BEHAVIOR OF KNOT INVARIANTS UNDER GENUS 2 MUTATION

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ABSTRACT. Genus 2 mutation is the process of cutting a 3-manifold along an embedded closed genus 2 surface, twisting by the hyper-elliptic involution, and gluing back. This paper compares genus 2 mutation with the better-known Conway mutation in the context of knots in the 3-sphere. Despite the fact that any Conway mutation can be achieved by a sequence of at most two genus 2 mutations, the invariants that are preserved by genus 2 mutation are a proper subset of those preserved by Conway mutation. In particular, while the Alexander and Jones polynomials are preserved by genus 2 mutation, the HOMFLY-PT polynomial is not. In the case of the sl_2-Khovanov homology, which may or may not be invariant under Conway mutation, we give an example where genus 2 mutation changes this homology. Finally, using these techniques, we exhibit examples of knots with the same same colored Jones polynomials, HOMFLY-PT polynomial, Kauffman polynomial, signature and volume, but different Khovanov homology.

CONTENTS

1. INTRODUCTION

In the 1980s, a plethora of new knot invariants were discovered, following the discovery of the Jones polynomial [J]. These powerful invariants were by construction *chiral*, i.e. they were often able to distinguish knots from their mirrors, as opposed to many of their classical counterparts. Soon after the appearance of these new quantum invariants of knots, many people studied their behavior under other kinds of involutions, and in particular under mutation. Chmutov, Duzhin, Lando, Lickorish, Lipson, Morton, Traczyk and others pioneered the behavior of the quantum knot invariants under mutation; see [CDL, LL, MC, MR1] and references therein. The quantum invariants come in two flavors: rationally valued Vassiliev invariants, and

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polynomially valued exact invariants (such as the Jones, HOMFLY, Kauffman, Alexander polynomials), see [Tu2]. Later on, abelian group valued invariants were constructed by Khovanov [Kh].

Here, we study the behavior of classical and quantum invariants of knots in S^3 under mutation, building on the above mentioned work. The notion of mutation was introduced by Conway in [Co], and has been used extensively in various generalized forms. Let us start by explaining what we mean by mutation. Roughly, mutation is modifying a 3-manifold by cutting it open along a certain kind of embedded surface, and then regluing in a different way. More precisely, consider one of the surfaces F from Figure 1.1, together with the specified involution τ; we will call the pair (F, τ) a *symmetric surface*. Suppose F is a symmetric surface

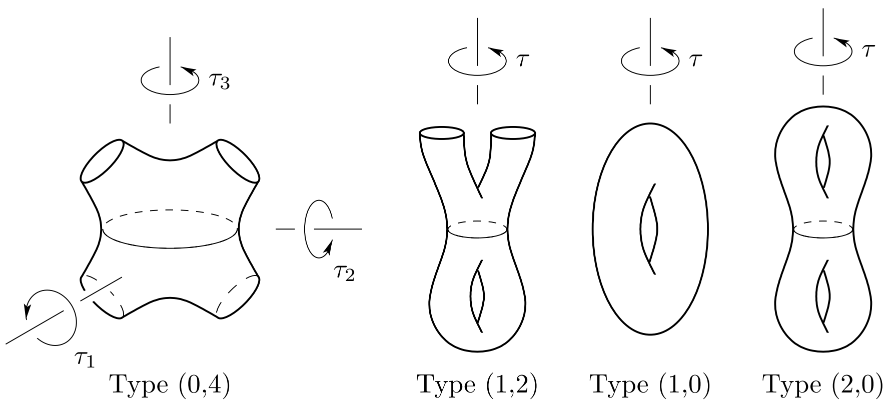

Figure 1.1. Symmetric surfaces of types $(0,4)$, $(1,2)$, $(1,0)$, and $(2,0)$ and their involutions. There are also symmetric surfaces of type $(1,1)$ and $(0,3)$ that are not pictured, since we will not need them here.

properly embedded in a compact orientable 3-manifold. The *mutant* of M along F is the result of cutting M open along F, and then regluing the two copies of F by the involution τ. The mutant manifold is denoted M^τ, and the operation is called *mutation*. When we want to distinguish the topological type of F, we refer to (g, s)-mutation where g is the genus and s is the number of boundary components.

The involutions used in mutation have very special properties, e.g. if γ is a non–boundary-parallel simple closed curve, then $\tau(\gamma)$ is isotopic to γ (neglecting orientations). As a result, while mutation is typically violent enough to change the global topology of M, it is simultaneously subtle enough that many invariants do not change. Studying this phenomenon has enriched our understanding of a number of invariants, be they classical, quantum, or geometric.

When studying knots in S^3, the most natural type of mutation is $(0,4)$-mutation, which has a simple interpretation in terms of a knot diagram, and is known to preserve a wide range of invariants. Here, we study the effects of $(2,0)$-mutation on knots in S^3. By this, we mean the following. If F is a closed 2-surface in S^3, then the mutant $(S^3)^\tau$ is always homeomorphic to S^3 (see Section 2.6). Thus if K is a knot in S^3 which is disjoint from F, it makes sense to talk about its mutant K^τ.

In this context, $(2,0)$-mutation is the most general type: any of the above mutations can be achieved by a sequence of at most two $(2,0)$-mutations (see Lemma 2.5 below). Given that any $(0,4)$-mutation can be implemented in this way, you might expect that an invariant unchanged by $(0,4)$-mutation would also be preserved by $(2,0)$-mutations. It turns out that this is not the case, as you can see from the following table; with the possible exception of Khovanov homology, all of the invariants listed there are preserved by $(0,4)$-mutation.

The results on the left-hand side are due either entirely or in large part to Ruberman [Ru], Cooper-Lickorish [CL] and Morton-Traczyk [MT], see below for details; the results on the right are new. One way of interpreting these results might be that the invariants on the left are more tied to the topology of $S^3 \setminus K$, whereas those on the right are more "diagrammatic" and tied to combinatorics of knot projections.

Preserved by $(2,0)$-mutation	Changed by $(2,0)$-mutation
Hyperbolic volume/Gromov norm of the knot exterior Alexander polynomial and generalized signature Colored Jones polynomial (for all colors)	HOMFLY-PT polynomial sl_2-Khovanov Homology

Table 1.2. Summary of known results on genus 2 mutation.

(Of course, this must be taken with more than a grain of salt, since knots are determined by their complements [GL1].) One of our original motivations for this work was to better understand the Volume Conjecture, which proposes a relationship between the colored Jones polynomials and the hyperbolic volume. The fact that both the colored Jones polynomials and hyperbolic volume are preserved by $(2,0)$-mutation is positive evidence for this conjecture.

One interesting open problem about $(0,4)$-mutation is whether this operation can change the sl_2-Khovanov homology introduced in [Kh]. For $(2,0)$-mutation, we settle the analogous question:

Theorem 1.3. *The sl_2-Khovanov Homology is not invariant under $(2,0)$-mutation of knots. In particular, the pair of $(2,0)$-mutant knots in Figure 1.5 have differing Khovanov homologies.*

For the odd variant of sl_2-Khovanov homology, Bloom recently showed that it is invariant under $(0,4)$-mutation [B]; as a consequence, the normal sl_2-Khovanov homology with mod 2 coefficients is also invariant. We do not whether either of these invariants is preserved by $(2,0)$-mutation.

Question 1.4. Is the odd sl_2-Khovanov homology preserved by genus 2 mutation?

The sl_n-homology introduced by Khovanov and Rozansky [KR] cannot be invariant under $(2,0)$-mutation, simply because the Euler characteristic need not be, since the HOMFLY-PT polynomial can change under $(2,0)$-mutation.

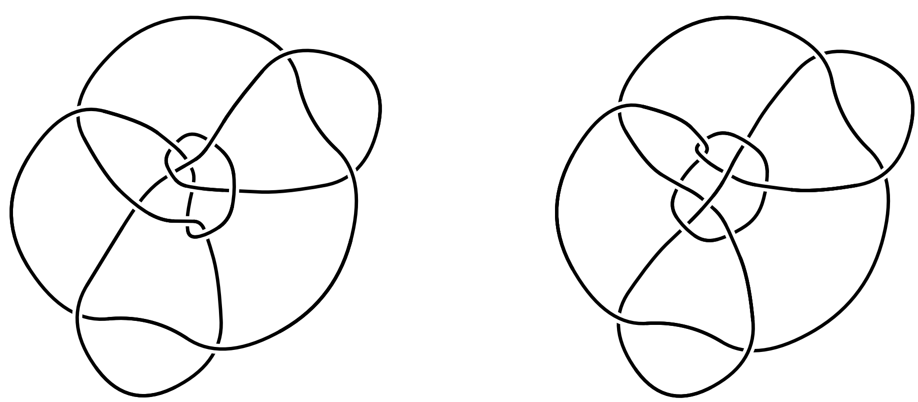

Figure 1.5. The pair of knots 14^n_{22185} (left) and 14^n_{22589} (right), in Knotscape notation.

One final result of this paper is

Proposition 1.6. *There exist knots with same colored Jones polynomials (for all colors), HOMFLY-PT and Kauffman polynomials, volume and signature, but different Khovanov (and reduced Khovanov) homology.*

The knots from Figure 1.5 are again examples here, and all the above claimed properties except for the Khovanov homology are consequences of the fact that they are $(2,0)$-mutant (see Figure 3.9.a). These same knots were studied by Stoimenow and Tanaka [ST1, ST2], who showed that these knots are not $(0,4)$-mutants, yet have the same colored Jones polynomials. (Stoimenow and Tanaka use notation 14_{41721} and 14_{42125} for what we denote 14^n_{22185} and 14^n_{22589}, respectively.)

There are other invariants whose behavior under genus 2 mutation it would be interesting to understand. In particular:

Question 1.7. Is the Kauffman polynomial invariant under genus 2 mutation? What about the property of having unknotting number one?

