# THE ÅRHUS INTEGRAL OF RATIONAL HOMOLOGY 3-SPHERES III: THE RELATION WITH THE LE-MURAKAMI-OHTSUKI INVARIANT

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Abstract. Continuing the work started in [Å-I] and [Å-II], we prove the relationship between the Århus integral and the invariant $\Omega$ (henceforth called LMO) defined by T.Q.T. Le, J. Murakami and T. Ohtsuki in [LMO]. The basic reason for the relationship is that both constructions afford an interpretation as "integrated holonomies". In the case of the Århus integral, this interpretation was the basis for everything we did in [Å-I] and [Å-II]. The main tool we used there was "formal Gaussian integration". For the case of the LMO invariant, we develop an interpretation of a key ingredient, the map $j_m$, as "formal negative-dimensional integration". The relation between the two constructions is then an immediate corollary of the relationship between the two integration theories.

## Contents



## 1. Introduction

This paper is the third in a four-part series on "the Århus integral of rational homology 3-spheres". In Part I of this series [Å-I], we gave the definition of a diagram-valued invariant $\mathring{A}$ of rational homology spheres.[1] In Part II ([Å-II]) we proved that $\mathring{A}$ is a well-defined

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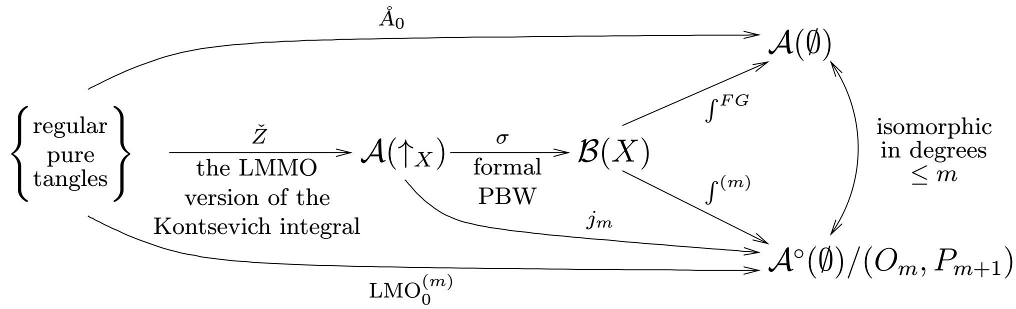
[1]A precise definition of $\mathring{A}$ appears in [Å-I]. It is a good idea to have [Å-I] as well as [LMO] handy while reading this paper, as many of the definitions introduced and explained in those articles will only be repeated here in a very brief manner.

invariant of rational homology 3-spheres and that it is universal in the class of finite type invariants of integral homology spheres. In this paper we show that $\AA$, when defined, is essentially equal to the invariant defined earlier by Le, Murakami, and Ohtsuki in [LMO]. Specifically, we have the following diagram:

(1)




In this diagram,

- "Regular pure tangles" are framed pure tangles whose linking matrix is non-degenerate, and whose components are colored in a bijective manner by the elements of some fixed finite set $X$ "of labels". (See [Å-I, Definition **??**]).
- $\check{Z}$ denotes the Kontsevich integral in its Le-Murakami-Murakami-Ohtsuki [LMMO] normalization (check [Å-I, Definition **??**] for the adaptation to pure tangles).
- $\mathcal{A}(\uparrow_X)$ denotes the completed graded space of chord diagrams for $X$-labeled pure tangles (modulo the usual $4T/STU$ relations). (See [Å-I, Definition **??**]).
- $\mathcal{B}(X)$ denotes the completed graded space of $X$-marked Chinese characters as in [B-N1, B-N2] and [Å-I, Definition **??**].
- $\sigma$ denotes the diagrammatic version of the Poincare-Birkhoff-Witt theorem (defined as in [B-N1, B-N2], but normalized slightly differently, as in [Å-I, Definition **??**]).
- $\mathcal{A}(\emptyset)$ denotes the completed graded space of manifold diagrams modulo the $AS$ and $IHX$ relations, as in [Å-I, Definition **??**].
- The partially defined map $\int^{FG}$ is "formal Gaussian integration", introduced and defined in [Å-I, Definition **??**]. In a sense explained there, it is a diagrammatic analogue of the usual notion of perturbed Gaussian integration. (This definition is recalled in Section 4.)
- $\mathcal{A}^\circ(\emptyset)$ is the space $\mathcal{A}(\emptyset)$, except that the diagrams may contain components with no vertices (closed circles).
- For any non-negative integer $m$, the relations $O_m$ and $P_{m+1}$ were introduced by Le, Murakami and Ohtsuki in [LMO] to make their invariant well-defined. $O_m$ says that disjoint union with a closed circle is equivalent to multiplication by $(-2m)$, and $P_{m+1}$ says that the sum of all ways of pairing up $2m+2$ stubs attaching to the rest of the diagram is 0 (see Figure 1). By [LMO, Lemma 3.3], when we restrict to the space $\mathcal{A}^\circ_{\leq m}(\emptyset)$ of diagrams of degree $\leq m$, the quotient by these relations is isomorphic to $\mathcal{A}_{\leq m}(\emptyset)$.

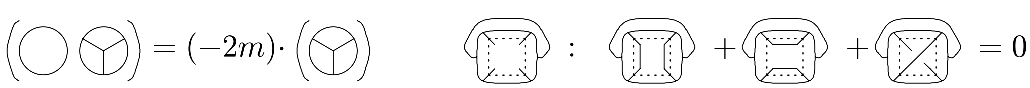



**Figure 1.** The relation $O_m$ and the relation $P_2$. In the figure for $P_2$, the dashed square marks the parts of the diagrams where the relation is applied.

