, for example, following the same line of reasoning as in Section 1.9 of [Kas23].

In the diagrammatic language, we denote the braiding morphism by



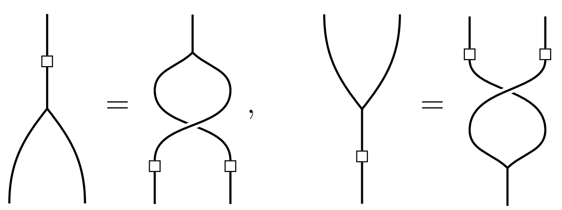
(43)

so that the compatibility relation (41) takes the graphical form (cf. (39))



(44)

and relations (42) become



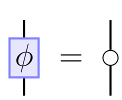
(45)

The second refinement that we need is the notion of a Hopf f-object or Hopf f-algebra which corresponds to a pair (H, ϕ) composed of a braided Hopf algebra H and a braided Hopf algebra automorphism $\phi : H \to H$. In addition to the axioms of a braided Hopf algebra for H, the pair (H, ϕ) satisfies the extra compatibility conditions between ϕ and all the structural morphisms of H:

$$\nabla(\phi \otimes \phi) = \phi\nabla, \quad (\phi \otimes \phi)\Delta = \Delta\phi, \tag{46}$$

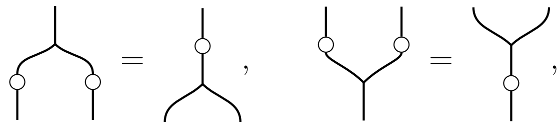
$$S\phi = \phi S, \quad \phi\eta = \eta, \quad \epsilon\phi = \epsilon. \tag{47}$$

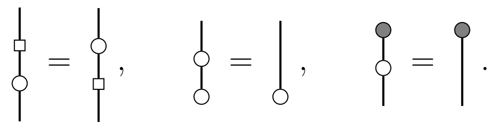
In the diagrammatic notation, we denote the automorphism ϕ by



(48)

so that the additional compatibility relations (46) and (47) take the form


(49)


(50)

4.2. **Diagrammatics of Yetter-Drinfel'd f-objects.** In this section we recall the definitions for Yetter–Drinfel'd f-objects over Hopf f-objects and provide the diagrammatic notation for them.

The original Yetter–Drinfel'd modules were defined by Yetter [Yet90] and they are essentially modules over Drinfel'd's quantum double of a Hopf algebra (hence the name of Drinfel'd). In the early literature, they were also called crossed modules; see eg. [Rad12, p. 385]. A detailed definition of these modules, their properties in the setting of braided Hopf algebras is given by Takeuchi [Tak00].

A left Yetter-Drinfel'd f-object over a Hopf f-object H is a triple (Y, λ_L, δ_L) where $\lambda_L\colon H\otimes Y \to Y$ and $\delta_L\colon Y \to H\otimes Y$ satisfy the left module and left comodule equations

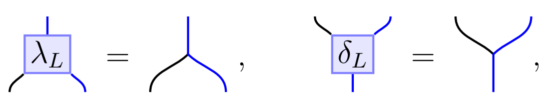
$$\text{left action}: \ \lambda_L(\mathrm{id}_H \otimes \lambda_L) = \lambda_L(\nabla\otimes \mathrm{id}_Y) \tag{51a}$$

$$\text{left action of unit}: \ \lambda_L(\eta\otimes \mathrm{id}_Y) = \mathrm{id}_Y \tag{51b}$$

$$\text{left coaction}: \ (\mathrm{id}_H \otimes \delta_L)\delta_L = (\Delta\otimes \mathrm{id}_Y)\delta_L \tag{51c}$$

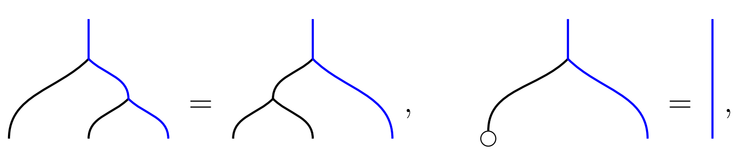
$$\text{left coaction of counit}: \ (\epsilon\otimes \mathrm{id}_Y)\delta = \mathrm{id}_Y \tag{51d}$$

In the diagrammatic setting, we will color the left f-objects by the blue color and the right f-objects by the red color. Using this coloring scheme, the morphisms λ_L and δ_L of the left Yetter-Drinfel'd f-objects are drawn graphically as

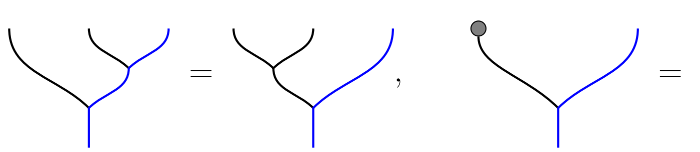


(52)

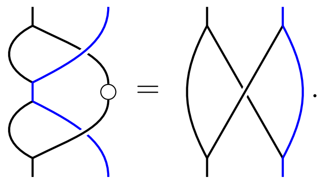
so that we obtain the graphical form of Equations (51a)–(51b)


(53)

of Equations (51c)–(51d)


(54)

and the compatibility relation (15)



(55)

Likewise, a right Yetter-Drinfel'd f-object over a Hopf f-object H is a triple (Y, λ_R, δ_R) where $\lambda_R \colon Y \otimes H \to Y$ and $\delta_R \colon Y \to Y \otimes H$ satisfy the right module and right comodule equations

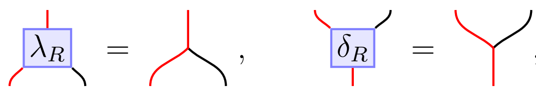
$$\text{right coaction}: \ (\delta_R \otimes \mathrm{id}_H)\delta_R = (\mathrm{id}_Y \otimes \Delta)\delta_R \tag{56a}$$

$$\text{right coaction of counit}: \ (\mathrm{id}_Y \otimes \epsilon)\delta_R = \mathrm{id}_Y \tag{56b}$$

$$\text{right action}: \ \lambda_R(\lambda_R \otimes \mathrm{id}_H) = \lambda_R(\mathrm{id}_Y \otimes \Delta) \tag{56c}$$

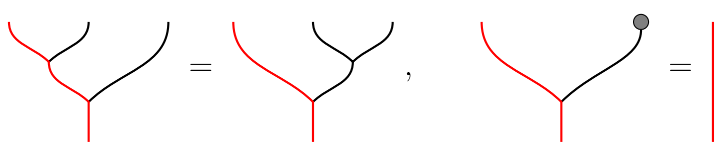
$$\text{right action of unit}: \ \lambda_R(\mathrm{id}_Y \otimes \eta) = \mathrm{id}_Y \tag{56d}$$

The corresponding maps λ_R and δ_R of the right Yetter-Drinfel'd f-objects are denoted by



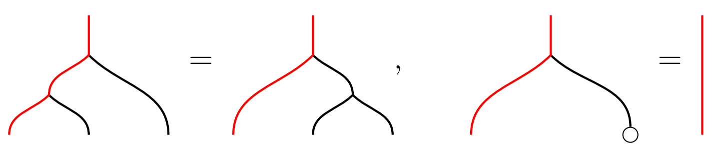
(57)

so that we have graphical form of Equations (56a) and (56b)



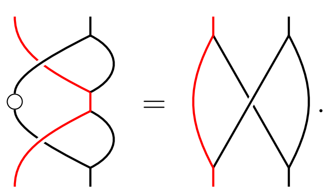
(58)

Equations (56c) and (56d)



(59)

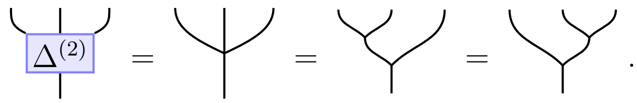
and the compatibility relation (16)



(60)

Note that the diagrammatic form of morphisms and relations for right Yetter-Drinfel'd f-objects are obtained from those of the left Yetter-Drinfel'd f-objects after rotating the diagram by 180 degrees and replacing the blue color by the red color.

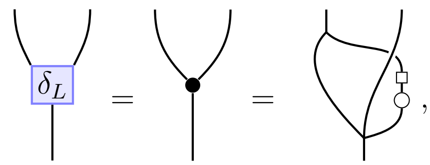
4.3. **Proof of Theorem 3.5.** In this section we prove Theorem 3.5 using the diagrammatic language that we have already described. Before doing so, we need the following diagrammatic notation for the double-iterated coproduct (17)



(61)

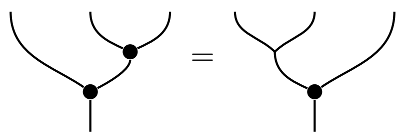
We will use similar multivalent vertices for higher-iterated coproducts and products.