Classical Conway $(0, 4)$-mutation preserves both these properties [L2, GL2]. As we discuss in Section 3.6 below, we expect that, in analogy with what happens with the HOMFLY-PT polynomial, genus 2 mutation should be able to change the Kauffman polynomial. **Addendum:** Morton and Ryder have confirmed this, showing that the Kauffman polynomial is *not* invariant under genus 2 mutation [MR2].

We now detail where the results in Table 1.2 come from. The invariance of the hyperbolic volume, or more generally the Gromov norm, was proven by Ruberman for all types of mutation [Ru]. The statement [Ru, Thm. 1.5] requires an additional hypothesis on F, but arguments elsewhere in [Ru] negate the need for this; see our discussion of Theorem 2.4 below. Cooper and Lickorish proved the invariance of the Alexander polynomial and generalized signature under a more limited class of $(2, 0)$-mutations than we consider here [CL]. This class, which we call handlebody mutations, turns out to be the main case anyway, and thus it is not hard to conclude the more general result; see Theorem 2.9 below. In the case of the colored Jones polynomials (for a definition see e.g. [J, Tu1]), the result essentially follows from Morton-Traczyk [MT], which we modify as Theorem 3.2. In the case of the non-invariance of the HOMFLY-PT polynomials, we give explicit examples based on the ideas of Section 3.4.

As usual, the presentation of our results does not follow the historical order by which they were discovered. The project started by running a computer program of A.Sh. (see [Sh]) to all knots with less than or equal to 16 crossings, taken from Knotscape [HTh]. The computer found a single pair of 14 crossing knots with the same HOMFLY-PT polynomial, Kauffman polynomial, signature, volume and different Khovanov Homology, and four pairs of 15 crossing knots with same behavior. The knots were isolated, redrawn, and a pattern was found. Namely, the knots in the above pairs have diagrams that differ by a so-called cabled mutation (see Section 2.10 for a definition). Cabled mutation can always be achieved by $(2, 0)$-mutation. This, together with a Kauffman bracket skein theory argument (which we later found in Morton-Traczyk's work [MT]) implies that these pairs have identical colored Jones polynomials, for all colors. At that time, the numerical equality of the volumes of these pairs was rather mysterious. Later on, we found that cabled mutation is a special case of $(2, 0)$-mutation. Ruberman's theorem explained why these pairs have equal volume. Once it was observed that Khovanov homology was not invariant under $(2, 0)$-mutation, we asked whether this was true for other well-known knot invariants, such as the colored Jones polynomials, the HOMFLY-PT and the Kauffman polynomials. Once we realized that the HOMFLY-PT and Kauffman polynomials ought to detect $(2, 0)$-mutation (and even cabled mutation), we tried to find examples of such knots.

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2. The topology of knot mutation

This section gives the basic topological lemmas about mutation that we will need. In addition to checking that $(2, 0)$-mutation of a knot in S^3 makes sense (i.e. mutating S^3 along such a surface always gives back S^3), we will show that one can usually reduce to the case where the mutation surface has a number of special properties. Finally, we introduce the notion of cabled mutation for knots in S^3, which is a special type of genus 2 mutation which is easy to realize diagrammatically.

We begin in the context of general 3-manifolds before specializing to the case of knots in S^3. From a topological point of view, it is often best to work with mutation surfaces that are incompressible. The following proposition is implicit in [Ru, Sec. 5], and explicit in a slightly weaker form in [Ka2, Lem. 2.2]; one application below will be to show that mutation makes sense for knots in S^3.

Proposition 2.1. *Let F be a closed genus 2 surface in a compact orientable 3-manifold M. Then either:*

(1) *F is incompressible, or*
(2) *M^τ can be obtained by mutating along one or two incompressible, non-boundary parallel tori, or*
(3) *$M^\tau \cong M$.*

Proof. The basic idea here is that if F is compressible, then M^τ is homeomorphic to the result of mutating M along any surface obtained by compressing F. So suppose D is an embedded compressing disc for F. Initially, let us suppose that ∂D is a non-separating curve in F. The key property of the hyper-elliptic involution τ is that if γ is any non-separating simple closed curve in F, then $\tau(\gamma)$ is isotopic to γ with the orientation reversed. Thus, we can isotope D so that $\tau(\partial D) = \partial D$, and the restriction of τ to ∂D is a reflection (that is, conjugate to reflecting a circle centered at the origin of $\mathbb{R}^2$ about the x-axis).

Now perform a surgery of F along D to obtain a surface T, which consists of the union of $F \setminus N(\partial D)$ with two parallel copies of D. Since ∂D is non-separating, T is a torus. There is a natural homomorphism σ of T which agrees with τ on $F \setminus N(\partial D)$ and permutes the two copies of D. We claim that

(1) The involution σ is just the elliptic involution of the torus shown in Figure 1.1.
(2) $M^\tau \cong M^\sigma$.

The first point is clear, and so turning to the second let us assume (for notational simplicity only) that F separates M. Denote by M_1 and M_2 the two pieces of M cut along F. Let X be the complement in M_2 of a product regular neighborhood N of D; we can then view our surface T as ∂X. Both M^τ and M^σ can be thought of as obtained by gluing together the pieces M_1, X, and N. Moreover, the way that M_1 and X are glued is exactly the same in both cases, since τ and σ agree on $F \setminus N$; hence M^τ and M^σ differ only in how the ball N is attached. Since there is a unique way of attaching a 3-ball to a 2-sphere up to homeomorphism, we have $M^\tau \cong M^\sigma$ as claimed. (You can also see the homeomorphism of N needed to build the map $M^\tau \to M^\sigma$ directly — thinking of N as a pancake, just flip it over.)

Thus in the case that ∂D is non-separating, we have shown that M^τ is homeomorphic to a mutant of M along a torus T. If ∂D is separating, then the picture is essentially the same. In this case, we can isotope ∂D so that τ fixes it pointwise. Proceed as above, the only difference being that now surgering F along D results in a disconnected surface consisting of two tori. Thus in either case, M^τ is homeomorphic to the result of mutating M along either one or two tori.

So to complete the proof of the proposition, we just need to show that if T is a torus in M with elliptic involution σ, then either

(1) T is incompressible and not boundary parallel.
(2) $M^\sigma \cong M$.

If T were boundary parallel, then mutating along it doesn't change the topology since the gluing map σ extends over the product region bounded by T and a component of ∂M. If T is compressible, then arguing as above we see that M^σ is homeomorphic to the result of mutating along a 2-sphere S in M, where the gluing map ϕ is just rotation of S about some axis through angle π; since ϕ is isotopic to the identity, we have that $M \cong M^\phi \cong M^\sigma$, as desired. □

Remark 2.2. Later, we will apply this proposition to a manifold M where ∂M is a torus, and need the following fact. As setup, note that since F is closed, there is a canonical identification of ∂M with ∂M^τ. The observation is that if we end up in case (3) where $M^\tau \cong M$, then the proof shows that there is a homeomorphism $f : M \to M^\tau$ where the restriction of f to ∂M is either the identity or the elliptic involution. (The later happens when part of F compresses to something parallel to the boundary torus.)

Remark 2.3. While Proposition 2.1 nominally concerns only genus 2 mutation, there are analogous statements for any of the symmetric surfaces, which follow from the same proof.

Ruberman proved that if M is hyperbolic, and F any symmetric surface in M, then M^τ is also hyperbolic and, moreover, M and M^τ have the same volume. This is stated in [Ru, Thm. 1.3] with the additional hypothesis that F is incompressible. However, as he observed in Section 5 of that same paper, this hypothesis can be dropped by appealing to Proposition 2.1 and Remark 2.3. Similarly, one has:

Theorem 2.4 ([Ru])**.** *Let M be a orientable 3-manifold, whose boundary, if any, consists of tori. Then the result of mutating M along any symmetric surface has the same Gromov norm as M itself.*

In the context of knots in S^3 that we consider below, we will be dealing with manifolds where ∂M is a single torus. In this case, Ruberman [Ru, Sec. 5] and Tillmann [Ti1, Rem. 1.3] observed that all of the types of mutations pictured in Figure 1.1 can be reduced to a sequence of genus 2 mutations, provided the mutation surface is separating.

Lemma 2.5 ([Ru, Ti1])**.** *Suppose M is a compact orientable 3-manifold whose boundary is a single torus. Let F be one of the symmetric surfaces depicted in Figure 1.1. Provided F is separating, mutation along F can always be accomplished by a composition of at most two $(2,0)$-mutations.*

The idea they used to prove this lemma is to tube copies of F along ∂F to build a closed genus 2 surface S. Mutating along S is the same as doing a certain mutation along the original surface F, for reasons similar to the proof of Proposition 2.1. In the case where F is a 4-punctured sphere, it may not be possible that the desired involution τ_i can be directly induced by mutation along a tubed surface S; however, in this case the needed mutation can be realized by mutating along the possible choices for S in succession.

2.6. **Genus 2 mutation of knots in S^3.** Suppose that F is a closed genus 2 surface in S^3. As S^3 is simply connected, the Loop Theorem implies that F, as well as any torus in S^3, is compressible. Therefore, the trichotomy of Proposition 2.1 forces $(S^3)^\tau$, the result of mutation along F, to again be homeomorphic to S^3. Thus if K is a knot in S^3 disjoint from F, then we can consider the resulting knot K^τ in $(S^3)^\tau \cong S^3$, which we call the mutant of K along F.

When the surface F bounds a genus 2 handlebody H in S^3, then the mutation operation is particularly simple to describe, since the hyper-elliptic involution τ extends to give a self-homeomorphism of H. When the knot K is contained in H, we say that K^τ is obtained from K by $(2,0)$-*handlebody mutation.* (If instead K is in the complement of H, then $K^\tau \cong K$.) Such $(2,0)$-handlebody mutation was studied by Cooper-Lickorish [CL], who were interested in how it affected the Alexander polynomial.