- $\int^{(m)}$ or "negative-dimensional formal integration" will be defined in Section 2.1. It is a minor variant of the map $j_m : \mathcal{A}(\uparrow_X) \to \mathcal{A}^\circ(\emptyset)$ defined in [LMO, Section 2]:

  **Lemma 1.1.** *(Proof in Section 2.1, and see also* [Le, Lemma 6.3]*) The composition* $\int^{(m)} \circ \sigma$ *is equal to* $j_m/(O_m, P_{m+1})$.

  This lemma will be proved in Section 2.1. The sense in which $\int^{(m)}$ is a negative-dimensional formal integral is explained in Section 3.
- $\mathring{A}_0$ is defined to be the composition $\int^{FG} \circ \sigma \circ \check{Z}$. It is precisely the pre-normalized Århus integral of [Å-I]. It is an invariant of regular links [Å-I, Definition **??**] which is also invariant under the second Kirby move [Å-II, Section **??**].
- As a consequence of Lemma 1.1, the arrow labeled $\mathrm{LMO}_0^{(m)}$, defined to be the composition $\int^{(m)} \circ \sigma \circ \check{Z}$, is the pre-normalized Le-Murakami-Ohtsuki invariant. This is a version of the $\Omega_m$ invariant of [LMO] without the prefactors which give invariance under the first Kirby move. Like $\mathring{A}_0$, it is an invariant of regular links which is also invariant under the second Kirby move.

The main tool in this paper is the following proposition, proved in Section 4. It says that the two "integrals" in (1) differ only by a normalization, and hence the same holds for $\mathring{A}$ and LMO:

**Proposition 1.2.** *(Proof in Section 4) If $G$ is a non-degenerate perturbed Gaussian, i.e.,*

$$G = P \exp\Big(\frac{1}{2} \sum_{x,y \in X} l_{xy} \, {}^{x}\frown^{y}\Big)$$

*for some invertible matrix $\Lambda = (l_{xy})$ and some $P \in \mathcal{B}^+(X) \subset \mathcal{B}(X)$, then*

$$\int^{(m)} G\, dX = (-1)^{m|X|} (\det \Lambda)^m \int^{FG} G\, dX.$$

Here and throughout this paper, the product on diagrams is the disjoint union product. $\mathcal{B}^+(X)$ is the space of "strutless" diagrams in which each component has at least one internal vertex (cf. [Å-II, Section **??**]). Note that $\int^{(m)}$ is defined in more cases than $\int^{FG}$, but when they are both defined, they are related in a simple way.

As an immediate consequence, we have the following theorem.

**Theorem 1.** *The $\mathring{A}$ integral is essentially equal to the* LMO *invariant. Specifically, for any regular link $L$ (with $n$ components and linking matrix $\Lambda$) and the rational homology 3-sphere $M$ obtained by surgery on $L$, we have*

$$\mathring{A}_0(L)/(O_m, P_{m+1}) = (-1)^{nm} (\det \Lambda)^{-m} \mathrm{LMO}_0^{(m)}(L) \tag{2}$$

$$\mathring{A}(M)/(O_m, P_{m+1}) = |H^1(M)|^{-m} \mathrm{LMO}^{(m)}(M) \tag{3}$$

$$\mathring{A}(M) = |H^1(M)|^{-\deg} \mathrm{LMO}(M) \tag{4}$$

*Here $\mathring{A}$ and $\mathrm{LMO}^{(m)}$ are $\mathring{A}_0$ and $\mathrm{LMO}_0^{(m)}$ normalized so that they are invariant under the first Kirby move, as described in* [LMO, Å-I], $\mathrm{LMO}(M)$ *is an assembly of the $LMO^{(m)}(M)$ into a single series, as described in* [LMO, Section 4.2], *and $|H^1(M)|^{-\deg}$ is the operation on diagrams which multiplies a diagram of degree $m$ by $|H^1(M)|^{-m}$.*

We note that equation (4) implies that $\mathring{A}(M) = \hat{\Omega}(M)$, where $\hat{\Omega}$ is the invariant defined in [LMO, Section 6.2].

*Proof of Theorem 1.* Equation (2) follows from Lemma 1.1 and Proposition 1.2. Applying (2) to $U_x^{\pm}$, the $x$-labeled unknot with $\pm 1$ framing, we find

$$\int^{FG} \sigma\check{Z}(U_x^+)\,dx = (-1)^m \int^{(m)} \sigma\check{Z}(U_x^+)\,dx \qquad \int^{FG} \sigma\check{Z}(U_x^-)\,dx = \int^{(m)} \sigma\check{Z}(U_x^-)\,dx \tag{5}$$

These are the renormalization factors used in the definition of $\mathring{A}$ and $\mathrm{LMO}^{(m)}$, respectively. Using them, we can compare $\mathring{A}(M)$ and $\mathrm{LMO}^{(m)}(M)$.