Using the following graphical representation of the left coaction (18)



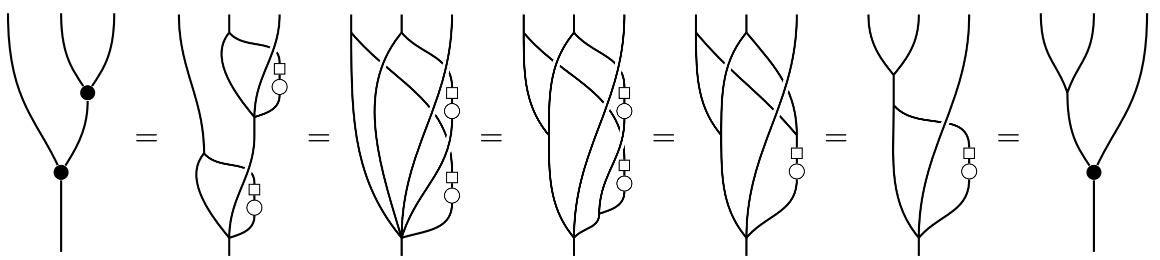
(62)

we can prove the left coaction property



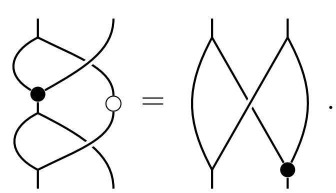
(63)

as follows:



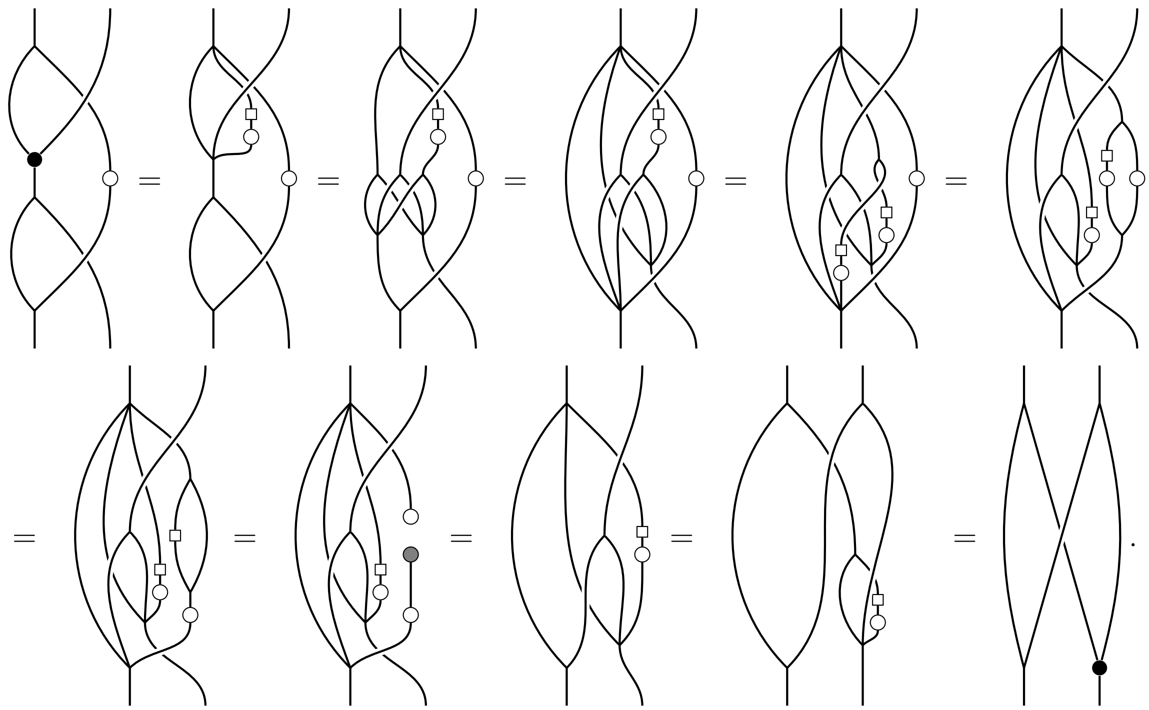
(64)

Similarly, we can prove the compatibility property (15), see (55) for its graphical form, which in this case takes the form



(65)

Indeed, with a bit longer graphical calculation, we have

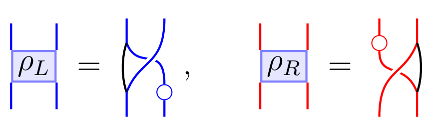


(66)

This completes the proof of part (a) of Theorem 3.5. The proof of part (b) is analogous, and is omitted. □

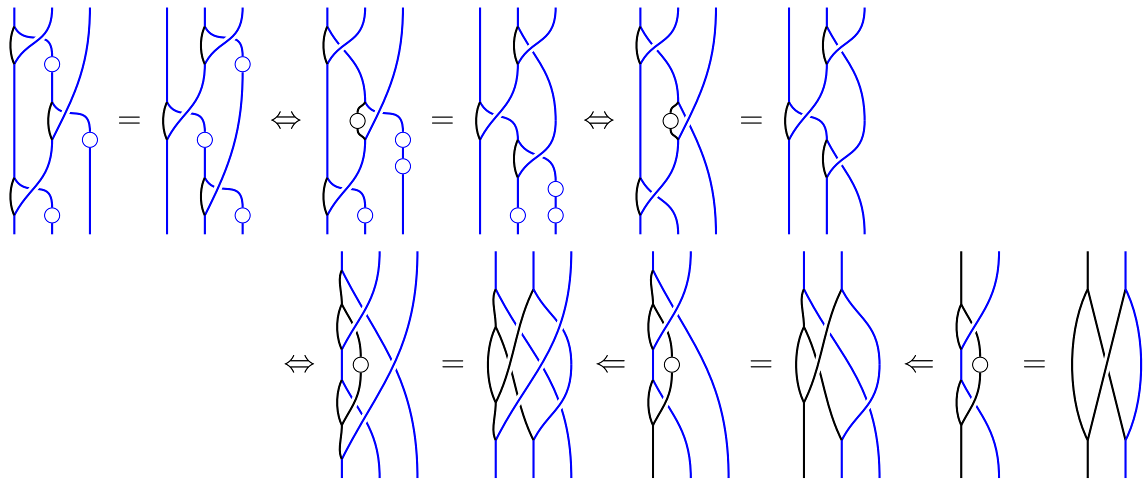
4.4. **Proof of Theorem 3.6.** In this section we show that the R-matrix (20) satisfies the Yang–Baxter equation (22), and omit the analogous proofs that the R-matrix (21) also satisfies the Yang–Baxter equation.

To begin with, the diagrammatic notation for the R-matrices ρ_L and ρ_R is given as follows



(67)

The proof of Theorem 3.6 is now given as follows:



(68)

where the last equality is the compatibility Equation (55). This completes the proof of Theorem 3.5 for left Yetter–Drinfel'd f-objects. The proof of the right Yetter–Drinfel'd f-objects is obtained by rotating the above diagrams by 180 degrees, followed by replacing the blue color by the red color. □

5. Braided tensor algebras and Nichols algebras

5.1. **Braided tensor algebras.** In this section we specialize the abstract language of Hopf f-objects and the Yetter–Drinfel'd f-objects to the context of a braided category $\mathcal{C}$ which, as a monoidal category, is a subcategory of the category $\mathbf{Vect}_{\mathbb{F}}$ of vector spaces over a field $\mathbb{F}$ with the monoidal structure given by the tensor product $\otimes_{\mathbb{F}}$. The objects of $\mathcal{C}$ will be called *braided vector spaces*. In this case, a Hopf f-object and a Yetter–Drinfel'd f-object will be respectively called a *Hopf f-algebra* and *Yetter–Drinfel'd f-module*.

Recall that it follows from the definition that a Hopf f-algebra is a pair (H, ϕ) of a braided Hopf algebra and an automorphism ϕ of it. There is an elementary universal construction of such pairs (H, ϕ) that we now discuss.

Fix a braided vector space V of finite dimension n. Then, the tensor algebra $T(V)$ has a unique structure of a braided Hopf algebra determined by declaring all elements of V to be primitive. We define the rank of $T(V)$ to be the dimension of V, and call the braided Hopf algebra $T(V)$ to be of diagonal type if the braiding on V is diagonal with respect a basis B of V.

In this case, $T(V)$ is $\mathbb{Z}^n_{\geq 0}$-graded and admits a rich Abelian group of braided Hopf algebra automorphisms. Namely, any map $t\colon B \to \mathbb{F}_{\neq 0}$ corresponds to a braided Hopf algebra automorphism ϕ_t of $T(V)$ uniquely determined by

$$\phi_t b = t_b b, \quad \forall b \in B, \tag{69}$$

where we denote by t_b the image $t(b)$. We call such automorphisms scaling automorphisms.

Summarizing, a finite dimensional vector space V with a diagonal braiding with respect to a basis B of V, together with a map $t\colon B \to \mathbb{F}_{\neq 0}$ determines a pair $(T(V), \phi_t)$ of a braided Hopf algebra and an automorphism of it. Using Theorems 3.5–3.6, we obtain multiparameter infinite-dimensional R-matrices over the vector space $T(V)$. Our interest is to find rigid R-matrices which correspond to finite-dimensional Yetter–Drinfel'd f-modules. We discuss this next.

5.2. **Nichols algebras.** It turns out that the braided tensor algebras $T(V)$ defined above have a canonical quotient called Nichols algebra which is a braided Hopf algebra. It can be finite or infinite dimensional.