As the next proposition shows, $(2,0)$-handlebody mutation is actually the main interesting case of genus 2 mutation, the only other case being $(1,0)$-*handlebody mutation*, which is defined analogously.

Proposition 2.7. *Let K be a knot in S^3 which is disjoint from a genus 2 surface F. Then either:*

- *K^τ is obtained from K by $(2,0)$-handlebody mutation, or*
- *K^τ is obtained from K by one or two $(1,0)$-handlebody mutations, or*
- *$K^\tau \cong K$.*

Proof. Let $M = S^3 \setminus N(K)$ be the exterior of K. Applying Proposition 2.1 to F thought of as a surface in M, we have three cases.

First, F may be incompressible in M; in this case, we claim this is actually a $(2,0)$-handlebody mutation. Let X and Y be the two pieces of S^3 cut along F, and suppose that K lies in X. Since F is incompressible in M, it is also incompressible as the boundary of Y. Thus any compressing disc for F in S^3 lies in X. Pick two such compressing discs, whose boundaries are disjoint non-parallel non-separating curves in F (by Dehn's Lemma, every embedded curve in F bounds a compressing disc as $\pi_1(S^3) = 1$). If we compress F along both these discs, we get a sphere which bounds a ball on *both* sides. This shows X is handlebody.

Second, suppose mutation along F in M can be achieved by one or two mutations along incompressible tori. The argument just given shows that those are $(1,0)$-handlebody mutations.

Finally, suppose that we are in the final case where $M^\tau \cong M$. This shows that the complements of K^τ and K are the same, but we need to show that the knots themselves are the same. Of course, knots are determined by their complements [GL1], but we now give an elementary argument. We can reconstruct K from M if we just mark the loop on ∂M which is the meridian for K, and the same for K^τ and M^τ. By Remark 2.2, the homeomorphism of $M^\tau \to M$ takes the meridian to the meridian, establishing $K^\tau \cong K$ as desired. □

A $(1,0)$-handlebody mutation may be realized by a $(2,0)$-handlebody mutation simply by adding a nugatory handle. Thus,

Corollary 2.8. *Any knot invariant which does not change under* $(2,0)$*-handlebody mutation, does not change under* $(2,0)$*-mutation.*

Using this, we can generalize [CL] to:

Theorem 2.9. *The Alexander polynomial and the generalized signatures of a knot in* S^3 *do not change under* $(2,0)$*-mutation.*

Proof. In [CL, Cor.8] Cooper-Lickorish prove that these invariants do not change under $(2,0)$-handlebody mutation. The result thus follows from Corollary 2.8. □

2.10. **Cabled mutation.** In this short section, we introduce the notion of cabled mutation, which is a special form of genus 2 mutation which we will use to construct examples where the HOMFLY-PT polynomial changes under mutation.

Consider a framed 2-2 tangle T in a ball, that is, a ball containing two disjoint properly embedded arcs (the *strings*), where each arc has a preferred framing. If T were part of a knot, then we could do $(0,4)$-mutation on it using one of the three involutions pictured in Figure 1.1. Let τ be one of these involutions which is string-preserving, that is, exchanges one of the endpoints of a fixed arc with the other. Let T^τ denote the image of T under the involution. Given natural numbers $n, m \geq 1$, let $T(n,m)$ (resp. $T^\tau(n,m)$) denote the tangle obtained by taking a n and m parallel of the strings of T (resp. T^τ).

Definition 2.11. Connected cabled mutation (or simply, cabled mutation) is the result of replacing $T(n,m)$ by $T^\tau(n,m)$ in some planar diagram of a knot of a knots in S^3.

When $n = m = 1$, cabled mutation is just usual $(0,4)$-mutation. One motivation for studying this notion is that $(0,4)$-mutation followed by connected cabling can be often be achieved by a connected cabled mutation.

Our next lemma discusses the relation between cabled mutation and genus 2 mutation.

Lemma 2.12. *Cabled mutation is a special form of genus 2 mutation.*

Proof. Starting with the boundary of the tangle T we can attach two tubes inside it, containing the strands of $T(n,m)$, to produce a closed genus 2 surface F. The cabled mutation on $T(n,m)$ can then be achieved by cutting along F and regluing; because the original involution on T is string preserving, the map we reglue F by is the hyper-elliptic involution τ pictured in Figure 1.1. (If τ was not strand preserving, then the regluing map for F is some other involution and this is not a mutation.) □

3. Behavior of quantum invariants under mutation

As mentioned in the introduction, many knot invariants are preserved under Conway $(0,4)$-mutation. Such invariants include the HOMFLY-PT (and, hence, Jones and Alexander) and Kauffman polynomials, see for example [L2, LL, MC, MT, CL]. In this section we analyze the behavior of several quantum invariants under $(2,0)$-mutation.

3.1. **Invariance of the colored Jones polynomials under** $(2,0)$**-mutation.** Morton and Traczyk showed that the colored Jones polynomials are invariant under Conway mutation [MT]. As we now describe, their approach easily generalizes:

Theorem 3.2. *The colored Jones polynomials of a knot are invariant under* $(2,0)$*-mutation for all colors.*

Proof. The theorem follows from the fact that the colored Jones polynomial can be defined via the Kauffman bracket skein theory, in the style of topological quantum field theory, see [Kf]. By Corollary 2.8 it suffices to consider genus 2 *handlebody* mutation.

The Kauffman bracket skein module of a genus 2 handlebody has a basis that consists of all the colored trivalent graphs $G(a,b,c)$, where a, b, and c are nonnegative integers with $c \leq 2\min\{a,b\}$ (see Figure 3.3). Indeed, a genus 2 handlebody is diffeomorphic to a (twice punctured disk) $\times I$, and a basis for the Kauffman bracket of the latter is given in [PS, Cor. 4.4]. Since this basis is clearly invariant under τ, it implies that the colored Jones polynomials are invariant under $(2,0)$-handlebody mutation, proving the theorem. □

Combining Theorem 3.2 with the Melvin-Morton-Rozansky Conjecture (settled in [B-NG]) gives an alternate proof of Theorem 2.9, namely that the Alexander polynomial of a knot is invariant under $(2,0)$-mutation.

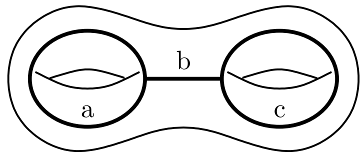
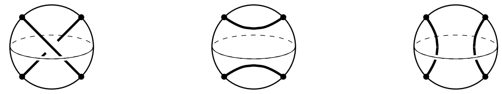


Figure 3.3. Basis of the Kauffman skein module of a closed genus 2 surface.

3.4. **Non-invariance of the HOMFLY-PT polynomial under** $(2,0)$**-mutation.** It is not hard to see that the HOMFLY-PT and Kauffman polynomials are invariant under $(0,4)$-mutation [L2]. This follows from the fact that the corresponding skein modules of a 3-ball with 4 marked points on the boundary have a basis consisting of the following three diagrams that are invariant under the involution in question:



In contrast, genus 2 mutation can change the HOMFLY-PT polynomial. In particular, we found a 75 crossing knot K_{75} which has a cabled mutant with differing HOMFLY-PT polynomials. This knot is depicted in Figure 3.5. As you can see, K_{75} contains a $(3,3)$-cabled tangle which is the region below the horizontal line; let K_{75}^{τ} be the cabled mutant of K_{75} with respect to a string-preserving involution τ of this tangle.

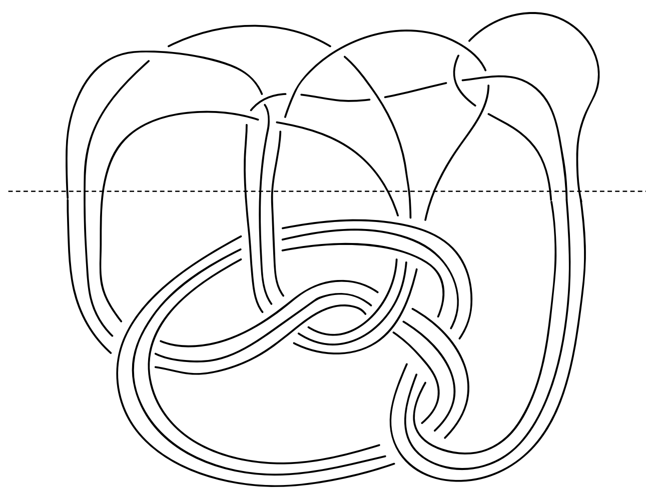


Figure 3.5. The knot K_{75}. It and its cabled mutant K_{75}^{τ} have different HOMFLY-PT polynomials

Direct computation with the Ewing-Millett computer program implemented in Knotscape shows that K_{75} and K_{75}^{τ} have different HOMFLY-PT polynomials. Coefficients of these polynomials are given in Table 3.15 on page 14 (with zero entries omitted). For example, the coefficient of the monomial $m^2 l^{-2}$ is 56 in both polynomials. On the other hand, the coefficients of $m^4 l^{-2}$ are -953 for K_{75} and -964 for K_{75}^{τ}.

Here is a heuristic reason why the HOMFLY-PT polynomial is not invariant under $(2,0)$-mutation, which explains how we came across our pair of 75 crossing knots. First, it was already known that there are $(2,0)$-mutant *links* with different HOMFLY-PT polynomials [CL]. In particular, start with the Kinoshita-Terasaka and Conway knots which are a famous pair of 11 crossing knots which differ by $(0,4)$-mutation. Morton and Traczyk showed (see [MC]) that a certain disconnected 3-cable on these knots have differing HOMFLY-PT polynomials; this gives a pair of cabled-mutant links with distinct HOMFLY-PT polynomials. (In contrast, Lickorish-Lipson showed [LL] that the HOMFLY-PT polynomial of 2-cables of mutant knots are always equal.) This suggests that we should have a good chance of getting a pair of connected cabled mutant knots with distinct HOMFLY-PT polynomials by the following procedure: take as a pattern tangle the one that appears in the Kinoshita-Terasaka and Conway pair, cable each of its components 3 times, and close it up to a knot in some fairly arbitrary way. This is exactly how we found the pair of knots with 75 crossings.