$$\begin{aligned}\mathring{A}(M) &= \left(\int^{FG} \sigma\check{Z}(U_x^+)\,dx\right)^{-\sigma_+} \left(\int^{FG} \sigma\check{Z}(U_x^-)\,dx\right)^{-\sigma_-} \int^{FG} \sigma\check{Z}(L)\,dX \\ &= (-1)^{(n-\sigma_+)m}(\det\Lambda)^{-m} \left(\int^{(m)} \sigma\check{Z}(U_x^+)\,dx\right)^{-\sigma_+} \left(\int^{(m)} \sigma\check{Z}(U_x^-)\,dx\right)^{-\sigma_-} \int^{(m)} \sigma\check{Z}(L)\,dX \\ &= |\det\Lambda|^{-m}\,\mathrm{LMO}^{(m)}(M) = |H^1(M)|^{-m}\,\mathrm{LMO}^{(m)}(M)\end{aligned}$$

This proves equation (3). Equation (4) follows from the fact that $\mathrm{LMO}(M)$ takes its degree $m$ part from $\mathrm{LMO}^{(m)}(M)$. □

1.1. **Plan of paper.** In Section 2, we define $\int^{(m)}$, prove Lemma 1.1, and give an alternate formulation of the $P_{m+1}$ relation. In Section 3 we prove some properties of $\int^{(m)}$ which justify the name "negative-dimensional formal integration". These properties are useful in Section 4, where we prove the central Proposition 1.2. Finally, Section 5 has some philosophy on negative-dimensional spaces, sign choices for diagrams, and the Rozansky-Witten invariants.

## 2. A reformulation of the Le-Murakami-Ohtsuki invariant

In this section we give a (minor) reformulation of the Le-Murakami-Ohtsuki invariant $\mathrm{LMO}^{(m)}$. In Section 2.1 we present our definition of $\int^{(m)}$ and prove equivalence with the definition of $j_m$ in [LMO]. In Section 2.2 we state and prove an alternate form $C_{2m+1}$ of the relation $P_{m+1}$.

### 2.1. Definition and notations.

**Definition 2.1.** Let "negative-dimensional integration" $\int^{(m)} : \mathcal{B}(X) \to \mathcal{A}(\emptyset)/(O_m, P_{m+1})$ be defined by

$$\int^{(m)} G\,dX = \left\langle \prod_{x\in X} \frac{1}{m!}\left(\frac{\partial_x\smile\partial_x}{2}\right)^m, G\right\rangle_X \Big/ (O_m, P_{m+1}) \tag{6}$$

Here the pairing $\langle\cdot\,,\cdot\rangle_X : \mathcal{B}(\partial_X)\otimes\mathcal{B}(X) \to \mathcal{A}^\circ(\emptyset)$ is defined (like in [Å-I, Definition **??**]) by

$$\langle D_1\,, D_2\rangle_X = \begin{pmatrix}\text{sum of all ways of gluing the } \partial_x\text{-marked legs of} \\ D_1 \text{ to the } x\text{-marked legs of } D_2\text{, for all } x\in X\end{pmatrix},$$

where, as there, $\partial_X = \{\partial_x : x \in X\}$ denotes a set of labels "dual" to the ones in $X$, and the sum is declared to be 0 if the numbers of appropriately marked legs don't match. If it is clear which legs are to be attached, the subscript $X$ may be omitted.

In other words, $\int^{(m)} G$ is the composition of:

- projection of $G$ to the component with exactly $2m$ legs of each color in $X$
- sum over all $\left((2m-1)!!\right)^{|X|} = \left(\frac{(2m)!}{2^m m!}\right)^{|X|}$ ways of pairing up the legs of each color in $X$
- quotient by the $O_m$ and $P_{m+1}$ relations.

Our definition of $\int^{(m)}$ is slightly different in appearance than the definition of the corresponding object, $j_m$, in [LMO]. For one, $j_m$ is defined on $\mathcal{A}(\uparrow_X)$ while $\int^{(m)}$ is defined on the different but isomorphic space $\mathcal{B}(X)$. We now prove Lemma 1.1, which says that this is the only difference between the two maps.

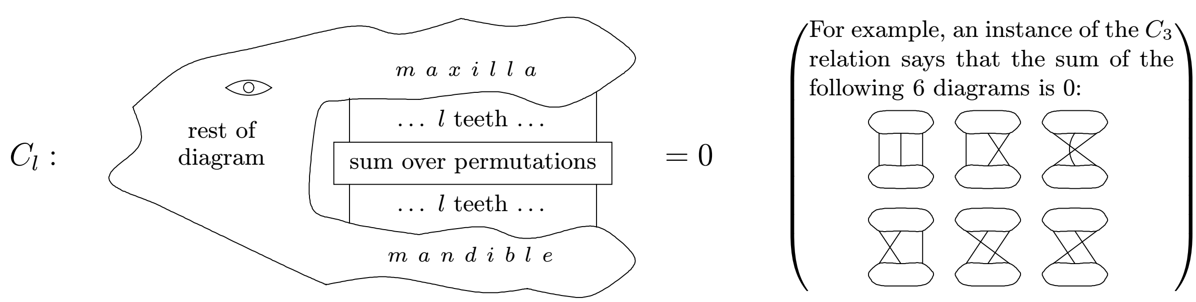
*Proof of Lemma 1.1.* We prove that $j_m \circ \chi = \int^{(m)}$, where $\chi : \mathcal{B}(X) \to \mathcal{A}(\uparrow_X)$ is the inverse of $\sigma$, defined by mapping every $X$-marked Chinese character in $\mathcal{B}(X)$ to the average of all the ways of ordering and attaching its legs to $n = |X|$ vertical arrows marked by the elements of $X$, respecting the markings ([B-N1, B-N2], [Å-I, Definition **??**]).