Recall that a *Nichols algebra* over a braided vector space V is the quotient braided Hopf algebra $\mathfrak{B}(V) = T(V)/J$ of the tensor algebra $T(V)$ over the maximal (braided) Hopf algebra ideal J intersecting trivially the part $\mathbb{F} \oplus V \subset T(V)$. In the case when the braiding is of diagonal type, the scaling automorphism ϕ_t of $T(V)$ decends to an automorphism of the braided Hopf algebra $\mathfrak{B}(V) = T(V)/J$ leading thereby to a Hopf f-algebra which we will call Nichols f-algebra. Thus, finite dimensional Nichols f-algebras can be used as an input to the construction of multiparameter knot invariants in (2).

Examples of finite dimensional Nichols algebras are the nilpotent Borel parts of Lustig's small quantum groups. A detailed description of finite dimensional Nichols algebras can be found for example in [AS00, AS02, And14]. Like quantum groups, Nichols algebras have PBW bases [Kha99] and the ones with diagonal braiding have been classified by Heckenberger [Hec06, Hec09] building on the work of Kharchenko [Kha99] and Andruskiewitsch–Schneider [AS00]. The list of diagonal Nichols algebras of rank (that is, dimension of V) at most 3 is given in Tables 1 and 2 of [Hec06], and from this, it follows that the majority of finite rank Nichols algebras do not come from quantum groups. A presentation of Nichols algebras of diagonal type in terms of generators and relations is given by Angiono [Ang15].

5.3. **Sub/quotient Yetter–Drinfel'd f-modules of Nichols f-algebras.** If a Nichols algebra is infinite dimensional, we cannot immediately proceed to the construction of knot invariants.

It turns out that a Nichols f-algebra $\mathfrak{B}(V)$, as a left/right Yetter–Drinfel'd f-module over itself, has a canonical quotient ${}^L\mathfrak{B}(V)$ and a canonical subspace $\mathfrak{B}(V)^R$ which are left and right Yetter–Drinfel'd f-modules over $\mathfrak{B}(V)$ respectively. The construction of these f-modules is as follows:

$$ {}^L\mathfrak{B}(V) = \mathfrak{B}(V)/\mathfrak{B}(V)W_{\delta_L}, \qquad W_{\delta_L} = \{x \in \mathfrak{B}(V) \setminus \mathbb{F} \mid \delta_L x = 1 \otimes x\} \tag{70} $$

where elements of W_{δ_L} are nothing else but the *coinvariant* elements in degree ≥ 1 with respect to the left coaction δ_L, see [Rad12, Def. 8.2.1]; and

$$ \mathfrak{B}(V)^R = U_{\lambda_R} \tag{71} $$

where $U_{\lambda_R} \subset \mathfrak{B}(V)$ is the smallest subspace of $\mathfrak{B}(V)$ that satisfies

$$ \Delta W_{\lambda_R} \subset U_{\lambda_R} \otimes \mathfrak{B}(V), \qquad W_{\lambda_R} = \{x \in \mathfrak{B}(V) \mid \lambda_R(x \otimes y) = x\epsilon y, \text{ for all } y \in \mathfrak{B}(V)\} \tag{72} $$

where elements of W_{λ_R} are nothing else but *invariant* elements with respect to the right action λ_R, see [Rad12, Def. 11.2.3]. Note that, by taking into account the fact that $\epsilon y = 0$ for all $y \in V$, the invariance condition takes the form $\lambda_R(x \otimes y) = 0$ for all $y \in V$.

If any one of the $\mathfrak{B}(V)$ f-modules ${}^L\mathfrak{B}(V)$ or $\mathfrak{B}(V)^R$ is finite dimensional, then, by Theorem 3.6, it can be used in (2) for construction of polynomial knot invariants.

In the next sections we illustate the quotients and the subspaces of a braided tensor algebra $T(V)$ when the dimension of V is 1 or 2.

6. The rank 1 tensor algebra

6.1. **Definition.** In this section we compute from first principles the R-matrices of Theorem 3.5 for the rank 1 tensor algebra, with no reference to Lie theory. As we will find out, the corresponding knot invariants are none other than the colored Jones and the ADO polynomials.

The rank 1 tensor algebra $T(\mathbb{F})$ is identified with the polynomial algebra $\mathbb{F}[x]$ in one indeterminate. It is an infinite dimensional $\mathbb{F}$-vector space with basis $B = \{x^k \mid k \in \mathbb{Z}_{\geq 0}\}$.

The Hopf algebra structure and the braided structure of $T(\mathbb{F})$ are determined by

$$\Delta x = x \otimes 1 + 1 \otimes x, \qquad \tau(x \otimes x) = q\, x \otimes x\,. \tag{73}$$

The above equations, together with the axioms of a braided Hopf algebra, and the choice of the basis, uniquely determine the braided Hopf algebra structure. The formulas involve the q-Pochhammer symbol $(x;q)_n$ and the q-binomial coefficients $\begin{bmatrix} k \\ m \end{bmatrix}_q$ defined by

$$(x;q)_n := \prod_{i=0}^{n-1}(1-xq^i), \qquad \begin{bmatrix} k \\ m \end{bmatrix}_q := \frac{(q;q)_k}{(q;q)_{k-m}(q;q)_m}\,. \tag{74}$$

Explicitly, we have the following.

Lemma 6.1. The coproduct, the antipode and the scaling automorphism ϕ_t of $T(\mathbb{F})$ are given by

$$\Delta x^k = \sum_{m=0}^{k} \begin{bmatrix} k \\ m \end{bmatrix}_q x^{k-m} \otimes x^m \tag{75}$$

$$S x^k = (-1)^k q^{k(k-1)/2} x^k \tag{76}$$

$$\phi_t x^k = t^k x^k \tag{77}$$

respectively.

Proof. The primitivity of x implies that

$$\Delta x = x_1 + x_2, \quad x_1 := x \otimes 1, \quad x_2 := 1 \otimes x, \tag{78}$$

and the braiding implies that

$$x_2 x_1 = q x_1 x_2\,. \tag{79}$$

This, combined with the q-binomial formula, gives

$$\Delta x^k = (x_1+x_2)^k = \sum_{m=0}^{k} \begin{bmatrix} k \\ m \end{bmatrix}_q x_1^{k-m} x_2^m = \sum_{m=0}^{k} \begin{bmatrix} k \\ m \end{bmatrix}_q x^{k-m} \otimes x^m\,. \tag{80}$$

This proves (75). To prove (76), apply (42) for $x^k \in T(\mathbb{F})$, use $\eta\,\epsilon\, x^k = \delta_{k,0}$ and compute

$$\begin{aligned}\nabla(\mathrm{id}_H \otimes S)\Delta x^k &= \nabla(\mathrm{id}_H \otimes S)\Big(\sum_{m=0}^{k} \begin{bmatrix} k \\ m \end{bmatrix}_q x^{k-m} \otimes x^m\Big) \\ &= \nabla\Big(\sum_{m=0}^{k} \begin{bmatrix} k \\ m \end{bmatrix}_q x^{k-m} \otimes Sx^m\Big) = \sum_{m=0}^{k} \begin{bmatrix} k \\ m \end{bmatrix}_q x^{k-m} Sx^m \,.\end{aligned}$$

This is a linear system of equations that uniquely determines Sx^k by induction on k. Since

$$\sum_{m=0}^{k} \begin{bmatrix} k \\ m \end{bmatrix}_q (-1)^m q^{m(m-1)/2} = \delta_{k,0} \tag{81}$$

Equation (76) follows. Finally (77) is clear since ϕ_t is an automorphism and $\phi_t x = tx$. □

6.2. **The left and right Yetter–Drinfel'd f-modules.** In this section we compute the R-matrices of Theorem 3.6 explicitly. We first compute the doubly iterated coproduct (17), the coaction (18) and the R-matrix (20). The formulas involve the q-multinomial coefficients defined by

$$\begin{bmatrix} k \\ m, n \end{bmatrix}_q := \frac{(q;q)_k}{(q;q)_{k-m-n}(q;q)_m(q;q)_n} \,. \tag{82}$$

Lemma 6.2. The doubly iterated coproduct $\Delta^{(2)}$, the left coaction δ_L and the R-matrix (20) ${}^L\rho$ are given by

$$\Delta^{(2)} x^k = \sum_{m=0}^{k} \sum_{n=0}^{k-m} \begin{bmatrix} k \\ m, n \end{bmatrix}_q x^{k-m-n} \otimes x^m \otimes x^n \tag{83}$$

$$\delta_L x^k = \sum_{m=0}^{k} \begin{bmatrix} k \\ m \end{bmatrix}_q (tq^m;q)_{k-m} x^{k-m} \otimes x^m \tag{84}$$

$$\rho_L(x^k \otimes x^l) = \sum_{m=0}^{k} \begin{bmatrix} k \\ m \end{bmatrix}_q (tq^{k-m};q)_m (tq^{k-m})^l x^{l+m} \otimes x^{k-m} \,. \tag{85}$$