3.6. **Expected non-invariance of the Kauffman polynomial under** $(2, 0)$**-mutation.** The heuristic reasons for the non-invariance of the HOMFLY-PT polynomial under $(2, 0)$-mutation applies equally well in the case of the Kauffman polynomial. For this reason, we expect that the Kauffman polynomial is not invariant under $(2, 0)$-mutation. To show this, it suffices to present a pair of cabled mutant knots with different Kauffman polynomials. However, the available computer programs for computing the Kauffman polynomial do not work well with knots with more than 50 crossings, and this has prevented us from examining any interesting examples.

3.7. **Proof of Proposition 1.6.** Now we show there exist knots with the same colored Jones, HOMFLY-PT, and Kauffman polynomials, the same volume and signature, but different Khovanov homology. Consider the tangles T and T^τ from Figure 3.8. Denote by $T(1, n)$ and $T^\tau(1, n)$ their $(1, n)$-cables, respectively (for some fixed n). Let K and K^τ be two knots that differ by replacement of $T(1, n)$ with $T^\tau(1, n)$. In particular, K and K^τ are connected cabled mutants and, thus, $(2, 0)$-mutant. Theorems 2.4 and 3.2 thus imply that K and K^τ have equal colored Jones polynomials and volume. A priori, K and K^τ could have different HOMFLY-PT and Kauffman polynomials. However, an elementary computation in the respective skein theories imply that K and K^τ also have equal HOMFLY-PT and Kauffman polynomials.

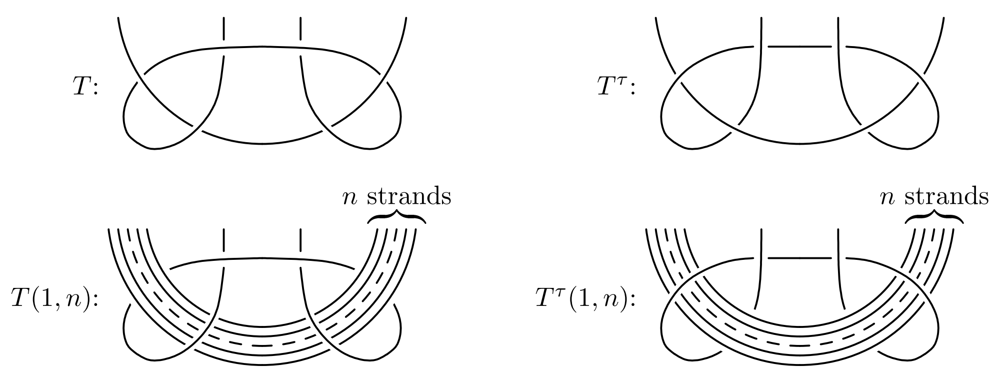
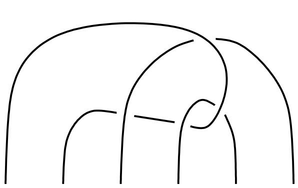


Figure 3.8. Cabling of a tangle and its mutant.

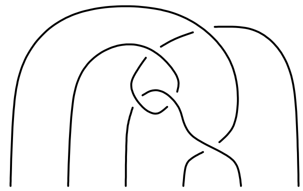
When $n = 2$, let us choose the closure of $T(1, 2)$ in one of the ways from Figure 3.9 to obtain five pairs of knots. In Knotscape notation [HTh], these pairs are $(14^n_{22185}, 14^n_{22589})$, $(15^n_{57606}, 15^n_{57436})$, $(15^n_{115375}, 15^n_{51748})$, $(15^n_{133697}, 15^n_{135711})$, and $(15^n_{148673}, \overline{15}^n_{151500})$, where the bar above the number of crossings means the mirror image of the corresponding knot. Computer calculations with `KhoHo` [Sh] show that knots from these pairs have different Khovanov Homology (see Section 3.10).

3.10. **Knots with few crossings.** We say that two knots are *almost mutant* if they have the same HOMFLY-PT and Kauffman polynomials, signature, and hyperbolic volume. This is an equivalence relation. Note that mutant knots are almost mutant.

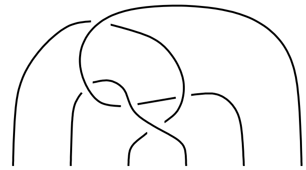
We can partition the set of knots with a bounded number of crossings according to the equivalence relation of being almost mutant. We worked out these equivalence classes for all knots with at most 16 crossings. As it turns out, almost mutant knots with at most 16 crossings always have the same number or crossings. As a consequence, two such knots are either both alternating or both non-alternating. This follows from the fact that the span of the Jones polynomial of a knot equals the number of crossings for this knot if and only if the knot is alternating. For non-alternating knots, Table 3.11 lists the number of such equivalence classes of a given size. We restrict the table to non-alternating knots only because we are



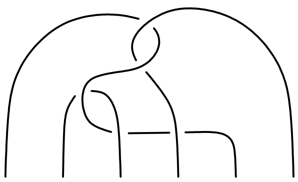
a. Knots 14^n_{22185} and 14^n_{22589}.



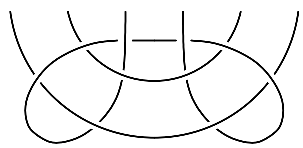
b. Knots 15^n_{57606} and 15^n_{57436}.



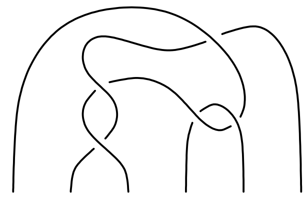
c. Knots 15^n_{115375} and 15^n_{51748}.



d. Knots 15^n_{133697} and 15^n_{135711}.



Common closure: tangle $T(1,2)$



e. Knots 15^n_{148673} and $\overline{15}^n_{151500}$.

Figure 3.9. Five pairs of cabled mutant knots with at most 15 crossings that have different Khovanov homology. They are closures of the $T(1,2)$ tangle.

interested primarily in the possibilities for the Khovanov homology of almost mutant pairs; for alternating knots, the Khovanov homology (at least the free part thereof) is completely determined by their Jones polynomials and signature [L1].

The number of knots in Table 3.11 is taken from Knotscape, which does not distinguish between mirror images. Therefore, we considered each knot twice: the knot itself and its mirror image. The number of amphicheiral knots can be found in [HThW]. The notation $a_1 : n_1, a_2 : n_2, \dots, a_k : n_k$ means that there are n_j equivalence classes of size a_j for $j = 1, 2, \dots, k$.

number of crossings	number of knots	counting mirror images	amphicheiral knots	size and number of almost mutant classes
≤ 13	6236	12468	4	2: 1028, 3: 54, 4: 42, 6: 2
14	27436	54821	51	2: 5349, 3: 298, 4: 359, 6: 30, 8: 10
15	168030	336059	1	2: 35423, 3: 1368, 4: 4088, 6: 290, 8: 136
16	1008906	2017322	490	2: 212351, 3: 6612, 4: 33156, 6: 2159, 7: 20, 8: 2229, 9: 4, 10: 8, 12: 201, 16: 22, 20: 2

Table 3.11. Sizes and numbers of almost mutant classes of non-alternating knots

It is remarkable that very few almost mutant knots have different Khovanov homology. There are only 5 pairs (10 if counted with mirror images) of such knots with at most 15 crossings. They are exactly the 5 cabled mutant pairs from Section 3.7 (see Figure 3.9)! We list values of various knots invariants for these knots below.

There are 27 pairs (54 with mirrors) of almost mutant knots with 16 crossings that have different Khovanov homology. Many of these pairs consist of cabled mutant knots, but we could not verify them all. The pairs are: $(16^n_{257474}, 16^n_{293658})$, $(16^n_{258027}, 16^n_{380926})$, $(16^n_{258035}, 16^n_{359938})$, $(16^n_{261803}, 16^n_{300395})$, $(16^n_{262535}, 16^n_{300387})$, $(16^n_{306846}, 16^n_{307597})$, $(16^n_{332130}, 16^n_{707045})$, $(16^n_{337388}, 16^n_{697474})$, $(16^n_{472161}, 16^n_{635329})$, $(16^n_{564024}, 16^n_{564036})$, $(16^n_{564059}, 16^n_{564068})$, $(16^n_{789164}, 16^n_{797712})$, $(16^n_{789206}, 16^n_{797688})$, $(16^n_{809314}, \overline{16}^n_{850490})$, $(16^n_{809334}, \overline{16}^n_{850512})$, $(16^n_{812818}, 16^n_{850972})$, $(16^n_{820956}, 16^n_{820968})$, $(16^n_{822219}, \overline{16}^n_{822229})$, $(16^n_{878609}, \overline{16}^n_{944604})$, $(16^n_{884231}, 16^n_{884268})$,

$(16^n_{885298}, 16^n_{885312})$, $(16^n_{885305}, 16^n_{885319})$, $(16^n_{885467}, 16^n_{885968})$, $(16^n_{890470}, \overline{16}^n_{944600})$, $(16^n_{937845}, \overline{16}^n_{947575})$, $(16^n_{939163}, 16^n_{945493})$, $(16^n_{943082}, 16^n_{943119})$.