First recall from [LMO] how $j_m$ is defined. If $D$ is a diagram representing a class in $\mathcal{A}(\uparrow_X)$, then $j_m(D)$ is computed by removing the $n$ arrows from $D$ so that $n$ groups of stubs remain, and then by gluing certain (linear combinations of) forests on these stubs, so that each tree in each forest gets glued only to the stubs within some specific group. It is not obvious that $j_m$ is well defined; it may not respect the $STU$ relation. With some effort, it is proven in [LMO] that for the specific combinations of forests used there, $j_m$ is indeed well defined.

Now every tree that has internal vertices has some two leafs that connect to the same internal vertex, and hence (modulo $AS$), every such tree is anti-symmetric modulo some transposition of its leafs. Thus gluing such a tree to symmetric combinations of diagrams, such as the ones in the image of $\chi$, we always get 0. Hence in the computation of $j_m \circ \chi$ it is enough to consider forests of trees that have no internal vertices; that is, forests of struts. Extracting the precise coefficients from [LMO] one easily sees that they are the same as in (6), and hence $j_m \circ \chi = \int^{(m)}$, as required. □

2.2. **The $C_l$ relations.** It will be convenient, for use in Proposition 3.1, to give another reformulation of the definitions of [LMO]. Instead of their $P_{m+1}$ relation, we may use another relation, the $C_{2m+1}$ relation. For motivation for this relation, see Section 3.2.

**Definition 2.2.** The $C_l$ relation[2] applies when we have a diagram with two sets of $l$ stubs (or teeth) each, and says that the sum of the diagrams obtained by attaching the two sets of stubs to each other in all $l!$ possible ways is 0, as in the following diagram.

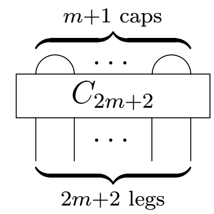



Note that both sets of relations, the $P_m$'s and the $C_l$'s, are decreasing in power. Namely, $P_m$ implies $P_{m+1}$ and $C_l$ implies $C_{l+1}$, for every $l$ and $m$ (one easily sees that $P_{m+1}$ is a sum

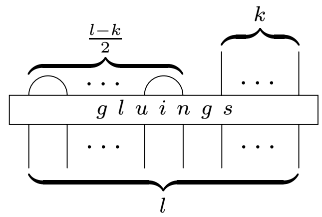
[2] '$C$' for 'Crocodile'.

of instances of $P_m$, and likewise for $C_{l+1}$ and $C_l$). The lemma below says that up to an index-doubling, the two chains of relations are equivalent.

**Lemma 2.3.** *The relations $C_{2m+1}$, $C_{2m+2}$, and $P_{m+1}$ are equivalent. (All of these relations may be applied inside any space of diagrams, regardless of the IHX, STU, or any other relations, so long as closed circles are allowed).*

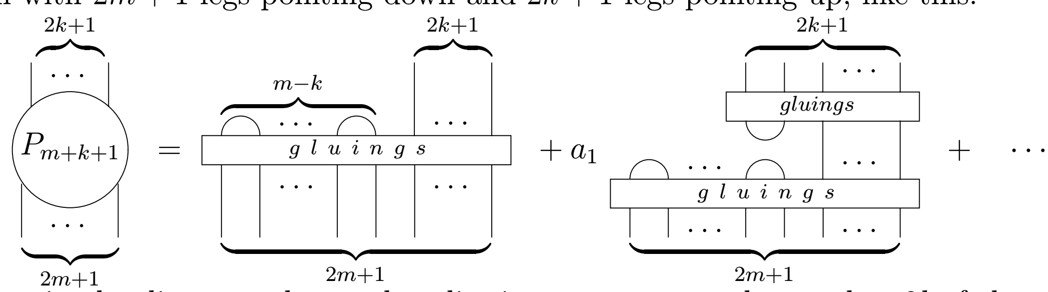



*Proof.* It was already noted that $C_{2m+1}$ implies $C_{2m+2}$. Next, it is easy to see that $C_{2m+2}$ implies $P_{m+1}$: just apply the relation $C_{2m+2}$ in the diagram shown on the right, and you get (a positive multiple of) the relation $P_{m+1}$.




The proof of the implication $P_{m+1} \Rightarrow C_{2m+1}$ is essentially the proof of Lemma 3.1 of [LMO], though the result is different. First, a definition: for $k \leq l$ and $l-k$ even, the diagram part $C_l^k$ has $k$ legs pointing up and $l$ legs pointing down, and it is the sum of all ways of attaching all of the $k$ legs to some of the $l$ legs and then pairing up the remaining $(l-k)$ legs, as illustrated on the left.

We now prove by induction that for all $0 \leq k \leq m$, $C_{2m+1}^{2k+1} = 0$ modulo $P_{m+1}$. For $k = 0$, $C_{2m+1}^1$ is just a version of $P_{m+1}$. For $k > 0$, apply $P_{m+k+1}$ (a consequence of $P_{m+1}$) to a diagram with $2m+1$ legs pointing down and $2k+1$ legs pointing up, like this:




As shown in the diagram, the result splits into a sum over the number $2l$ of the upwards pointing legs that get paired with each other; for $l = 0$, we just get $C_{2m+1}^{2k+1}$; for $l > 0$, the result can be considered to split into two diagrams, a (positive multiple of a) reversed $C_{2k+1}^{2(k-l)+1}$ on top of a $C_{2m+1}^{2(k-l)+1}$. But, by the induction hypothesis, this latter term $C_{2m+1}^{2(k-l)+1}$ vanishes modulo $P_{m+1}$, and we are left with just the first term, which is therefore also a consequence of $P_{m+1}$. This completes the inductive proof. To conclude the proof of Lemma 2.3, note that $C_{2m+1} = C_{2m+1}^{2m+1}$. □

## 3. Negative-dimensional formal integration

In this section, we give several justifications of the name "negative-dimensional formal integration" for the map $\int^{(m)}$ defined above. While doing this, we prove several properties of $\int^{(m)}$ (Propositions 3.1 and 3.2) that are used in the proof of Proposition 1.2 in Section 4.