Proof. We compute

$$\begin{aligned}\Delta^{(2)} x^k &= \sum_{m=0}^{k} \begin{bmatrix} k \\ m \end{bmatrix}_q x^{k-m} \otimes \Delta x^m = \sum_{m=0}^{k} \begin{bmatrix} k \\ m \end{bmatrix}_q x^{k-m} \otimes \sum_{n=0}^{m} \begin{bmatrix} m \\ n \end{bmatrix}_q x^{m-n} \otimes x^n \\ &= \sum_{0 \le n \le m \le k} \begin{bmatrix} k \\ m \end{bmatrix}_q \begin{bmatrix} m \\ n \end{bmatrix}_q x^{k-m} \otimes x^{m-n} \otimes x^n \\ &= \sum_{n=0}^{k} \sum_{m=0}^{k-n} \begin{bmatrix} k \\ m+n \end{bmatrix}_q \begin{bmatrix} m+n \\ n \end{bmatrix}_q x^{k-m-n} \otimes x^m \otimes x^n \\ &= \sum_{n=0}^{k} \sum_{m=0}^{k-n} \begin{bmatrix} k \\ m, n \end{bmatrix}_q x^{k-m-n} \otimes x^m \otimes x^n = \sum_{m=0}^{k} \sum_{n=0}^{k-m} \begin{bmatrix} k \\ m, n \end{bmatrix}_q x^{k-m-n} \otimes x^m \otimes x^n\end{aligned} \tag{86}$$

and, using Equations (77) and (76),

$$
\begin{aligned}
\delta_L x^k &= \sum_{m=0}^{k}\sum_{n=0}^{k-m} \begin{bmatrix} k \\ m,n \end{bmatrix}_q (\nabla\otimes \mathrm{id}_H)(\mathrm{id}_H \otimes \tau)(x^{k-m-n}\otimes x^m \otimes S\phi_t\, x^n) \\
&= \sum_{m=0}^{k}\sum_{n=0}^{k-m} q^{mn} \begin{bmatrix} k \\ m,n \end{bmatrix}_q (\nabla\otimes \mathrm{id}_H)(x^{k-m-n}\otimes S\phi_t x^n \otimes x^m) \\
&= \sum_{m=0}^{k}\sum_{n=0}^{k-m} q^{mn} \begin{bmatrix} k \\ m,n \end{bmatrix}_q x^{k-m-n} S\phi_t x^n \otimes x^m \\
&= \sum_{m=0}^{k}\sum_{n=0}^{k-m} q^{mn}(-t)^n q^{n(n-1)/2} \begin{bmatrix} k \\ m,n \end{bmatrix}_q x^{k-m}\otimes x^m \\
&= \sum_{m=0}^{k} \begin{bmatrix} k \\ m \end{bmatrix}_q \sum_{n=0}^{k-m} \begin{bmatrix} k-m \\ n \end{bmatrix}_q (-tq^m)^n q^{n(n-1)/2} x^{k-m}\otimes x^m \\
&= \sum_{m=0}^{k} \begin{bmatrix} k \\ m \end{bmatrix}_q (tq^m;q)_{k-m} x^{k-m}\otimes x^m
\end{aligned}
\tag{87}
$$

where the last equality follows from the q-binomial theorem. Thus, the R-matrix (20) for $Y = H$ is given by

$$
\begin{aligned}
\rho_L(x^k\otimes x^l) &= (\nabla\otimes \mathrm{id}_H)(\mathrm{id}_H \otimes \tau_{H,H})(\delta_L\otimes \phi_H)(x^k\otimes x^l) \\
&= \sum_{m=0}^{k} \begin{bmatrix} k \\ m \end{bmatrix}_q (tq^m;q)_{k-m} t^l (\nabla\otimes \mathrm{id}_H)(\mathrm{id}_H \otimes \tau_{H,H})(x^{k-m}\otimes x^m\otimes x^l) \\
&= \sum_{m=0}^{k} \begin{bmatrix} k \\ m \end{bmatrix}_q (tq^m;q)_{k-m} (tq^m)^l x^{k+l-m}\otimes x^m \\
&= \sum_{m=0}^{k} \begin{bmatrix} k \\ m \end{bmatrix}_q (tq^{k-m};q)_m (tq^{k-m})^l x^{l+m}\otimes x^{k-m}\,.
\end{aligned}
\tag{88}
$$

□

We next compute the doubly iterated product (17), the right action (19) and the R-matrix (21).

Lemma 6.3. The doubly iterated product $\nabla^{(2)}$, the right action λ_R and the R-matrix (21) are given by

$$\nabla^{(2)}(x^k\otimes x^l\otimes x^m) = x^{k+l+m} \tag{89}$$

$$\lambda_R(x^k\otimes x^l) = (tq^k;q)_l x^{k+l} \tag{90}$$

$$\rho_R(x^k\otimes x^l) = \sum_{m=0}^{l} \begin{bmatrix} l \\ m \end{bmatrix}_q (tq^k)^{l-m}(tq^k;q)_m x^{l-m}\otimes x^{k+m}\,. \tag{91}$$

Proof. Equation (89) is clear. To calculate the right action $\lambda_R(x^k\otimes x^l)$, we start with the case $l=1$:

$$\begin{aligned}\lambda_R(x^k\otimes x) &= \nabla^{(2)}(S\phi_t\otimes \mathrm{id}_{H\otimes H})(\tau\otimes \mathrm{id}_G)(x^k\otimes x\otimes 1+x^k\otimes 1\otimes x)\\ &= \nabla^{(2)}(S\phi_t\otimes \mathrm{id}_{H\otimes H})(q^k x\otimes x^k\otimes 1+1\otimes x^k\otimes x)\\ &= \nabla^{(2)}(-tq^k x\otimes x^k\otimes 1+1\otimes x^k\otimes x) = -tq^kx^{k+1}+x^{k+1}\\ &= (1-tq^k)x^{k+1}\,.\end{aligned} \tag{92}$$

Now, we have

$$\begin{aligned}\lambda_R(x^k\otimes x^l) &= \lambda_R(\lambda_R(x^k\otimes x)\otimes x^{l-1}) = (1-tq^k)\lambda_R(x^{k+1}\otimes x^{l-1})\\ &= (1-tq^k)(1-tq^{k+1})\lambda_R(x^{k+2}\otimes x^{l-2}) = \cdots = (tq^k;q)_l\lambda_R(x^{k+l}\otimes x^{l-l})\\ &= (tq^k;q)_l x^{k+l}\,.\end{aligned} \tag{93}$$

Thus, the R-matrix (21) for $Y=H$ is given by

$$\begin{aligned}\rho_R(x^k\otimes x^l) &= \sum_{m=0}^{l}\begin{bmatrix} l\\ m\end{bmatrix}_q (\phi_H\otimes\lambda_R)(\tau_{H,H}\otimes \mathrm{id}_H)(x^k\otimes x^m\otimes x^{l-m})\\ &= \sum_{m=0}^{l}\begin{bmatrix} l\\ m\end{bmatrix}_q q^{km}t^m(tq^k;q)_{l-m}x^m\otimes x^{k+l-m}\\ &= \sum_{m=0}^{l}\begin{bmatrix} l\\ m\end{bmatrix}_q (tq^k)^{l-m}(tq^k;q)_m x^{l-m}\otimes x^{k+m}\,.\end{aligned} \tag{94}$$

□

The R-matrices (85) and (91) depend on two variables t and q, and using the basis $B=\{x^k\,|\,k\in\mathbb{Z}_{\geq 0}\}$, their entries are in $\mathbb{Z}[t^{\pm1},q^{\pm1}]$ and satisfy the Yang–Baxter equation on an infinite dimensional space $T(\mathbb{F})$.

However, to define knot invariants as state-sums, we need to have rigid R-matrices over finite dimensional vector spaces. In the remaining subsections we give several solutions to this problem and identify the corresponding knot invariants.

6.3. q **a root of unity: the ADO polynomials.** The Nichols f-algebra $\mathfrak{B}(\mathbb{F})$ is finite-dimensional if $q=\omega$ is a root of unity of order $\mathrm{ord}(\omega)=N>1$ (for $N=1$, see Remark 6.4 below). In this case, it follows that $\left[\begin{smallmatrix}N\\k\end{smallmatrix}\right]_\omega=0$ for $0<k<N$ and Equation (75) implies that x^N is primitive and thus generates a Hopf ideal of $\mathbb{F}[x]$ with finite-dimensional Nichols algebra $\mathbb{F}[x]/(x^N)$.

In this case, the left R-matrix (85) coincides with the R-matrix of Akutsu–Deguchi–Ohtsuki [ADO92] and the knot invariant of Theorem 3.7 is the ADO polynomial times the identity matrix.

Remark 6.4. When $N=1$, the Nichols algebra $\mathfrak{B}(\mathbb{F})=\mathbb{F}[x]$ is infinite dimensional. In this exceptional case, and we will replace it with the 1-dimensional algebra obtained by imposing the relation $x=0$ in the list of finite dimensional Nichols algebras, despite the fact that $x\in V$.

6.4. q **generic: colored Jones polynomials.** When q is not a root of unity, the Nichols f-algebra $\mathfrak{B}(\mathbb{F}) = \mathbb{F}[x]$ is infinite-dimensional. However, it turns out that one can extract finite dimensional left or right Yetter–Drinfel'd f-modules if

$$t = q^{1-n}, \qquad n \in \mathbb{Z}_{>0}\,. \tag{95}$$

Indeed, under this assumption for the scaling automorphism, Equation (84), implies that x^n is a coinvariant element, $\delta_L x^n = 1 \otimes x^n$. This gives a quotient left Yetter–Drinfel'd f-module $\mathbb{F}[x]/(x^n)$ of dimension n. The corresponding R-matrix is the one of the n-colored Jones polynomial.