We used Knotscape [HTh] to list all almost mutant knots with at most 16 crossings. Khovanov homology was computed using `KhoHo` [Sh] for all knots with at most 15 crossings and `JavaKh` [B-NGr] for non-alternating knots with 16 crossings. It is worth noticing that Knotscape only computes hyperbolic volume with the precision of 12 significant digits. We used program Snap [G] to compute the volume with the precision of 180 significant digits to verify our data. As it turns out, there are no knots with at most 16 crossings that have non-zero difference in hyperbolic volumes that is less than 10^{-13}. Only 132 pairs of knots have difference in volumes less than 10^{-9} and, hence, are considered as having the same volume by Knotscape. None of these pairs are almost mutants.

To end this section we list values of some quantum and hyperbolic invariants for the almost mutant knots with at most 15 crossings that have different Khovanov homology. The data are presented in Tables 3.12–3.14. They were computed using Knotscape [HTh] and `KhoHo` [Sh]. HOMFLY-PT and Kauffman polynomials are given by the tables of their coefficients. Our notation for Khovanov homology is borrowed from [B-N2]. An expression a^i_j in the "ranks" string means that the multiplicity of $\mathbb{Z}$ in the Khovanov homology group with homological grading i and q-grading j is a. Negative grading is shown with underlined numbers. A similar convention is used for 2-torsion as well (this is the only torsion that appears for these knots). In this case, a is the multiplicity of $\mathbb{Z}_2$. For example, the homology group of 14^n_{22185} with homological grading 0 and q-grading (-1) is $\mathbb{Z}^2 \oplus \mathbb{Z}_2^2$ (see Table 3.12).

$\mathbf{14^n_{22185}}$ and $\mathbf{14^n_{22589}}$

Alexander: 1

Jones: $-t^{-6}+t^{-5}+t^{-2}-t^{-1}+2-t-t^4+t^5$

HOMFLY-PT:

	l^{-4}	l^{-2}	1	l^2	l^4
1	−3	8	−5		1
m^2	−4	14	−11		1
m^4	−1	7	−6		
m^6		1	−1		

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Kauffman:

	a^{-4}	a^{-3}	a^{-2}	a^{-1}	1	a	a^2	a^3	a^4	a^5
1	1				−5		−8		−3	
z^2		5		11		11		9		4
z^4	−8		−10		15		32		15	
z^6		−18		−30		−22		−27		−17
z^8	14		18		−15		−51		−32	
z^{10}		20		27		13		26		20
z^{12}	−7		−8		7		35		27	
z^{14}		−8		−9		−2		−9		−8
z^{16}	1		1		−1		−10		−9	
z^{18}		1		1				1		1
z^{20}							1		1	

Khovanov Homology for $\mathbf{14^n_{22185}}$:

ranks: $1^{\underline{7}}_{\underline{13}}\, 1^{\underline{6}}_{\underline{9}}\, 1^{\underline{4}}_{\underline{7}}\, 1^{\underline{3}}_{\underline{7}}\, 1^{\underline{3}}_{\underline{3}}\, 1^{\underline{2}}_{\underline{5}}\, 1^{\underline{2}}_{\underline{3}}\, 1^{\underline{1}}_{\underline{3}}\, 1^{\underline{1}}_{\underline{1}}\, 1^{0}_{\underline{3}}\, 2^{0}_{\underline{1}}\, 2^{0}_{1}\, 2^{1}_{1}\, 1^{1}_{3}\, 1^{2}_{1}\, 1^{2}_{3}\, 1^{2}_{5}\, 1^{3}_{3}\, 1^{3}_{5}\, 1^{3}_{7}\, 1^{4}_{7}\, 1^{5}_{7}\, 1^{6}_{11}$

2-torsion: $1^{\underline{6}}_{\underline{11}}\, 1^{\underline{4}}_{\underline{9}}\, 1^{\underline{4}}_{\underline{7}}\, 1^{\underline{3}}_{\underline{7}}\, 2^{\underline{3}}_{\underline{5}}\, 1^{\underline{2}}_{\underline{5}}\, 2^{\underline{1}}_{\underline{3}}\, 1^{\underline{1}}_{\underline{1}}\, 2^{0}_{\underline{1}}\, 1^{0}_{1}\, 1^{1}_{\underline{1}}\, 1^{1}_{1}\, 1^{2}_{3}\, 1^{3}_{3}\, 1^{3}_{5}\, 1^{4}_{5}\, 1^{6}_{9}$

Khovanov Homology for $\mathbf{14^n_{22589}}$:

ranks: $1^{\underline{7}}_{\underline{13}}\, 1^{\underline{6}}_{\underline{9}}\, 1^{\underline{5}}_{\underline{9}}\, 1^{\underline{4}}_{\underline{9}}\, 1^{\underline{4}}_{\underline{7}}\, 1^{\underline{4}}_{\underline{5}}\, 1^{\underline{3}}_{\underline{7}}\, 1^{\underline{3}}_{\underline{5}}\, 1^{\underline{3}}_{\underline{3}}\, 1^{\underline{2}}_{\underline{5}}\, 2^{\underline{2}}_{\underline{3}}\, 1^{\underline{1}}_{\underline{3}}\, 1^{\underline{1}}_{\underline{1}}\, 1^{\underline{1}}_{1}\, 2^{0}_{\underline{1}}\, 2^{0}_{1}\, 1^{1}_{1}\, 1^{1}_{3}\, 1^{2}_{1}\, 1^{2}_{5}\, 1^{3}_{5}\, 1^{5}_{7}\, 1^{6}_{11}$

2-torsion: $1^{\underline{6}}_{\underline{11}}\, 1^{\underline{4}}_{\underline{7}}\, 1^{\underline{3}}_{\underline{7}}\, 1^{\underline{3}}_{\underline{5}}\, 1^{\underline{2}}_{\underline{5}}\, 1^{\underline{1}}_{\underline{3}}\, 1^{\underline{1}}_{\underline{1}}\, 1^{0}_{\underline{3}}\, 2^{0}_{\underline{1}}\, 1^{1}_{\underline{1}}\, 2^{1}_{1}\, 1^{2}_{3}\, 2^{3}_{3}\, 1^{3}_{5}\, 1^{4}_{5}\, 1^{4}_{7}\, 1^{6}_{9}$

Table 3.12. Various invariants of almost mutant knots with different Khovanov Homology (part 1)

$\mathbf{15^n_{57436}}$ and $\mathbf{15^n_{57606}}$

Alexander: $-t^{-2}+3-t^{2}$

Jones: $t^{-7}-t^{-6}+t^{-4}-2t^{-3}+2t^{-2}-2t^{-1}+2-t^{2}+2t^{3}-2t^{4}+t^{5}$

HOMFLY-PT:

	$\mathbf{l^{-6}}$	$\mathbf{l^{-4}}$	$\mathbf{l^{-2}}$	$\mathbf{1}$	$\mathbf{l^{2}}$	$\mathbf{l^{4}}$
$\mathbf{1}$	2	−4	3		−1	1
$\mathbf{m^2}$	1	−5	3	−1	−3	1
$\mathbf{m^4}$		−1	1		−1	

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Kauffman:

	$\mathbf{a^{-4}}$	$\mathbf{a^{-3}}$	$\mathbf{a^{-2}}$	$\mathbf{a^{-1}}$	$\mathbf{1}$	$\mathbf{a}$	$\mathbf{a^{2}}$	$\mathbf{a^{3}}$	$\mathbf{a^{4}}$	$\mathbf{a^{5}}$	$\mathbf{a^{6}}$
$\mathbf{1}$	1		1				−3		−4		−2
$\mathbf{z^2}$		2		2		−4		−10		−6	
$\mathbf{z^4}$	−5		−5		8		23		30		15
$\mathbf{z^6}$		−14		−18		13		44		27	
$\mathbf{z^8}$	10		4		−19		−40		−59		−32
$\mathbf{z^{10}}$		24		31		−15		−69		−47	
$\mathbf{z^{12}}$	−6		7		20		21		41		27
$\mathbf{z^{14}}$		−13		−15		7		43		34	
$\mathbf{z^{16}}$	1		−6		−8		−3		−11		−9
$\mathbf{z^{18}}$		2		2		−1		−11		−10	
$\mathbf{z^{20}}$			1		1				1		1
$\mathbf{z^{22}}$								1		1	

Khovanov Homology for $\mathbf{15^n_{57436}}$:

ranks: $1^{8}_{\underline{15}}\,1^{7}_{\underline{11}}\,1^{5}_{\underline{9}}\,2^{4}_{\underline{9}}\,1^{4}_{\underline{5}}\,1^{3}_{\underline{7}}\,2^{3}_{\underline{5}}\,2^{2}_{\underline{5}}\,1^{2}_{\underline{3}}\,1^{1}_{\underline{5}}\,2^{1}_{\underline{3}}\,2^{1}_{\underline{1}}\,1^{0}_{\underline{3}}\,3^{0}_{\underline{1}}\,3^{0}_{1}\,1^{1}_{\underline{1}}\,3^{1}_{1}\,1^{1}_{3}\,2^{2}_{1}\,1^{2}_{3}\,2^{2}_{5}\,1^{3}_{3}\,2^{3}_{5}\,1^{4}_{5}\,1^{4}_{7}\,1^{5}_{7}\,1^{5}_{9}\,1^{6}_{11}$

2-torsion: $1^{7}_{\underline{13}}\,1^{5}_{\underline{11}}\,1^{5}_{\underline{9}}\,1^{4}_{\underline{9}}\,2^{4}_{\underline{7}}\,2^{3}_{\underline{7}}\,2^{2}_{\underline{5}}\,1^{2}_{\underline{3}}\,3^{1}_{\underline{3}}\,1^{1}_{\underline{1}}\,1^{0}_{\underline{3}}\,2^{0}_{\underline{1}}\,1^{1}_{\underline{1}}\,1^{1}_{1}\,1^{2}_{1}\,2^{2}_{3}\,2^{3}_{3}\,1^{4}_{5}\,1^{5}_{7}\,1^{6}_{9}$