3.1. **Why integration?** First, why should $\int^{(m)}$ be called an integral? An *integral* is (more or less) a linear map from some space of functions to some space of scalars. In our case, the appropriate space of "functions" is $\mathcal{B}(X)$ and the appropriate space of "scalars" is $\mathcal{A}(\emptyset)$; more on the interpretation of diagrams as functions and/or scalars appears in [Å-I, Section **??**] and [Å-II, Section **??**]. The linearity of $\int^{(m)}$ is immediate.

But $\int^{(m)}$ is not just any integral; it is a Lebesgue integral. The defining property of the usual Lebesgue integral on $\mathbb{R}^n$ is translation invariance by a vector $(\bar{x}^i)$. We show that the parallel property holds for $\int^{(m)}$:

**Proposition 3.1** (Translation Invariance). *For any diagram $D \in \mathcal{B}(X)$, we have*

$$\int^{(m)} D\,dX = \int^{(m)} D/(x \mapsto x + \bar{x})\,dX \tag{7}$$

The notation $D/(x \mapsto x + \bar{x})$ means (as in [Å-II, Section **??**]), for each leg of $D$ colored $x$ for $x \in X$, sum over coloring the leg by $x$ or by $\bar{x}$. (So we end up with a sum of $2^t$ terms, where $t$ is the number of $X$ colored legs in $D$.) The set $\bar{X} = \{\bar{x} \mid x \in X\}$ is an independent set of variables for the formal translation.

*Proof.* By the relation $P_{m+1}$ in the definition of $\int^{(m)}$ (or rather, by $C_{2m+1}$), any diagram $D \in \mathcal{B}(X)$ with more than $2m$ legs on any component gets mapped to 0 on either side of (7), so we may assume that $D$ has $2m$ legs or fewer of any color. But, for the right hand side to be non-zero, $D/(x \mapsto x + \bar{x})$ must have exactly $2m$ legs colored $x$ for each $x \in X$; this can only happen from diagrams in $D$ with exactly $2m$ legs of each color and when none of them get converted to $\bar{x}$. But these are exactly the diagrams appearing in the integral on the left hand side. □

3.2. **The relations $O_m$ and $C_{2m+1}$.** Now that you're convinced that $\int^{(m)}$ is an integral, you must be wondering why we called it a "negative-dimensional" integral. Recall what $\int^{(m)}$ is: it is the sum over all ways of gluing in some struts, followed by the quotient by the $O_m$ and $P_{m+1}$ relations. This quotient is crucial; otherwise $\int^{(m)}$ is some random map without particularly nice properties. But what are these relations?

The relation $O_m$ is simple. Recall [Å-II, Section **??**] that we like to think of diagrams as representing tensors and/or functions in/on some vector space $V$. Since a strut corresponds to the identity tensor in $V^\star \otimes V$ (cf. [Å-II, Figure **??**]), its closure, a circle, should correspond to the trace of the identity, or the dimension of $V$. Hence $O_m$, which says that a circle is equivalent to the constant $(-2m)$, is the parallel of saying "$\dim V = -2m$".

The relation $P_{m+1}$ is a little more subtle. It is easier to look at the equivalent relation $C_{2m+1}$, which implies the relation $C_l$ for every $l > 2m$. If a single vertex corresponds to some space $V$, then a collection of $l$ vertices corresponds to $V^{\otimes l}$; and, when we sum over all permutations without signs, we get (a multiple of) the projection onto the symmetric subspace, $S^l(V)$. The relation $C_l$ says that this projection (and hence the target, $S^l(V)$) is 0. Compare this with the following statement about $\mathbb{R}^k$ for $k \geq 0$:

$$\dim S^l(\mathbb{R}^k) = \begin{pmatrix} l+k-1 \\ l \end{pmatrix} = \frac{(k+l-1)(k+l-2)\cdots(k+1)k}{l(l-1)\cdots 2\cdot 1}.$$

We can see from this formula that if a space $V$ formally has a dimension $k = -2m$, then $\dim S^l(V)$ vanishes precisely when $l > 2m$. This is in complete agreement with what we just found about $P_{m+1}$.