Moreover, under the same assumption (95), Equation (92) implies that x^{n-1} is an invariant element, whose coproduct generates the n-dimensional space with basis x^k, $0 \leq k \leq n-1$, and this gives an n-dimensional right Yetter–Drinfel'd f-submodule of $\mathbb{F}[x]$. The corresponding knot invariant is the identity matrix times the n-th colored Jones polynomial.

Summarising, in the rank 1 case, the corresponding matrix-valued knot invariants are the identity times the ADO and the colored Jones polynomials.

7. A rank 2 tensor algebra

In this section we discuss the case of Nichols f-algebras of rank 2 of diagonal type. In order to keep the construction as simple as possible, we consider the Nichols algebra $\mathfrak{B}(V, c)$ associated with two-dimensional vector space V with basis $B = \{x_1, x_2\}$ and diagonal braiding

$$\tau(x_i \otimes x_j) = q_{ij} x_j \otimes x_i, \qquad c = \begin{pmatrix} -1 & q_{12} \\ q_{21} & -1 \end{pmatrix}. \tag{96}$$

Denote by H_c the corresponding Nichols algebra. In Heckenberger's list [Hec08, Table 1] the isomorphism type of H_c is determined by the parameter $q := q_{12}q_{21}$ of the generalized Dynkin diagram (see also [Hec07, Defn. 3.1]).

For generic values of q, H_c, is an infinite-dimensional quotient of the free noncommutative algebra in x_1 and x_2 by the 2-sided ideal generated by x_1^2, x_2^2, and a basis of H_c is given by alternating words in letters x_1 and x_2.

7.1. q **a root of unity: two-variable knot polynomials over cyclotomic fields.** When $q = \omega$ is a root of unity of order $N \geq 1$, we have $(x_2x_1)^N + (-q_{21}x_1x_2)^N = 0 \in H_c$, and the Nichols algebra H_c is $4N$-dimensional with a basis given by all alternating words in x_1 and x_2 of length $\leq 2N$, excluding $(x_2x_1)^N$. H_c is a Nichols f-algebra with scaling automorphism defined by $\phi_t x_i = t_i x_i$ for $i = 1, 2$ where t_1 and t_2 are two independent invertible elements. To emphasize the dependence of this f-algebra on the braiding c and the automorphism ϕ_t, we will denote it by $H_{c,t}$. Theorem 3.6 constructs a left R-matrix $R_{c,t}$ on $H_{c,t}$ with entries in $\mathbb{Z}[t_1, t_2, q_{12}^{\pm 1}, q_{21}^{\pm 1}]$ (keeping in mind that $q_{12}q_{21} = \omega$) which is invertible with determinant $(t_1t_2)^{N^2}$ for all of its rotated versions.

Using this R-matrix together with Theorem 3.7 we arrive at the following knot invariants.

Definition 7.1. For a root of unity ω, we have the knot invariant

$$K \mapsto J_{R_{c,t}}(K) \in \mathrm{End}(H_{c,t}) \tag{97}$$

and denote by $\Lambda_{\omega,K}(t_1,t_2) \in \mathbb{Z}[\omega, t_1^{\pm 1}, t_2^{\pm 1}]$ the $(1,1)$-entry of $J_{R_{c,t}}(K)$ with respect to the above basis of $H_{c,t}$.

A priori, the knot invariant depends on the Nichols f-algebra $H_{c,t}$, and hence on q_{12}, q_{21}, t_1 and t_2. However, the polynomial invariant depends only on t_1, t_2 and on the product $\omega = q = q_{12}q_{21}$. This follows from the fact that the isomorphism class of the Nichols algebra depends on q alone (see [Hec07, Defn. 3.1]). What's more, the invariant Λ_ω satisfies the symmetry

$$\Lambda_{\omega,K}(t_1,t_2) = \Lambda_{\omega,K}(t_2,t_1)\,. \tag{98}$$

This follows from the existence of a unique isomorphism

$$\sigma : H_{c,t} \to H_{\bar{c},\bar{t}}, \qquad \bar{c} = c^t, \qquad \overline{(t_1,t_2)} = (t_2,t_1) \tag{99}$$

that satisfies $\sigma(x_1) = x_2$ and $\sigma(x_2) = x_1$, and consequently, the $R_{c,t}$-matrix satisfies $(\sigma \otimes \sigma) R_{c,t} (\sigma^{-1} \otimes \sigma^{-1}) = R_{\bar{c},\bar{t}}$.

Conjecture 7.2. For every knot K, we have

$$J_{R_{c,t}}(K) = \Lambda_{\omega,K}(t_1,t_2)\, \mathrm{id}_{H_{c,t}}\,. \tag{100}$$

Remark 7.3. The above knot invariant Λ_ω was defined using the left R-matrix of the Nichols f-algebra. Of course, we could consider the invariant from the right R-matrix, but the two appear to be equal.

7.2. **The cases $q = \pm 1$.** The first invariant from the above definition occurs when $\omega = 1$, thus $N = 1$, where the corresponding Nichols f-algebra $H_{c,t}$ is 4-dimensional. Its left and right R-matrices are easy to analyze, and the corresponding knot invariant satisfies

$$\Lambda_{1,K}(t_1,t_2) = \Delta_K(t_1)\Delta_K(t_2) \tag{101}$$

where $\Delta_K(t) \in \mathbb{Z}[t^{\pm 1}]$ is the canonically normalized Alexander polynomial.

The next invariant occurs when $\omega = -1$, thus $N = 2$ and the Nichols algebra is 8-dimensional. We now discuss the properties of this invariant.

To begin with, the Nichols f-algebra $H_{c,t}$ is 8-dimensional and isomorphic to the nilpotent Borel subalgebra of the small quantum group $u_q(\mathfrak{sl}_3)$ with $q = \sqrt{-1}$. A basis for $H_{c,t}$ is

$$\{1, x_1, x_2, x_1x_2, x_2x_1, x_1x_2x_1, x_2x_1x_2, x_1x_2x_1x_2\}$$

and the corresponding 64×64 left R-matrix $R_{c,t}$ has entries in $\mathbb{Z}[t_1, t_2, q_{21}^{\pm 1}]$ and can be computed explicitly. It is a sparse matrix with only 157 out of 4096 entries (about 3.8%) nonzero. Due to its size, we do not present these entries here, but give a sample value

$$\begin{aligned} R(x_1x_2x_1x_2 \otimes x_2x_1) = {}& s^2t^2\, x_2x_1 \otimes x_1x_2x_1x_2 - q_{21}^{-1}s^2t(1+t)\, x_1x_2x_1 \otimes x_2x_1x_2 \\ & + q_{21}^2(-1+s)st^2\, x_2x_1x_2 \otimes x_1x_2x_1 + q_{21}^{-1}(-1+s)st\, x_1x_2x_1x_2 \otimes x_2x_1\,, \end{aligned} \tag{102}$$

where for simplicity we abbreviate t_1 and t_2 by s and t. The determinant of this R-matrix and all of its rotated versions is $(st)^{64}$.

For all knots for which we computed the invariant, we confirmed that Conjecture 7.2 holds, and the Laurent polynomial $\Lambda_{-1,K}(s^2,t^2)$ coincides with Harper's polynomial $\Delta_{\mathfrak{sl}_3}(s,t)$ [Har].

• **Symmetry.** In addition to the symmetry (98) for Λ_ω, it turns out that the polynomial $\Lambda_{-1,K}(t,s)$ is invariant under the involutions

$$(s,t)\mapsto(t,s), \qquad (s,t)\mapsto(s,-1/(st)), \qquad (s,t)\mapsto(1/s,1/t) \tag{103}$$

which generate a group G of order 12. The existence of these additional symmetries follow from the identification of the Nichols algebra with the Borel part of the small quantum group, whose Weyl group is the symmetric group of order 3, and together with σ, generates the group G of order 12. The invariant polynomial ring can be identified with

$$\mathbb{Z}[t_1^{\pm1},t_2^{\pm1}]^G=\mathbb{Z}[u,v], \tag{104}$$

where

$$u=\langle s\rangle+\langle t\rangle-\langle st\rangle-2, \qquad v=\langle s^2t\rangle+\langle st^2\rangle-\langle s/t\rangle-2, \qquad \langle x\rangle=x+x^{-1}\,. \tag{105}$$

Thus, we can write the polynomial $\Lambda_{-1,K}$ in the form

$$\Lambda_{-1,K}(s,t)=\tilde\Lambda_K(u,v), \quad \tilde\Lambda_K(u,v)\in\mathbb{Z}[u,v]\,. \tag{106}$$

• **Chirality and mutation.** Experimentally, it appears that the polynomial $\Lambda_{-1,K}$ does not distinguish a knot from its mirror image. Regarding mutation, it distinguishes the pair $(11n34, 11n42)$ of mutant knots but not the pair $(11n73, 11n74)$ of mutant knots.
• **Specialization.** Experimentally, it appears that the specialization $u=0$ reproduces the Alexander–Conway polynomial $\nabla_K(z)$

$$\tilde\Lambda_{-1,K}(0,z^2)=\nabla_K(z). \tag{107}$$

• **Comparison with other knot polynomials.** Regarding the independence of Λ_{-1} from other knot polynomials, we have the following observations.