Khovanov Homology for $\mathbf{15^n_{57606}}$:

ranks: $1^{8}_{\underline{15}}\,1^{7}_{\underline{11}}\,1^{6}_{\underline{11}}\,1^{5}_{\underline{11}}\,1^{5}_{\underline{9}}\,1^{5}_{\underline{7}}\,2^{4}_{\underline{9}}\,1^{4}_{\underline{7}}\,1^{4}_{\underline{5}}\,1^{3}_{\underline{7}}\,3^{3}_{\underline{5}}\,2^{2}_{\underline{5}}\,1^{2}_{\underline{3}}\,1^{2}_{\underline{1}}\,2^{1}_{\underline{3}}\,2^{1}_{\underline{1}}\,1^{0}_{\underline{3}}\,2^{0}_{\underline{1}}\,3^{0}_{1}\,1^{1}_{\underline{1}}\,2^{1}_{1}\,1^{1}_{3}\,1^{2}_{1}\,1^{2}_{3}\,1^{2}_{5}\,1^{3}_{3}\,1^{3}_{5}\,1^{4}_{5}\,1^{4}_{7}\,1^{5}_{7}\,1^{5}_{9}\,1^{6}_{11}$

2-torsion: $1^{7}_{\underline{13}}\,1^{5}_{\underline{9}}\,1^{4}_{\underline{9}}\,1^{4}_{\underline{7}}\,2^{3}_{\underline{7}}\,1^{2}_{\underline{5}}\,1^{2}_{\underline{3}}\,1^{1}_{\underline{5}}\,3^{1}_{\underline{3}}\,1^{0}_{\underline{3}}\,3^{0}_{\underline{1}}\,1^{1}_{\underline{1}}\,1^{1}_{1}\,2^{2}_{1}\,2^{2}_{3}\,2^{3}_{3}\,1^{3}_{5}\,1^{4}_{5}\,1^{5}_{7}\,1^{6}_{9}$

$\mathbf{15^n_{133697}}$ and $\mathbf{15^n_{135711}}$

Alexander: $t^{-3}-t^{-2}-t^{-1}+3-t-t^{2}+t^{3}$

Jones: $-t^{-6}+2t^{-5}-2t^{-4}+t^{-3}-t^{-1}+3-2t+2t^{2}-t^{3}+t^{5}-t^{6}$

HOMFLY-PT:

	$\mathbf{l^{-4}}$	$\mathbf{l^{-2}}$	$\mathbf{1}$	$\mathbf{l^{2}}$	$\mathbf{l^{4}}$
$\mathbf{1}$	−2	5	−6	7	−3
$\mathbf{m^2}$	−3	10	−12	13	−4
$\mathbf{m^4}$	−1	6	−6	7	−1
$\mathbf{m^6}$		1	−1	1	

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Kauffman:

	$\mathbf{a^{-5}}$	$\mathbf{a^{-4}}$	$\mathbf{a^{-3}}$	$\mathbf{a^{-2}}$	$\mathbf{a^{-1}}$	$\mathbf{1}$	$\mathbf{a}$	$\mathbf{a^{2}}$	$\mathbf{a^{3}}$	$\mathbf{a^{4}}$	$\mathbf{a^{5}}$
$\mathbf{1}$		−3		−7		−6		−5		−2	
$\mathbf{z^2}$	6		11		6		2		4		3
$\mathbf{z^4}$		12		28		30		23		9	
$\mathbf{z^6}$	−18		−35		−10		6		−12		−11
$\mathbf{z^8}$		−28		−48		−39		−48		−29	
$\mathbf{z^{10}}$	20		32		−1		−26		2		15
$\mathbf{z^{12}}$		26		35		20		46		35	
$\mathbf{z^{14}}$	−8		−10		5		26		12		−7
$\mathbf{z^{16}}$		−9		−10		−3		−17		−15	
$\mathbf{z^{18}}$	1		1		−1		−9		−7		1
$\mathbf{z^{20}}$		1		1				2		2	
$\mathbf{z^{22}}$							1		1		

Khovanov Homology for $\mathbf{15^n_{133697}}$:

ranks: $1^{7}_{\underline{13}}\,1^{6}_{\underline{11}}\,1^{6}_{\underline{9}}\,1^{5}_{\underline{9}}\,1^{5}_{\underline{7}}\,1^{4}_{\underline{7}}\,1^{4}_{\underline{5}}\,1^{3}_{\underline{7}}\,1^{3}_{\underline{5}}\,1^{3}_{\underline{3}}\,1^{2}_{\underline{5}}\,2^{2}_{\underline{3}}\,1^{2}_{\underline{1}}\,2^{1}_{\underline{3}}\,1^{1}_{\underline{1}}\,1^{1}_{1}\,3^{0}_{\underline{1}}\,3^{0}_{1}\,1^{1}_{\underline{1}}\,1^{1}_{1}\,2^{1}_{3}\,3^{2}_{3}\,1^{2}_{5}\,1^{3}_{3}\,1^{3}_{5}\,2^{3}_{7}\,1^{4}_{5}\,1^{4}_{7}\,1^{4}_{9}\,1^{5}_{9}\,1^{6}_{9}\,1^{7}_{13}$

2-torsion: $1^{6}_{\underline{11}}\,1^{5}_{\underline{9}}\,1^{4}_{\underline{7}}\,1^{3}_{\underline{7}}\,2^{3}_{\underline{5}}\,2^{2}_{\underline{5}}\,2^{2}_{\underline{3}}\,1^{1}_{\underline{3}}\,1^{1}_{\underline{1}}\,3^{0}_{\underline{1}}\,1^{0}_{1}\,3^{1}_{1}\,1^{1}_{3}\,1^{2}_{1}\,1^{2}_{3}\,2^{3}_{5}\,1^{4}_{5}\,1^{4}_{7}\,1^{5}_{7}\,1^{7}_{11}$

Khovanov Homology for $\mathbf{15^n_{135711}}$:

ranks: $1^{7}_{\underline{13}}\,1^{6}_{\underline{11}}\,1^{6}_{\underline{9}}\,1^{5}_{\underline{9}}\,1^{5}_{\underline{7}}\,2^{4}_{\underline{7}}\,1^{4}_{\underline{5}}\,2^{3}_{\underline{7}}\,1^{3}_{\underline{5}}\,2^{3}_{\underline{3}}\,1^{2}_{\underline{5}}\,3^{2}_{\underline{3}}\,1^{2}_{\underline{1}}\,2^{1}_{\underline{3}}\,2^{1}_{\underline{1}}\,1^{1}_{1}\,3^{0}_{\underline{1}}\,3^{0}_{1}\,1^{0}_{3}\,1^{1}_{1}\,2^{1}_{3}\,2^{2}_{3}\,1^{2}_{5}\,1^{3}_{3}\,2^{3}_{7}\,1^{4}_{7}\,1^{6}_{9}\,1^{7}_{13}$

2-torsion: $1^{6}_{\underline{11}}\,1^{5}_{\underline{9}}\,1^{4}_{\underline{7}}\,2^{3}_{\underline{5}}\,2^{2}_{\underline{5}}\,1^{2}_{\underline{3}}\,1^{1}_{\underline{3}}\,1^{1}_{\underline{1}}\,2^{0}_{\underline{1}}\,1^{0}_{1}\,1^{1}_{\underline{1}}\,3^{1}_{1}\,1^{2}_{1}\,2^{2}_{3}\,2^{3}_{5}\,2^{4}_{5}\,1^{4}_{7}\,1^{5}_{7}\,1^{5}_{9}\,1^{7}_{11}$

Table 3.13. Various invariants of almost mutant knots with different Khovanov Homology (part 2)

$\mathbf{15^n_{115375}}$ and $\mathbf{15^n_{51748}}$

Alexander: 1

Jones: $t^{-7}-t^{-6}-t^{-3}+t^{-2}-t^{-1}+2+t^3-t^4$

HOMFLY-PT:

	$\mathbf{l^{-6}}$	$\mathbf{l^{-4}}$	$\mathbf{l^{-2}}$	$\mathbf{1}$	$\mathbf{l^2}$
$\mathbf{1}$	1		−6	9	−3
$\mathbf{m^2}$	1		−11	14	−4
$\mathbf{m^4}$			−6	7	−1
$\mathbf{m^6}$			−1	1	

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Kauffman:

	$\mathbf{a^{-3}}$	$\mathbf{a^{-2}}$	$\mathbf{a^{-1}}$	$\mathbf{1}$	$\mathbf{a}$	$\mathbf{a^2}$	$\mathbf{a^3}$	$\mathbf{a^4}$	$\mathbf{a^5}$	$\mathbf{a^6}$
$\mathbf{1}$		3		9		6				−1
$\mathbf{z^2}$	−3		−9		−17		−19		−8	
$\mathbf{z^4}$		−10		−21		−4		19		12
$\mathbf{z^6}$	9		17		37		62		33	
$\mathbf{z^8}$		14		21		−7		−45		−31
$\mathbf{z^{10}}$	−6		−8		−28		−78		−52	
$\mathbf{z^{12}}$		−7		−8		6		34		27
$\mathbf{z^{14}}$	1		1		9		44		35	
$\mathbf{z^{16}}$		1		1		−1		−10		−9
$\mathbf{z^{18}}$					−1		−11		−10	
$\mathbf{z^{20}}$								1		1
$\mathbf{z^{22}}$							1		1	