3.3. **An example: Gaussian integration.** Enough of generalities, let's compute! Consider the well known Gaussian integral over $\mathbb{R}^n$,

$$\int_{\mathbb{R}^n} e^{q(x,x)/2}\, d^n x = \frac{(2\pi)^{\frac{n}{2}}}{(\det\, -q)^{\frac{1}{2}}}$$

where $q$ is an arbitrary negative-definite quadratic form $q$. The factor $(2\pi)^{n/2}$ is just a normalization factor that could be absorbed into the measure $d^n x$. (Recall that we identified $\int^{(m)}$ as Lebesgue integration by translation invariance, which only determines the measure up to an overall scale factor.) The remaining factor, $(\det\, -q)^{-1/2}$, is the more fundamental one. Does a similar result hold for $\int^{(m)}$? In order to answer this question, one would first have to know what a "determinant" of a quadratic form on a negative-dimensional space is. While there is a good answer to this question (called the "superdeterminant" or the "Berezinian"; see, e.g., [Be, page 82]), it would take us too far afield to discuss it in full. Instead, let us take a slightly different tack. Fix a negative-definite quadratic form $\Lambda$ on $\mathbb{R}^n$ once and for all, and consider the quadratic form $q = \Lambda \otimes (\delta_{ij})$ on $\mathbb{R}^n \otimes \mathbb{R}^k \cong \mathbb{R}^{nk}$. We have $\det q = (\det \Lambda)^k$ and so we find that

$$\int_{\mathbb{R}^{nk}} e^{q(x,x)/2}\, d^{nk} x = C(\det\, -\Lambda)^{-k/2} \tag{8}$$

for some constant $C$.

Consider now the sum $\sum_{x,y\in X} l_{xy}\, {}^{x}\!\frown\!{}^{y}$ in $\mathcal{B}(X)$. According to the voodoo of diagrammatic calculus [Å-II, Section **??**], it is in analogy with a quadratic form on $V^{\otimes n}$, where $n = |X|$ and $V$ is some vector space that plays a role similar to $\mathbb{R}^k$ in the above discussion. The proposition below is then the diagrammatic analog of (8), taking $k = \dim V = -2m$.

**Proposition 3.2.** *For any set $X$ with $|X| = n$ and $\Lambda = (l_{xy})$ a symmetric matrix on $\mathbb{R}^X$,*

$$\int^{(m)} \exp\left(\frac{1}{2}\sum_{x,y\in X} l_{xy}\, {}^{x}\!\frown\!{}^{y}\right) = (\det\, -\Lambda)^m = (-1)^{nm}(\det \Lambda)^m$$

Note that other than symmetry, there is no restriction on the matrix $\Lambda$. This proposition is a consequence of Lemma 4.2 of [LMO] and a computation for $m = 1$ (given in [LMMO]), but we give our own proof for completeness, and also to provide a more direct link to typical determinant calculations.

*Proof.* We are to calculate the reduction modulo $O_m$ and $P_{m+1}$ of

$$D_1 := \left\langle \prod_{x\in X} \frac{1}{m!}\left(\frac{\partial_x\!\smile\!\partial_x}{2}\right)^m, \exp\left(\frac{1}{2}\sum_{x,y\in X} l_{xy}\, {}^{x}\!\frown\!{}^{y}\right) \right\rangle_X.$$

The only terms that can appear in $D_1$ are closed loops. The relation $O_m$ replaces each of these by a number, reducing the result to $\mathbb{Q}$. The relation $P_{m+1}$ is irrelevant and will be ignored in the remainder of the proof.

Introduce a new set of variables $A$ (and dual variables $\partial_A$) with $|A| = m$, and consider

$$D_2 := \left\langle \prod_{x\in X} \frac{1}{m!}\left(\sum_{a\in A} \overset{(x,a)}{\underset{(\partial_x,\partial_a)}{\downarrow}}\right)^m, \exp\left(\sum_{x,y\in X}\sum_{a\in A} l_{xy} \overset{(\partial_x,\partial_a)}{\underset{(y,a)}{\uparrow}}\right) \right\rangle_{X\times A}. \tag{9}$$

Let us compare $D_1$ and $D_2$. After all relevant gluings, $D_1$ becomes a sum (with coefficients) of disjoint unions of unoriented loops, each of which is a polygon of struts whose vertices are colored by elements of $X$. Similarly, $D_2$ is also a sum (with coefficients) of a disjoint union of loops, only that now the loops are oriented and the struts they are made of are colored by the elements of $X \times A$, keeping the $A$ part of the coloring constant along each loop. In both cases the coefficients come from the same simple rule, which involves only the the $X$ part of the coloring. We see that each term in $D_1$ with $c$ circles corresponds to $(2m)^c$ terms of $D_2$: for each loop in a given term of $D_1$, choose a color $a \in A$ and an orientation, and you get a term in $D_2$. So we find that

$$D_2 \big/ (\bigcirc = -1) \;=\; D_1 \big/ (\bigcirc = -2m).$$

Recall that $\frac{1}{4!}(a+b+c+d)^4 = abcd + (\text{non-multi-linear terms})$. Similarly,

$$\prod_{x \in X} \frac{1}{m!} \left( \sum_{a \in A} \underset{(\partial_x, \partial_a)}{\overset{(x,a)}{\downarrow}} \right)^m = \prod_{(x,a) \in X \times A} \underset{(\partial_x, \partial_a)}{\overset{(x,a)}{\downarrow}} + (\text{terms with strut repetitions}).$$

We assert that terms with strut repetitions can be ignored in the computation of $D_2 \big/ (\bigcirc = -1)$. Indeed, for some fixed $x_0 \in X$ and $a_0 \in A$ set $\alpha = (x_0, a_0)$ and $\partial_\alpha = (\partial_{x_0}, \partial_{a_0})$, and suppose a strut repetition like $\downarrow^{\alpha}_{\partial_\alpha} \downarrow^{\alpha}_{\partial_\alpha}$ occurs within the left operand of a pairing as in (9). Then, as illustrated in Figure 2, the gluings in the evaluation of the pairings come in pairs. One easily sees that the number of cycles differs by $\pm 1$ for the gluings within each pair, and hence modulo $(\bigcirc = -1)$ the whole sum of gluings vanishes.