(a) The knots $\overline{7_4}$ and 9_2 (where $\overline{K}$ is the mirror image of K) have equal Knot Floer Homology (a well-known fact; see Manolescu [Man16]) and confirmed by SnapPy, thus have Seifert genus 1 and none is fibered [CDGW]. On the other hand, the two knots have different Λ_{-1}-polynomials; see Table 1.
(b) The colored Jones and the ADO polynomials do not distinguish mutant pairs of knots, since the corresponding tensor product of representations is multiplicity-free. This fact was pointed out to us by J. Murakami and T. Ohtsuki. On the other hand, the Λ_{-1}-polynomial sometimes detects mutation, and sometimes does not. In particular, it distinguishes the mutant pair of knots $11n42$ (Kinoshita–Terasaka knot) and $11n34$ (Conway knot) — the fact that was pointed out by Harper in [Har] for his polynomial $\Delta_{\mathfrak{sl}_3}(s,t)$. On the other hand, it does not distinguish the mutant pair of knots $11n74$ (a fibered, Seifert genus 2 knot) and $11n73$ (a non-fibered, Seifert genus 3 knot).
(c) The knots 8_8 and 10_{129} have isomorphic Khovanov homology [BN02], yet different Λ_{-1}-polynomials.

• **Values.** Table 1 gives the result of computer calculation for all knots of up to 6 crossings, and few higher crossing knots that appear in the above discussion.

K	$\tilde{\Lambda}_K(u,v)$
3_1	$1+4u+u^2+v$
4_1	$1-6u+u^2-v$
5_1	$1+12u+19u^2+8u^3+u^4+(3+7u+3u^2)v+v^2$
5_2	$1+10u+6u^2+2v$
6_1	$1-10u+6u^2-2v$
6_2	$1-8u-15u^2+2u^3+u^4+(-1-9u+u^2)v-v^2$
6_3	$1+2u+15u^2+6u^3+u^4+(1+9u+u^2)v+v^2$
7_1	$1+24u+86u^2+104u^3+53u^4+12u^5+u^6$ $+(6+35u+60u^2+33u^3+5u^4)v+(5+10u+6u^2)v^2+v^3$
7_4	$(1+2u)(1+18u)+4v$
8_8	$1+10u+36u^2+28u^3+6u^4+2(1+9u+3u^2)v+2v^2$
8_{17}	$1-14u-23u^2-38u^3+10u^4+8u^5+u^6$ $+(-1-17u-44u^2+5u^3+2u^4)v+(-2-13u+u^2)v^2-v^3$
9_2	$1+20u+24u^2+4v$
10_2	$1-150u^2-380u^3-279u^4-44u^5+25u^6+10u^7+u^8$ $+(2-55u-260u^2-274u^3-58u^4+19u^5+5u^6)v$ $+(-5-62u-91u^2-24u^3+5u^4)v^2+(-5-15u-2u^2)v^3-v^4$
10_{129}	$1+10u+32u^2+36u^3+6u^4+2(1+8u+2u^2)v+2v^2$
$11n34$	$1+12u+8u^2+60u^3+48u^4+8u^5+2u(1+2u)(-1+6u)v+2u^2v^2$
$11n42$	$1+12u+8u^2-12u^3-2uv$
$11n73$ $11n74$	$1+20u+10u^2+4u^3+u^4+2(1+4u+u^2)v+v^2$

TABLE 1. The polynomial $\tilde{\Lambda}_K(u,v)$ for some knots K.

7.3. q **generic: two-variable polynomials.** In this section we classify all finite-dimensional right Yetter–Drinfel'd f-modules by classifying all invariant vectors. Recall that for generic q, the Nichols f-algebra $H_{c,t}$ has a basis that consists of all alternating words in the letters x_1 and x_2, where $x_1^2 = x_2^2 = 0$. Thus, every basis element is one of the following forms

$$(x_1x_2)^a, \quad (x_2x_1)^b, \quad x_2(x_1x_2)^c, \quad x_1(x_2x_1)^d \tag{108}$$

with $a,b,c,d \in \mathbb{Z}_{\geq 0}$. Moreover, $H_{c,t}$ is $\mathbb{Z}^2_{\geq 0}$-graded, and thus also $\mathbb{Z}_{\geq 0}$-graded where the $\mathbb{Z}_{\geq 0}$-degree is the sum of the components of the $\mathbb{Z}^2_{\geq 0}$-degree.

It follows from (108) that the degree $2n-1$ part of $H_{c,t}$ is the direct sum of two bi-degrees $(n,n-1)$ and $(n-1,n)$, each of them being one-dimensional. This implies that there are no invariant vectors of degree $2n-1$. Indeed, the only vector of bi-degree $(n-1,n)$ is

$$x := (x_2x_1)^{n-1}x_2 = x_2(x_1x_2)^{n-1}$$

which is λ_R-annihilated by x_2 but not by x_1:

$$\lambda_R(x\otimes x_1) = (x_2x_1)^n + (-q_{21})^n t_1(x_1x_2)^n \tag{109}$$

which never vanishes since the vectors $(x_2x_1)^n$ and $(x_1x_2)^n$ are linearly independent.

Thus, invariant vectors can only be of even degree $2n$ coming from bi-degree (n,n). The corresponding subspace is two-dimensional with the basis vectors $(x_1x_2)^n$ and $(x_2x_1)^n$. Taking a vector of the form

$$v_{n,\alpha} = (x_1x_2)^n + \alpha(x_2x_1)^n, \quad \alpha \in \mathbb{F}, \tag{110}$$

we calculate the λ_R-action on it of the generating elements:

$$\lambda_R(v_{n,\alpha}\otimes x_1) = (1-\alpha t_1(-q_{21})^n)(x_1x_2)^n x_1, \quad \lambda_R(v_{n,\alpha}\otimes x_2) = (\alpha - t_2(-q_{12})^n)(x_2x_1)^n x_2. \tag{111}$$

Thus, $v_{n,\alpha}$ is an invariant vector if

$$\alpha = t_2(-q_{12})^n \tag{112}$$

and

$$t_1t_2q^n = 1 \tag{113}$$

where $q = q_{12}q_{21}$.

Proposition 7.4. Assume that q is not a root of unity, and the parameters (q, t_1, t_2) satisfy (113) for some $n \geq 1$, and $(1-t_1)(1-t_2) \neq 0$. Then, the right Yetter–Drinfel'd f-module Y_n generated by the element $v_{n,\alpha}$ defined in (110) with α given by (112) is $4n$-dimensional and it is the linear span of the vector $v_{n,\alpha}$ and all vectors of degree less or equal to $2n-1$.

Proof. The coproduct of $v_{n,\alpha}$ always contains the term $1\otimes v_{n,\alpha}$ so that $1 \in Y_n$. The λ_R-action of the generating elements on 1 gives

$$\lambda_R(1\otimes x_i) = (1-t_i)x_i, \qquad i = 1, 2\,. \tag{114}$$

By the assumption on t_1 and t_2, we conclude that vectors x_1 and x_2 are both in Y_n. Assume by induction that both vectors of odd degree $2k-1$ are contained in Y_n where $1 \leq k < n$. Then, the λ_R-action on them of the generating elements produces two vectors in degree $2k$

$$\lambda_R((x_1x_2)^{k-1}x_1 \otimes x_2) = (x_1x_2)^k + t_2(-q_{12})^k(x_2x_1)^k \tag{115}$$

and

$$\lambda_R((x_2x_1)^{k-1}x_2 \otimes x_1) = (x_2x_1)^k + t_1(-q_{21})^k(x_1x_2)^k \tag{116}$$

which are linearly independent if $t_1t_2q^k \neq 1$. Thus, the λ_R-action of the generating elements on all vectors of degree $2k$ produces all vectors of degree $2k+1$. We conclude that all vectors of degree $\leq 2n-1$ are in Y_n. Now, equations (115) and (116) at $k = n$ imply that both vectors are proportional to $v_{n,\alpha}$. □

We can parametrize the variables (t_1, t_2, q) that satisfy $t_1t_2q^n = 1$ by $t_1 = 1/(q^{n/2}t)$ and $t_2 = t/q^{n/2}$. Theorem 3.6 defines an R-matrix T_n on the right Yetter–Drinfel'd f-module Y_n, and combined with Theorem 3.7 we arrive at the following knot invariants.

Definition 7.5. For every integer $n \geq 1$ we have the knot invariant

$$K \mapsto J_{T_n}(K) \in \mathrm{End}(Y_n) \tag{117}$$

and denote by $V_{n,K}(t,q) \in \mathbb{Z}[q^{\pm 1/2}, t^{\pm 1}]$ the $(1,1)$-entry of $J_{T_n}(K)$.

Conjecture 7.6. For every knot K, we have

$$J_{T_n}(K) = V_{n,K}(t,q)\,\mathrm{id}_{Y_n}\,. \tag{118}$$

Note that the symmetry σ from (99) corresponds under the above parametrizations to the symmetry $t \leftrightarrow t^{-1}$, and as a result, we have

$$V_{n,K}(t,q) = V_{n,K}(t^{-1},q)\,. \tag{119}$$

7.4. **A rank-2 analogue of the Jones polynomial.** In this section we discuss the knot invariants coming from the 4 and 8-dimensional DY-modules Y_1 and Y_2.