Khovanov Homology for $\mathbf{15^n_{115375}}$:

ranks: $1^{\underline{8}}_{\underline{15}}\,1^{\underline{7}}_{\underline{11}}\,1^{\underline{5}}_{\underline{9}}\,1^{\underline{4}}_{\underline{9}}\,1^{\underline{4}}_{\underline{5}}\,1^{\underline{3}}_{\underline{7}}\,1^{\underline{3}}_{\underline{5}}\,1^{\underline{2}}_{\underline{5}}\,1^{\underline{2}}_{\underline{3}}\,1^{\underline{1}}_{\underline{5}}\,1^{\underline{1}}_{\underline{3}}\,1^{\underline{1}}_{\underline{1}}\,3^{0}_{\underline{1}}\,2^{0}_{1}\,1^{1}_{\underline{1}}\,1^{1}_{1}\,1^{1}_{3}\,1^{2}_{1}\,1^{2}_{3}\,1^{2}_{5}\,1^{3}_{5}\,1^{4}_{5}\,1^{5}_{9}$

2-torsion: $1^{\underline{7}}_{\underline{13}}\,1^{\underline{5}}_{\underline{11}}\,1^{\underline{5}}_{\underline{9}}\,1^{\underline{4}}_{\underline{9}}\,2^{\underline{4}}_{\underline{7}}\,1^{\underline{3}}_{\underline{7}}\,2^{\underline{2}}_{\underline{5}}\,1^{\underline{2}}_{\underline{3}}\,2^{\underline{1}}_{\underline{3}}\,1^{\underline{1}}_{\underline{1}}\,1^{0}_{\underline{3}}\,1^{0}_{\underline{1}}\,1^{1}_{1}\,1^{2}_{1}\,1^{2}_{3}\,1^{3}_{3}\,1^{5}_{7}$

Khovanov Homology for $\mathbf{15^n_{51748}}$:

ranks: $1^{\underline{8}}_{\underline{15}}\,1^{\underline{7}}_{\underline{11}}\,1^{\underline{6}}_{\underline{11}}\,1^{\underline{5}}_{\underline{11}}\,1^{\underline{5}}_{\underline{9}}\,1^{\underline{5}}_{\underline{7}}\,1^{\underline{4}}_{\underline{9}}\,1^{\underline{4}}_{\underline{7}}\,1^{\underline{4}}_{\underline{5}}\,1^{\underline{3}}_{\underline{7}}\,2^{\underline{3}}_{\underline{5}}\,1^{\underline{2}}_{\underline{5}}\,1^{\underline{2}}_{\underline{3}}\,1^{\underline{2}}_{\underline{1}}\,1^{\underline{1}}_{\underline{3}}\,1^{\underline{1}}_{\underline{1}}\,2^{0}_{\underline{1}}\,2^{0}_{1}\,1^{1}_{\underline{1}}\,1^{1}_{3}\,1^{2}_{3}\,1^{4}_{5}\,1^{5}_{9}$

2-torsion: $1^{\underline{7}}_{\underline{13}}\,1^{\underline{5}}_{\underline{9}}\,1^{\underline{4}}_{\underline{9}}\,1^{\underline{4}}_{\underline{7}}\,1^{\underline{3}}_{\underline{7}}\,1^{\underline{2}}_{\underline{5}}\,1^{\underline{2}}_{\underline{3}}\,1^{\underline{1}}_{\underline{5}}\,2^{\underline{1}}_{\underline{3}}\,1^{0}_{\underline{3}}\,2^{0}_{\underline{1}}\,1^{1}_{1}\,2^{2}_{1}\,1^{2}_{3}\,1^{3}_{3}\,1^{3}_{5}\,1^{5}_{7}$

$\mathbf{15^n_{148673}}$ and $\overline{\mathbf{15}}^{\mathbf{n}}_{\mathbf{151500}}$

Alexander: $t^{-4}-2t^{-3}+t^{-2}+3t^{-1}-5+3t+t^2-2t^3+t^4$

Jones: $-t^{-3}+2t^{-2}-2t^{-1}+1+t-t^2+3t^3-3t^4+2t^5-t^6+t^8-t^9$

HOMFLY-PT:

	$\mathbf{1}$	$\mathbf{l^2}$	$\mathbf{l^4}$	$\mathbf{l^6}$	$\mathbf{l^8}$
$\mathbf{1}$	−2	5	−2	1	−1
$\mathbf{m^2}$	−6	13	−2	1	−1
$\mathbf{m^4}$	−5	15	−1		
$\mathbf{m^6}$	−1	7			
$\mathbf{m^8}$		1			

Signature: −2 | Volume: 13.081220984

Kauffman:

	$\mathbf{a^{-9}}$	$\mathbf{a^{-8}}$	$\mathbf{a^{-7}}$	$\mathbf{a^{-6}}$	$\mathbf{a^{-5}}$	$\mathbf{a^{-4}}$	$\mathbf{a^{-3}}$	$\mathbf{a^{-2}}$	$\mathbf{a^{-1}}$	$\mathbf{1}$	$\mathbf{a}$
$\mathbf{1}$		−1		−1		−2		−5		−2	
$\mathbf{z^2}$	5		5		−2		−2		2		2
$\mathbf{z^4}$		10		12		13		22		11	
$\mathbf{z^6}$	−18		−22		10		16		−9		−11
$\mathbf{z^8}$		−28		−32		−22		−48		−30	
$\mathbf{z^{10}}$	20		25		−13		−32		1		15
$\mathbf{z^{12}}$		26		28		13		46		35	
$\mathbf{z^{14}}$	−8		−9		7		27		12		−7
$\mathbf{z^{16}}$		−9		−9		−2		−17		−15	
$\mathbf{z^{18}}$	1		1		−1		−9		−7		1
$\mathbf{z^{20}}$		1		1				2		2	
$\mathbf{z^{22}}$							1		1		

Khovanov Homology for $\mathbf{15^n_{148673}}$:

ranks: $1^{\underline{5}}_{\underline{7}}\,1^{\underline{4}}_{\underline{5}}\,1^{\underline{4}}_{\underline{3}}\,1^{\underline{3}}_{\underline{3}}\,1^{\underline{3}}_{\underline{1}}\,1^{\underline{2}}_{\underline{1}}\,1^{\underline{2}}_{1}\,1^{\underline{1}}_{\underline{1}}\,1^{\underline{1}}_{1}\,1^{\underline{1}}_{3}\,2^{0}_{1}\,3^{0}_{3}\,1^{0}_{5}\,2^{1}_{3}\,1^{1}_{5}\,1^{1}_{7}\,3^{2}_{5}\,2^{2}_{7}\,1^{3}_{5}\,1^{3}_{7}\,3^{3}_{9}\,3^{4}_{9}\,1^{4}_{11}\,1^{5}_{9}\,1^{5}_{11}\,2^{5}_{13}\,1^{6}_{11}\,1^{6}_{13}\,1^{6}_{15}\,1^{7}_{15}\,1^{8}_{15}\,1^{9}_{19}$

2-torsion: $1^{\underline{4}}_{\underline{5}}\,1^{\underline{3}}_{\underline{3}}\,1^{\underline{2}}_{\underline{1}}\,1^{\underline{1}}_{\underline{1}}\,2^{\underline{1}}_{1}\,2^{0}_{1}\,2^{0}_{3}\,1^{1}_{3}\,1^{1}_{5}\,3^{2}_{5}\,1^{2}_{7}\,4^{3}_{7}\,1^{3}_{9}\,1^{4}_{7}\,1^{4}_{9}\,2^{5}_{11}\,1^{6}_{11}\,1^{6}_{13}\,1^{7}_{13}\,1^{9}_{17}$

Khovanov Homology for $\overline{\mathbf{15}}^{\mathbf{n}}_{\mathbf{151500}}$:

ranks: $1^{\underline{5}}_{\underline{7}}\,1^{\underline{4}}_{\underline{5}}\,1^{\underline{4}}_{\underline{3}}\,1^{\underline{3}}_{\underline{3}}\,1^{\underline{3}}_{\underline{1}}\,2^{\underline{2}}_{\underline{1}}\,1^{\underline{2}}_{1}\,2^{\underline{1}}_{\underline{1}}\,1^{\underline{1}}_{1}\,2^{\underline{1}}_{3}\,2^{0}_{1}\,4^{0}_{3}\,1^{0}_{5}\,2^{1}_{3}\,2^{1}_{5}\,1^{1}_{7}\,3^{2}_{5}\,2^{2}_{7}\,1^{2}_{9}\,1^{3}_{7}\,3^{3}_{9}\,2^{4}_{9}\,1^{4}_{11}\,1^{5}_{9}\,2^{5}_{13}\,1^{6}_{13}\,1^{8}_{15}\,1^{9}_{19}$

2-torsion: $1^{\underline{4}}_{\underline{5}}\,1^{\underline{3}}_{\underline{3}}\,1^{\underline{2}}_{\underline{1}}\,2^{\underline{1}}_{1}\,2^{0}_{1}\,1^{0}_{3}\,1^{1}_{3}\,1^{1}_{5}\,2^{2}_{5}\,1^{2}_{7}\,1^{3}_{5}\,4^{3}_{7}\,1^{4}_{7}\,2^{4}_{9}\,2^{5}_{11}\,2^{6}_{11}\,1^{6}_{13}\,1^{7}_{13}\,1^{7}_{15}\,1^{9}_{17}$

Table 3.14. Various invariants of almost mutant knots with different Khovanov Homology (part 3)