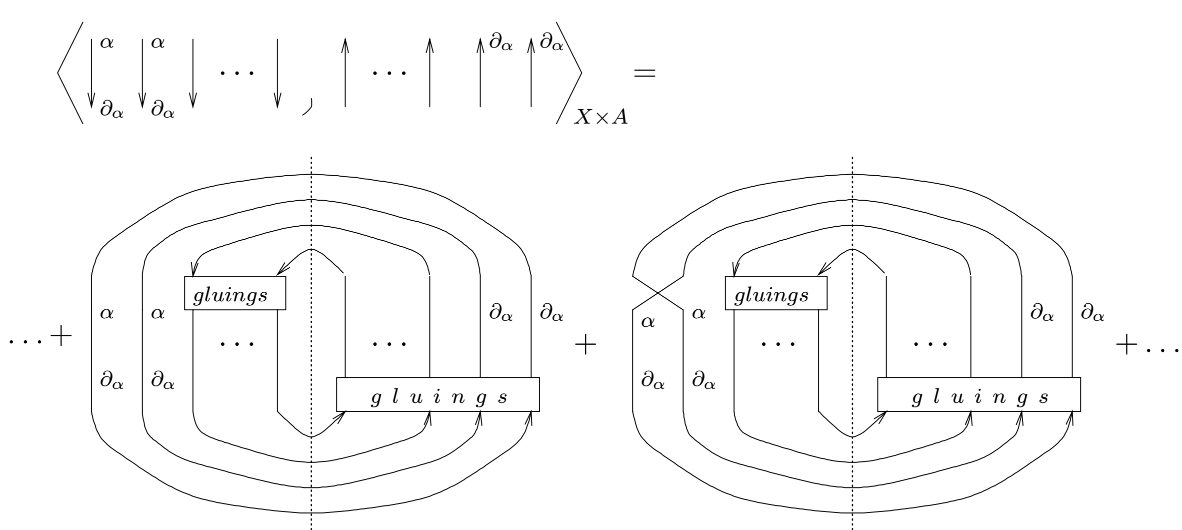



**Figure 2.** Two ways of gluing a repeating strut.

Now compare $D_2$ to

$$D_3 := \left\langle \prod_{(x,a) \in X \times A} \underset{(\partial_x, \partial_a)}{\overset{(x,a)}{\downarrow}} , \exp\left( \sum_{x,y \in X} \sum_{a \in A} l_{xy} \underset{(y,a)}{\overset{(\partial_x, \partial_a)}{\uparrow}} \right) \right\rangle_{X \times A}.$$

By our assertion, $D_2 = D_3$ modulo $(\bigcirc = -1)$. But $D_3$ looks very much like the usual formula for the determinant: it reduces to

$$D_3/(\bigcirc = -1) \;=\; \sum_{\pi\in S(X\times A)} \prod_{(x,a)\in X\times A} l_{xa,\,\pi(xa)}(-1)^{\mathrm{cycles}(\pi)}$$

where $(l_{xa,yb}) = \Lambda \otimes (\delta_{ab})$. Using the relationship between the number of cycles of a permutation $\pi \in S(X \times A)$ and its signature, $(-1)^{\mathrm{cycles}(\pi)} = (-1)^{nm}\,\mathrm{sgn}(\pi)$, we find that

$$\int^{(m)} G = (-1)^{nm}\det(l_{xa,yb}) = (-1)^{nm}(\det\Lambda)^m$$

as required. □

## 4. Relating the two integration theories

The classical computation of perturbed Gaussian integration uses only translation invariance, and a single non-perturbed computation to determine the normalization coefficient. For negative dimensional integration, translation invariance was proven in Proposition 3.1, and the non-perturbed computation is in Section 3.3. So the proof of Proposition 1.2 proceeds just as in the classical computation of perturbed Gaussian integration:

*Proof of Proposition 1.2.* Recall that we are comparing $\int^{(m)}$ to $\int^{FG}$ on a perturbed Gaussian with variables in $X$ and a non-degenerate quadratic part $\Lambda = (l_{xy})$:

$$G = P\exp\left(\frac{1}{2}\,l_{xy}\,{}^{x}\!\frown{}^{y}\right).$$

(Here and throughout this proof, repeated variables should be summed over $X$.) From [Å-I, Definition **??**] we have

$$\int^{FG} G\,dX = \left\langle \exp\left(-\frac{1}{2}\,l^{xy}\,{}_{\partial_x}\!\smile{}_{\partial_y}\right), P\right\rangle,$$

with, as usual, $(l^{xy}) = \Lambda^{-1}$. We need to evaluate $\int^{(m)} G\,dX$. First separate out the strutless part, $P$, using a standard trick:

$$\begin{aligned}\int^{(m)} G\,dX &= \int^{(m)} \left\langle P/(x\mapsto\partial_{\bar x})\,,\,\exp\left(\frac{1}{2}\,l_{xy}\,{}^{x}\!\frown{}^{y} + \overset{x}{\underset{\bar x}{|}}\right)\right\rangle_{\bar X} dX\\ &= \left\langle P/(x\mapsto\partial_{\bar x})\,,\,\int^{(m)}\exp\left(\frac{1}{2}l_{xy}\,{}^{x}\!\frown{}^{y} + \overset{x}{\underset{\bar x}{|}}\right) dX\right\rangle_{\bar X}\end{aligned}$$