Calculations for $n=1$ indicate that Conjecture 7.6 holds true and that $V_{1,K}(t,q)$ coincides with the Links–Gould two-variable knot polynomial coming from the quantum superalgebra $\mathrm{U}_q(\mathfrak{gl}(2|1))$ [LG92].

When $n=2$, the knot polynomial $V_{2,K}(t,q) \in \mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$ is in a sense a rank 2 analogue of the Jones polynomial. We discuss the properties of this invariant now.

• **Symmetry.** The symmetry (119) implies that V_2 can be written in the form

$$V_{2,K}(t,q) = \tilde{V}_{2,K}(u,q) \in \mathbb{Z}[q^{\pm 1}, u], \quad u = t + t^{-1} - q - q^{-1}\,. \tag{120}$$

• **Chirality and mutation.** The 2-variable polynomial invariant V_2 detects chirality, for instance, it distinguishes the left handed and right handed trefoils. Indeed, $V_{2,\overline{3_2}}(t,q) = V_{3_2}(t,q^{-1})$ (a property that conjecturally holds for all knots) and

$$\tilde{V}_{2,\overline{3_1}}(u,q) = 1 + (q + 2q^3 - q^4 + q^5 - q^6)u + (q^2 + q^4 - q^5)u^2\,. \tag{121}$$

The V_2-polynomial distinguishes the Conway-KT pair $(11n34, 11n42)$ of mutant knots, as well as the mutant pair $(11n73, 11n74)$ where the t-degree is $(12,8)$ in both cases.

• **Specialization.** Experimentally, it appears that for all knots we have

$\tilde{V}_{2,K}(0,q) = 1$ (equivalently, $V_{2,K}(q,q) = 1$).
$\tilde{V}_{2,K}(z^2,1) = \nabla_K(z)^2$ (equivalently, $V_{2,K}(t,1) = \Delta_K(t)^2$) where $\nabla_K(z)$ is the Alexander–Conway polynomial.

• **Relation with the genus of a knot.** Experimentally, for all the knots in Table 1, as well as for a few 12 and 13 crossing knots, and for the 3-strand pretzel knots, we have:

$$\deg_t V_{2,K} = 4g(K) \tag{122}$$

where the Seifert genus $g(K)$ is the smallest genus of a spanning surface of a knot. Here, by t-degree of a Laurent polynomial of t we mean the difference between the highest and the lowest power of t. Following the ideas in the works [NvdV, KT], we expect that $4g(K)$ is an upper bound for the degrees in (122). Further computations are needed to see to which extent the equality holds.

• **Comparison with other knot polynomials.** Regarding the independence of V_2 from other knot polynomials, since it detects mutation, it is not determined neither by the colored Jones polynomials, nor by the sequence of the ADO polynomials. Moreover,

The knots $\overline{7_4}$ and 9_2 have equal Knot Floer Homology but different V_2-polynomials.
The knots 8_8 and 10_{129} have isomorphic Khovanov homology, yet different V_2-polynomial.

• **Values.** The explicit values of the V_2 polynomials (even in the shorter form $\tilde{V}_2$) are considerably more complicated than those of Table 1, but we have computed them for all the knots that appear in Table 1 and for several knots with 12 and 13 crossings. To give an idea of the complexity involved, for the knots with at most 5 crossings we have:

$$
\begin{aligned}
\tilde{V}_{2,\overline{3_1}} =& 1 + (q + 2q^3 - q^4 + q^5 - q^6)u + (q^2 + q^4 - q^5)u^2, \\
\tilde{V}_{2,4_1} =& 1 + (-q^{-3} + q^{-2} - 2q^{-1} + 2 - 2q + q^2 - q^3)u + (q^{-2} - q^{-1} + 1 - q + q^2)u^2, \\
\tilde{V}_{2,\overline{5_1}} =& 1 + (2q + 3q^3 - q^4 + 3q^5 - q^6 + 2q^7 - q^8 + q^9 - 2q^{10} + q^{11} - q^{12})u \\
&+ (4q^2 + 7q^4 - 3q^5 + 10q^6 - 6q^7 + 6q^8 - 7q^9 + 3q^{10} - 3q^{11})u^2 \\
&+ (3q^3 + 6q^5 - 3q^6 + 6q^7 - 6q^8 + 3q^9 - 3q^{10})u^3 + (q^4 + q^6 - q^7 + q^8 - q^9)u^4, \\
\tilde{V}_{2,\overline{5_2}} =& 1 + (q + 3q^3 - q^4 + 3q^5 - 2q^6 + 2q^7 - 2q^8 + q^9 - q^{10})u + (3q^2 - 2q^3 + 6q^4 - 3q^5 + 3q^6 - 3q^7 + q^8 - q^9)u^2\,.
\end{aligned}
$$

For the knots that appear in the comparison section, we have

$$
\begin{aligned}
\tilde{V}_{2,\overline{7_4}} =& 1 + (q + 3q^3 + 4q^5 - q^6 + 5q^7 - 3q^8 + 4q^9 - 4q^{10} + 2q^{11} - 3q^{12} + q^{13} - q^{14})u \\
&+ (9q^2 - 12q^3 + 22q^4 - 12q^5 + 26q^6 - 17q^7 + 15q^8 - 14q^9 + 5q^{10} - 6q^{11} + q^{12} - q^{13})u^2, \\
\tilde{V}_{2,9_2} =& 1 + (q + 3q^3 + 4q^5 - q^6 + 3q^7 - 2q^8 + 2q^9 - 2q^{10} + 2q^{11} - 2q^{12} + 2q^{13} - 2q^{14} + 2q^{15} - 2q^{16} + q^{17} - q^{18})u \\
&+ (7q^2 - 2q^3 + 12q^4 - 6q^5 + 10q^6 - 7q^7 + 9q^8 - 7q^9 + 7q^{10} - 7q^{11} + 5q^{12} - 5q^{13} + 3q^{14} - 3q^{15} + q^{16} - q^{17})u^2
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{V}_{2,8_8} =& 1 + (-q^{-6} + q^{-5} + 2q^{-3} - q^{-2} + 2q^{-1} - 2 + q + q^3 + q^4 + q^7 - 2q^8 + 2q^9 - q^{10})u \\
&+ (-q^{-5} + q^{-4} - q^{-3} + 3q^{-2} - 4q^{-1} + 7 - 4q + 12q^2 - 10q^3 + 11q^4 - 9q^5 + 7q^6 - 5q^7 + 2q^8 - q^9)u^2 \\
&+ (-3q^{-4} + 5q^{-3} - 10q^{-2} + 18q^{-1} - 17 + 25q - 19q^2 + 19q^3 - 14q^4 + 9q^5 - 6q^6 + 2q^7 - q^8)u^3 \\
&+ (-3q^{-3} + 5q^{-2} - 7q^{-1} + 13 - 10q + 12q^2 - 9q^3 + 7q^4 - 5q^5 + 2q^6 - q^7)u^4, \\
\tilde{V}_{2,10_{129}} =& 1 + (-q^{-10} + q^{-9} - q^{-8} + q^{-7} + 2q^{-4} + q^{-1} - 2 + 3q - 2q^2 + 3q^3 - 2q^4 + q^5)u \\
&+ (q^{-9} - 3q^{-8} + q^{-7} + q^{-6} - 4q^{-5} + 8q^{-4} - 9q^{-3} + 11q^{-2} - 8q^{-1} + 11 - 5q + 6q^2 - 3q^3 + 3q^4 - 3q^5 + 2q^6 - q^7)u^2 \\
&+ (2q^{-8} - 4q^{-7} + 2q^{-6} - 10q^{-4} + 20q^{-3} - 30q^{-2} + 42q^{-1} - 38 + 40q - 26q^2 + 18q^3 - 10q^4 + 4q^5 - 2q^6)u^3 \\
&+ (-3q^{-5} + 5q^{-4} - 8q^{-3} + 12q^{-2} - 12q^{-1} + 16 - 10q + 8q^2 - 5q^3 + 2q^4 - q^5)u^4\,,
\end{aligned}
$$