$\mathbf{K_{75}}$	$\mathbf{l^{-6}}$	$\mathbf{l^{-4}}$	$\mathbf{l^{-2}}$	$\mathbf{1}$	$\mathbf{l^{2}}$	$\mathbf{l^{4}}$	$\mathbf{l^{6}}$	$\mathbf{l^{8}}$	$\mathbf{l^{10}}$	$\mathbf{l^{12}}$	$\mathbf{l^{14}}$
$\mathbf{1}$			-2	-1	3	3	-1	-1			
$\mathbf{m^{2}}$	7	56	139	135	25	-23	-11	-22	-21	-5	
$\mathbf{m^{4}}$	-211	-953	-1458	-523	454	-151	-622	-90	128	-14	-26
$\mathbf{m^{6}}$	1579	5441	4719	-3552	-4992	5025	8085	1863	-426	586	365
$\mathbf{m^{8}}$	-5299	-14273	77	30645	15926	-36529	-42026	-8337	844	-4408	-2181
$\mathbf{m^{10}}$	9130	16660	-36120	-94856	-14197	132597	123772	19005	-21	16978	7239
$\mathbf{m^{12}}$	-7427	-370	97671	154882	-38199	-291772	-234023	-25052	-4083	-39284	-14827
$\mathbf{m^{14}}$	-161	-22770	-125295	-136384	136281	425309	301818	17636	9947	59198	19943
$\mathbf{m^{16}}$	6309	29798	83571	43754	-205222	-430970	-273994	-1817	-12278	-60701	-18164
$\mathbf{m^{18}}$	-6442	-19849	-17860	35183	188278	311912	177631	-8519	9103	43287	11276
$\mathbf{m^{20}}$	3412	7996	-16095	-50831	-115309	-163389	-82376	8513	-4248	-21639	-4732
$\mathbf{m^{22}}$	-1083	-2018	16221	30148	48742	62044	27027	-4244	1254	7544	1313
$\mathbf{m^{24}}$	207	312	-7152	-10708	-14310	-16893	-6106	1265	-227	-1794	-230
$\mathbf{m^{26}}$	-22	-27	1859	2424	2871	3209	901	-228	23	277	23
$\mathbf{m^{28}}$	1	1	-293	-344	-376	-403	-78	23	-1	-25	-1
$\mathbf{m^{30}}$			26	28	29	30	3	-1		1	
$\mathbf{m^{32}}$			-1	-1	-1	-1					

$\mathbf{K_{75}^{\tau}}$	$\mathbf{l^{-6}}$	$\mathbf{l^{-4}}$	$\mathbf{l^{-2}}$	$\mathbf{1}$	$\mathbf{l^{2}}$	$\mathbf{l^{4}}$	$\mathbf{l^{6}}$	$\mathbf{l^{8}}$	$\mathbf{l^{10}}$	$\mathbf{l^{12}}$	$\mathbf{l^{14}}$
$\mathbf{1}$			-2	-1	3	3	-1	-1			
$\mathbf{m^{2}}$	7	56	139	135	25	-23	-11	-22	-21	-5	
$\mathbf{m^{4}}$	-211	-964	-1533	-740	111	-466	-783	-125	131	-12	-26
$\mathbf{m^{6}}$	1579	5507	5179	-2207	-2871	6936	9016	2038	-451	577	366
$\mathbf{m^{8}}$	-5299	-14405	-1058	27131	10403	-41352	-44238	-8694	894	-4402	-2181
$\mathbf{m^{10}}$	9130	16781	-34668	-89806	-6231	139236	126581	19388	-56	16977	7239
$\mathbf{m^{12}}$	-7427	-425	96604	150503	-45229	-297275	-236105	-25284	-4073	-39284	-14827
$\mathbf{m^{14}}$	-161	-22758	-124827	-134003	140223	428170	302742	17715	9946	59198	19943
$\mathbf{m^{16}}$	6309	29797	83450	42938	-206629	-431908	-274235	-1831	-12278	-60701	-18164
$\mathbf{m^{18}}$	-6442	-19849	-17843	35354	188587	312100	177665	-8518	9103	43287	11276
$\mathbf{m^{20}}$	3412	7996	-16096	-50851	-115347	-163410	-82378	8513	-4248	-21639	-4732
$\mathbf{m^{22}}$	-1083	-2018	16221	30149	48744	62045	27027	-4244	1254	7544	1313
$\mathbf{m^{24}}$	207	312	-7152	-10708	-14310	-16893	-6106	1265	-227	-1794	-230
$\mathbf{m^{26}}$	-22	-27	1859	2424	2871	3209	901	-228	23	277	23
$\mathbf{m^{28}}$	1	1	-293	-344	-376	-403	-78	23	-1	-25	-1
$\mathbf{m^{30}}$			26	28	29	30	3	-1		1	
$\mathbf{m^{32}}$			-1	-1	-1	-1					

Table 3.15. Coefficients of the HOMFLY-PT polynomials of the knot K_{75} and its cabled mutant K_{75}^{τ}.

References

[B-N1] D. Bar-Natan, `Knot atlas`, `www.math.toronto.edu/~drorbn/wiki/`

[B-N2] ———, *On Khovanov's categorification of the Jones polynomial*, Algebr. Geom. Topol. **2** (2002), 337–370.

[B-NG] ———, S. Garoufalidis, *On the Melvin-Morton-Rozansky conjecture*, Invent. Math., **125** (1996) 103–133.

[B-NGr] D. Bar-Natan and J. Green, `JavaKh` — a fast program for computing Khovanov homology, part of the `KnotTheory` Mathematica Package, `http://katlas.math.utoronto.ca/wiki/Khovanov_Homology`

[B] J. Bloom, *Odd Khovanov homology is mutation invariant*, preprint 2009 `arXiv:0903.3746`.

[CDL] S.V. Chmutov, S.V. Duzhin and S.K. Lando, *Vassiliev knot invariants. I. Introduction*, Singularities and bifurcations, Adv. Soviet Math. **21** (1994) 117–126.

[Co] J.H. Conway, *An enumeration of knots and links, and some of their algebraic properties*, in Computational Problems in Abstract Algebra (Proc. Conf., Oxford, 1967) (1970) 329–358.

[CL] D. Cooper and W.B.R. Lickorish, *Mutations of links in genus 2 handlebodies*, Proc. Amer. Math. Soc. **127** (1999) 309–314.

[G] O. Goodman, `Snap`, a computer program for studying arithmetic invariants of hyperbolic 3-manifolds, `http://www.ms.unimelb.edu.au/~snap`, also available from `http://www.math.columbia.edu/~neumann`.

[GL1] C. McA. Gordon and J. Luecke, *Knots are determined by their complements*, J. Amer. Math. Soc. **2** (1989), no. 2, 371–415.

[GL2] C. McA. Gordon and J. Luecke, *Knots are determined by their complements*, J. Amer. Math. Soc. **2** (1989) 371–415.

[HTh] J. Hoste and M. Thistlethwaite, `Knotscape`, `http://www.math.utk.edu/~morwen/knotscape.html`

[HThW] ———, J. Weeks, *The first 1701936 knots*, Math. Intelligencer **20** (1998) 33–48.

[J] V. Jones, *Hecke algebra representation of braid groups and link polynomials*, Ann. of Math. **126** (1987) p. 335–388.

[Ka1] A. Kawauchi, *Topological imitation, mutation and the quantum* SU(2) *invariants*, J. Knot Theory Ramifications **3** (1994) 25–39.

[Ka2] ———, *Almost identical imitations of* $(3,1)$*-dimensional manifold pairs and the manifold mutation*, J. Austral. Math. Soc. Ser. A **55** (1993) 100–115.

[Kf] L. Kauffman, *On knots*, Ann. of Math. Studies, **115**, Princeton University Press, 1987.

[Kh] M. Khovanov, *A categorification of the Jones polynomial*, Duke Math. J. **101** (2000) 359–426.

[KR] M. Khovanov and L. Rozansky, *Matrix factorizations and link homology*, Fund. Math. **199** (2008) 1–91.

[L1] E. S. Lee, *An endomorphism of the Khovanov invariant*, Adv. Math. **197** (2005) 554–586.

[L2] W. B. R. Lickorish, *Linear skein theory and link polynomials*, Topology Appl. **27** (1987), no. 3, 265–274.

[LL] W.B.R. Lickorish and A.S. Lipson, *Polynomials of* 2*-cable-like links*, Proc. Amer. Math. Soc. **100** (1987) 355–361.

[MC] H.R. Morton and P.R. Cromwell, *Distinguishing mutants by knot polynomials*, J. Knot Theory Ramifications **5** (1996) 225–238.

[MR1] H.R. Morton and H.J. Ryder, *Mutants and* $\mathrm{SU}(3)_q$ *invariants*, Geom. Topol. Monogr. **1** (1998) 365–381.

[MR2] H.R. Morton and H.J. Ryder, *Invariantsof genus 2 mutants*, Preprint 2007, `arXiv:0708.0514`.

[MT] H.R. Morton and P. Traczyk, *The Jones polynomial of satellite links around mutants*, Braids (Santa Cruz, CA, 1986), Contemp. Math. **78** (1988) 587–592.

[PS] J. Przytycki and A.S. Sikora, *On skein algebras and* $\mathrm{Sl}_2(C)$*-character varieties*, Topology **39** (2000) 115–148.

[Ru] D. Ruberman, *Mutation and volumes of knots in* S^3, Invent. Math. **90** (1987) 189–215.

[Sh] A. Shumakovitch, `KhoHo` — a program for computing and studying Khovanov homology, `http://www.geometrie.ch/KhoHo`

[ST2] A. Stoimenow and T. Tanaka, *Deciding mutation with the colored Jones polynomial*, to appear in proceedings of the "Topology of Knots VIII" conference.

[ST1] A. Stoimenow and T. Tanaka, *Mutation and the colored Jones polynomial*, preprint 2006.

[Thu] W. Thurston, *The geometry and topology of 3-manifolds*, 1979 notes, available from MSRI.

[Ti1] S. Tillmann, *On the Kinoshita-Terasaka knot and generalised Conway mutation*, J. Knot Theory Ramifications **9** (2000) 557–575.

[Ti2] *Character varieties of mutative 3-manifolds*, Algebr. Geom. Topol. **4** (2004) 133–149.

[Tu1] V. Turaev, *The Yang-Baxter equation and invariants of links*, Inventiones Math. **92** (1988) 527–553.

[Tu2] ———, *Quantum invariants of knots and 3-manifolds*, de Gruyter Studies in Mathematics **18** (1994).

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