Now we complete the square in the integral:

$$\int^{(m)}\exp\left(\frac{1}{2}\,l_{xy}\,{}^{x}\!\frown{}^{y} + \overset{x}{\underset{\bar x}{|}}\right) dX = \exp\left(-\frac{1}{2}\,l^{xy}\,{}_{\bar x}\!\smile{}_{\bar y}\right)\int^{(m)}\exp\left(\frac{1}{2}\,l_{xy}\,{}^{x}\!\frown{}^{y} + \overset{x}{\underset{\bar x}{|}} + \frac{1}{2}\,l^{xy}\,{}_{\bar x}\!\smile{}_{\bar y}\right) dX$$

A short computation (left to the reader) shows that we have, indeed, completed the square:

$$\exp\left(\frac{1}{2}\,l_{xy}\,{}^{x}\!\frown{}^{y} + \overset{x}{\underset{\bar x}{|}} + \frac{1}{2}\,l^{xy}\,{}_{\bar x}\!\smile{}_{\bar y}\right) = \exp\left(\frac{1}{2}\,l_{xy}\,{}^{x}\!\frown{}^{y}\right)\Big/(x\mapsto x + l^{xy}\bar y).$$

But now, by Propositions 3.1 and 3.2,

$$\int^{(m)} \exp\left(\frac{1}{2}\, l_{xy}\; {}^{x}\!\frown^{y} + \underset{\bar{x}}{\overset{x}{|}} + \frac{1}{2}\, l^{xy}\; {}_{\bar{x}}\!\smile_{\bar{y}}\right) dX = \int^{(m)} \exp\left(\frac{1}{2}\, l_{xy}\; {}^{x}\!\frown^{y}\right) dX = (-1)^{nm}(\det\Lambda)^m$$

and so

$$\begin{aligned}\int^{(m)} G\, dX &= \left\langle P/(x \to \partial_{\bar{x}})\, , \exp\left(-\frac{1}{2}\, l^{xy}\; {}_{\bar{x}}\!\smile_{\bar{y}}\right) \cdot (-1)^{nm}(\det\Lambda)^m \right\rangle_{\bar{X}} \\ &= (-1)^{nm}(\det\Lambda)^m \int^{FG} G\, dX \Big/ (O_m, C_{2m+1})\end{aligned}$$

□

*Remark* 4.1. As in the classical case (see, e.g., the Appendix of [Å-I]), this proof can be recast in the language of Laplace (or Fourier) transforms.

## 5. Some philosophy

The impatient mathematical reader may skip this section; there is nothing with rigorous mathematical content here. For the moment, the material in this section is purely philosophy, and not very well-developed philosophy at that.

This interpretation of $\int^{(m)}$ as negative-dimensional formal integration probably seems somewhat strange. After all, Chern-Simons theory, the basis for the theory of trivalent graphs (the spaces $\mathcal{A}$ and $\mathcal{B}$), and much of the theory of Vassiliev invariants, takes place very definitely in positive dimensions: the vector space associated to the a vertex is some Lie algebra $\mathfrak{g}$. To integrate over these positive dimensional spaces, a different theory is necessary, as developed in Part II of this series [Å-II].

One potential answer to this problem is to forget about the Lie algebra for the moment and just look at the structure of diagrams. There are at least three different reasonably natural sign conventions for the diagrams under consideration [Ko1, Th]. The standard choice is to give an orientation (ordering up to even permutations) of the edges around each vertex. But another natural (though usually less convenient) choice is to leave the edges around a vertex unordered and, instead, give a direction on each edge and a sign ordering of the *set* of all vertices.[3] But now look what happens to the space $\mathcal{B}$: because of the ordering on the vertices, the diagrams are no longer completely symmetric under the action of permuting the legs; they are now completely *anti-symmetric*. This anti-symmetry of legs is exactly what we would expect for functions of fermionic variables or functions on a negative-dimensional space. Furthermore, the integration map $\int^{(m)}$ is quite suggestive from this point of view: it looks like evaluation against a top exterior power of a symplectic form on a vector space, which is a correct analogue of integration.

Alternatively, we could try to keep the connection with physics, but try to find a physical theory that exhibits this negative-dimensional behavior. Fortunately, such a theory has been found: it is the Rozansky-Witten theory [RW, Ko2, Ka]. In this theory weight systems are constructed from a Hyper-Kähler manifold $Y$ of dimension $4m$. The really interesting thing for present purposes is that the factors assigned to the vertices are holomorphic one-forms

[3] In this discussion, we assume that all vertices of the graphs have odd valency (as holds for all diagrams considered in this paper). See [Ko1, Th] for details on dealing with diagrams with vertices of even valency.

on $Y$, which anti-commute (using the wedge product on forms). (In keeping with the above remarks about signs, each edges is assigned a symplectic form on $Y$, which is anti-symmetric.) So in this case, there *is* a kind of $(-2m)$-dimensional space associated to vertices. (But note that this space is "spread out" over $Y$: it is the parity-reversed holomorphic tangent bundle.)

Finally, it's interesting to note that the definition of $\int^{(m)}$ is more general than that of $\int^{FG}$, and the proofs are equally simple. On the other hand, $\int^{FG}$ has some advantages. Its philosophical meaning is much clearer and it takes values in $\mathcal{A}(\emptyset)$ directly, rather than in some quotient.

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