as well as

$$
\begin{aligned}
\tilde{V}_{2,11n34} =& 1 + (-q^{-10} + 2q^{-9} - 2q^{-7} + 4q^{-5} - 2q^{-4} - 4q^{-3} + 6q^{-2} - 2q^{-1} - 4 + 6q - 6q^2 + 6q^3 - 7q^4 + 8q^5 - 3q^6 - 4q^7 \\
&+ 5q^8 - q^9 - 2q^{10} + 2q^{12} - q^{13})u + (-2q^{-7} + 5q^{-6} - 5q^{-5} + 4q^{-4} - 5q^{-3} + 11q^{-2} - 13q^{-1} + 9 - 4q + 2q^2 - q^3 \\
&- 2q^4 + 5q^5 - 7q^6 + 2q^7 + 2q^8 - q^9 - 2q^{10} + 4q^{11} - 2q^{12})u^2 + (2q^{-8} - 5q^{-7} + 3q^{-6} + 4q^{-5} - 15q^{-4} + 26q^{-3} \\
&- 33q^{-2} + 41q^{-1} - 48 + 53q - 53q^2 + 49q^3 - 44q^4 + 36q^5 - 27q^6 + 13q^7 - 5q^9 + 5q^{10} - 2q^{11})u^3 + (3q^{-7} - 9q^{-6} \\
&+ 12q^{-5} - 9q^{-4} - 4q^{-3} + 23q^{-2} - 44q^{-1} + 69 - 85q + 85q^2 - 69q^3 + 44q^4 - 23q^5 + 4q^6 + 9q^7 - 12q^8 + 9q^9 \\
&- 3q^{10})u^4 + (3q^{-6} - 9q^{-5} + 12q^{-4} - 11q^{-3} + 17q^{-1} - 28 + 38q - 38q^2 + 28q^3 - 17q^4 + 11q^6 - 12q^7 + 9q^8 - 3q^9)u^5 \\
&+ (q^{-5} - 3q^{-4} + 3q^{-3} - 1q^{-2} - 2q^{-1} + 5 - 5q + 5q^2 - 5q^3 + 2q^4 + q^5 - 3q^6 + 3q^7 - q^8)u^6, \\
\tilde{V}_{2,11n42} =& 1 + (q^{-10} + 2q^{-9} - 2q^{-7} + 4q^{-5} - 2q^{-4} - 4q^{-3} + 6q^{-2} - 2q^{-1} - 4 + 6q - 6q^2 + 6q^3 - 7q^4 + 8q^5 - 3q^6 - 4q^7 + 5q^8 \\
&- q^9 - 2q^{10} + 2q^{12} - q^{13})u + (-2q^{-7} + 4q^{-6} - q^{-5} - 2q^{-4} + 8q^{-2} - 15q^{-1} + 17 - 13q + 11q^2 - 9q^3 + 8q^5 - 12q^6 \\
&+ 8q^7 - 2q^8 - 2q^{10} + 4q^{11} - 2q^{12})u^2 + (q^{-8} - 2q^{-7} + q^{-6} - q^{-5} + 2q^{-4} - 4q^{-3} + 7q^{-2} - 8q^{-1} + 10 - 10q + 10q^2 \\
&- 9q^3 + 5q^4 - 4q^5 + 3q^6 - 4q^7 + 5q^8 - 3q^9 + 2q^{10} - q^{11})u^3 + (q^{-5} - 3q^{-4} + 3q^{-3} - q^{-2} - q^{-1} + 3 - 4q + 4q^2 \\
&- 3q^3 + q^4 + q^5 - 3q^6 + 3q^7 - q^8)u^4\,,
\end{aligned}
$$

and finally the values for the pair $(K11n73, K11n74)$ which are not distinguished by the polynomial Λ_{-1} of Section 7.1

$$\begin{aligned}
\tilde{V}_{2,11n73} =&1+(-q^{-13}+2q^{-12}-2q^{-10}+q^{-9}+2q^{-8}-4q^{-7}+4q^{-5}-5q^{-4}+6q^{-3}-6q^{-2}+8q^{-1}-6+3q+3q^{2}-2q^{3}-q^{4}\\
&+4q^{5}-2q^{6}+q^{9}-q^{10})u+(-3q^{-12}+7q^{-11}-5q^{-10}+2q^{-9}+q^{-8}-2q^{-7}-2q^{-6}+2q^{-5}-4q^{-4}+5q^{-3}-5q^{-2}\\
&+5q^{-1}+2-5q+13q^{2}-8q^{3}+6q^{4}-4q^{5}+4q^{6}-3q^{7}+q^{8}-q^{9})u^{2}+(-5q^{-11}+14q^{-10}-18q^{-9}+22q^{-8}-21q^{-7}\\
&+14q^{-6}-12q^{-5}+6q^{-4}-8q^{-2}+14q^{-1}-15+22q-18q^{2}+20q^{3}-14q^{4}+6q^{5}-2q^{6}-2q^{7}+q^{8})u^{3}+(-6q^{-10}\\
&+18q^{-9}-26q^{-8}+36q^{-7}-38q^{-6}+26q^{-5}-15q^{-4}-10q^{-3}+32q^{-2}-40q^{-1}+46-33q+20q^{2}-9q^{3}-2q^{4}+5q^{5}\\
&-6q^{6}+3q^{7})u^{4}+(-4q^{-9}+12q^{-8}-16q^{-7}+20q^{-6}-17q^{-5}+2q^{-4}+6q^{-3}-17q^{-2}+26q^{-1}-20+16q-6q^{2}\\
&-5q^{3}+6q^{4}-6q^{5}+3q^{6})u^{5}+(-q^{-8}+3q^{-7}-3q^{-6}+2q^{-5}-q^{-4}-2q^{-3}+3q^{-2}-3q^{-1}+4-2q+q^{3}-2q^{4}+q^{5})u^{6},\\
\tilde{V}_{2,11n74} =&1+(-q^{-13}+2q^{-12}-2q^{-10}+q^{-9}+2q^{-8}-4q^{-7}+4q^{-5}-5q^{-4}+6q^{-3}-6q^{-2}+8q^{-1}-6+3q+3q^{2}-2q^{3}-q^{4}\\
&+4q^{5}-2q^{6}+q^{9}-q^{10})u+(-2q^{-12}+4q^{-11}-2q^{-10}+q^{-8}+4q^{-7}-10q^{-6}+8q^{-5}-10q^{-4}+5q^{-3}+q^{-2}-q^{-1}\\
&+10-11q+13q^{2}-6q^{3}+3q^{4}-q^{5}+3q^{6}-3q^{7}+q^{8}-q^{9})u^{2}+(-q^{-11}+2q^{-10}-2q^{-9}+2q^{-8}+2q^{-7}-4q^{-6}\\
&+4q^{-5}-7q^{-4}+q^{-3}+6-q+4q^{2}-q^{4}+q^{5}-2q^{6})u^{3}+(q^{-6}-2q^{-5}+2q^{-4}-3q^{-3}+2q^{-2}-q^{-1}+2+q-q^{2}\\
&+q^{3}-q^{4})u^{4}\,.
\end{aligned}$$

8. Summary of rank 1 and rank 2 polynomials

Here, we summarise the polynomial invariants that we have considered in the form of two tables, one at roots of unity and another at generic values of q. Within each table, the underlying R-matrix is a linear automorphism of $V \otimes V$ for a finite dimensional vector space V given by a Nichols algebra when q is a root of unity and by a Yetter–Drinfel'd f-module when q is generic. We denote the normalized Alexander polynomial by $\Delta(t)$, the ADO polynomial at a root of unity ω by $\mathrm{ADO}_\omega(t)$, and the n-colored Jones polynomial (with $n = 2$ equal to the Jones polynomial) by $J_n(q)$. All these polynomials are normalized to take the value 1 at the unknot.

	rank 1	rank 2
$\dim(V)$	N	$4N$
Invariant	$\mathrm{ADO}_\omega(t)$	$\Lambda_\omega(t_1, t_2)$
$\omega = 1$	1	$\Lambda_1 \overset{?}{=} \Delta(t_1)\Delta(t_2)$
$\omega = -1$	$\Delta(t)$	$\Lambda_{-1} \overset{?}{=}$ Harper polynomial

Table 2. Knot polynomials at primitive N-th roots of unity ω.

	rank 1	rank 2
$\dim(V)$	n	$4n$
Invariant	$J_n(q)$	$V_n(t, q)$
$n = 1$	1	$V_1 \overset{?}{=}$ Links–Gould polynomial
$n = 2$	$J(q)$	$V_2(t, q)$

Table 3. Knot polynomials at generic values of q.

The invariants at roots of unity should be dual to those at generic values of q in the sense of equivalences

$$\{\mathrm{ADO}_\omega(t) \mid \omega \text{ is a root of unity}\} \longleftrightarrow \{J_n(q) \mid n \geq 1\}, \tag{123a}$$

$$\{\Lambda_\omega(t_1, t_2) \mid \omega \text{ is a root of unity}\} \longleftrightarrow \{V_n(t, q) \mid n \geq 1\}. \tag{123b}$$

Mathematically, the meaning of these equivalences is that for each knot, the set of its polynomial invariants on the left determines the set of its invariants on the right and vice-versa. The equivalence (123a) is known, and it follows from Habiro's expansion of the colored Jones polynomial [Hab08, Wil22]. Explicitly, the left hand side determines the right hand side of the above equivalences by

$$\mathrm{ADO}_\omega(\omega^{1-n}) = J_n(\omega), \tag{124a}$$

$$\Lambda_\omega(t\omega^{-n/2}, t^{-1}\omega^{-n/2}) = V_n(t, \omega) \tag{124b}$$

valid for all positive integers $n > 0$ and all complex roots of unity ω. A proof of these equalities will be given in a subsequent publication. Equation (124a) (with our normalization of the ADO and colored Jones polynomials) appears in Murakami–Nagatomo [MN08].

As a sanity check, Equation (124b) for $\omega = -1$ is a consistency relation with the values of the Λ_{-1} polynomials given in Table 1 with the polynomials V_2 given in this section.

Finally, we mention that the interpretation of (123a) and (123b) in mathematical physics is a kind of duality or correspondence arising in supersymmetric quantum field theories; see for example [GHN⁺21].

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