

PERIODS, THE MEROMORPHIC 3D-INDEX AND THE TURAEV–VIRO INVARIANT

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ABSTRACT. The 3D-index of Dimofte–Gaiotto–Gukov is an interesting collection of q -series with integer coefficients parametrised by a pair of integers and associated to a 3-manifold with torus boundary. In this note, we explain the structure of the asymptotic expansions of the 3D-index when $q = e^{2\pi i\tau}$ and τ tends to zero (to all orders and with exponentially small terms included), and discover two phenomena: (a) when τ tends to zero on a ray near the positive real axis, the horizontal asymptotics of the meromorphic 3D-index match to all orders with the asymptotics of the Turaev–Viro invariant of a knot, in particular explaining the Volume Conjecture of Chen–Yang from first principles, (b) when $\tau \rightarrow 0$ on the positive imaginary axis, the vertical asymptotics of the 3D-index involves periods of a plane curve (the A -polynomial), as opposed to algebraic numbers, explaining some predictions of Hodgson–Krickner–Siejakowski and leading to conjectural identities between periods of the A -polynomial of a knot and integrals of the Euler beta-function.

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1. INTRODUCTION

1.1. **The 3D-index.** Quantum invariants attached to 3-dimensional objects, whether defined as functions at complex roots of unity, q -series, or by analytic functions in the cut plane $\mathbb{C}' = \mathbb{C} \setminus (-\infty, 0]$, have many interesting and surprising connections with each other. Although they are well-defined topological invariants, their relations are largely conjectural and lead to startling statements and numerical predictions explained in detail in work of Gukov, Mariño and collaborators [27, 26, 13, 15, 16] and in two papers of Zagier and the first author [24, 25].

In this paper, we will focus on the asymptotic properties of a very interesting quantum knot invariant, the 3D-index of Dimofte–Gaiotto–Gukov [10, 11] and its meromorphic version of [21]. The 3D-index is a collection of q -series with integer coefficients associated to an ideally triangulated 3-manifold with torus boundary components [10, 11]. These q -series occur in mathematical physics as BPS counts of supersymmetric field theories and have fascinating properties. They are topological invariants [19, 21] that are conjecturally related to other quantum invariants such as the complex Chern–Simons partition function (also known as the state-integrals of Andersen–Kashaev [1]) and even the Kashaev invariant itself [30].

One of the discoveries of [25] is that the radial asymptotics of q -hypergeometric series in general depend on the ray in which q approaches 1 or any complex root of unity. A further discovery of [15, 16] is that as we move the ray, the asymptotic series changes by linear combinations of power series in $\tilde{q} = e^{-2\pi i/\tau}$ (when $q = e^{2\pi i\tau}$) with integer coefficients.

1.2. **Two discoveries for the asymptotics of the 3D-index of the 4_1 knot.** The paper concerns two numerical discoveries concerning the horizontal and the vertical asymptotics of the meromorphic 3D-index:

- (a) the horizontal asymptotics of the meromorphic 3D-index $I^{\text{mer}}(0,0)(q)$ matches those of the Turaev–Viro invariant of a knot and involves algebraic coefficients,
- (b) the vertical asymptotics of the meromorphic 3D-index involve periods (in the sense of Kontsevich–Zagier [32]) of the $\text{PSL}_2(\mathbb{C})$ -character variety.

Both discoveries stem from the fact that the meromorphic 3D-index is a sum over the integers of the rotated 3D-index, the latter being a bilinear combination of colored holomorphic blocks. The first discovery explains the Chen–Yang volume conjecture [5] from first principles and refines it to all orders in perturbation theory and the second explains some predictions of Hodgson–Krieger–Siejakowski for the asymptotics of the meromorphic 3D-index [28].

We will present these discoveries in the order they were found for the 4_1 knot. Our first experiment was to compute the first few terms of the asymptotics of the meromorphic 3D-index for the 4_1 knot,

$$I_{4_1}^{\text{mer}}(0,0)(q) = 1 - 4q - q^2 + 36q^3 + 70q^4 + 100q^5 + 34q^6 - 116q^7 - 410q^8 - 808q^9 + \dots, \quad (1)$$

and found out that

$$I_{4_1}^{\text{mer}}(0,0)(q) \sim e^{\frac{2\text{Vol}(4_1)}{2\pi\tau}} \frac{1}{3^{\frac{3}{4}} 2^{\frac{1}{2}} \sqrt{\tau}} \times \left(1 - \frac{19}{24\sqrt{-3}^3} 2\pi i\tau + \frac{1333}{1152\sqrt{-3}^6} (2\pi i\tau)^2 - \frac{1601717}{414720\sqrt{-3}^9} (2\pi i\tau)^3 + \dots \right) \quad (2)$$

when $q = e^{2\pi i\tau}$ and $\tau \rightarrow 0$ on a ray sufficiently close to the positive real axis. We then recognised that the number 1333 appeared in the asymptotics of the Turaev–Viro invariant $\text{TV}_{4_1, m+1/2}$ (computed years ago in unpublished work of the first author), and in fact the above series agreed with the asymptotics of the Turaev–Viro invariant $\text{TV}_{4_1, m+1/2}$, after merely replacing τ by $m+1/2$. Surely, this was not a coincidence, and the matching was then checked for the next 7 terms, and still held. The Turaev–Viro invariant is a complex-valued function at roots of unity defined by a state-sum using an ideal triangulation of the knot complement, placing representations of $\mathfrak{sl}_2(\mathbb{C})$ at the edges, using the quantum 6j-symbols at each tetrahedron, and summing over the labels.

The observed matching horizontal asymptotics of the meromorphic 3D-index and of the Turaev–Viro invariant, aside from explaining a volume conjecture, suggests that the two invariants are realisations of the same quantum invariant, one being a q -series with integer coefficients, and another being a function at roots of unity. Moreover, it hints that the Turaev–Viro invariant is one entry of a matrix-valued quantum knot invariant whose rows and columns are labeled by boundary-parabolic $\text{PSL}_2(\mathbb{C})$ -representations of the knot complement. A third realisation of this invariant is the bilinear combination of asymptotic series, and the three realisations are related by an underlying holomorphic function on a cut plane $\mathbb{C}' := \mathbb{C} \setminus (-\infty, 0]$, much in the theme of [24] and [25].

Our second discovery is about the vertical asymptotics of the meromorphic 3D-index $I_{4_1}^{\text{mer}}(0,0)(q)$ of the 4_1 knot. This time as $\tau \in i\mathbb{R}_+$ tends to 0 on the imaginary axis, we found

out that

$$\begin{aligned}
I_{4_1}^{\text{mer}}(0,0)(q) &\sim e^{\frac{2\text{Vol}(4_1)}{2\pi\tau}} \frac{1}{3^{\frac{3}{4}} 2^{\frac{1}{2}} \sqrt{\tau}} \left(1 - \frac{19}{24\sqrt{-3}^3} 2\pi i\tau + \frac{1333}{1152\sqrt{-3}^6} (2\pi i\tau)^2 - \frac{1601717}{414720\sqrt{-3}^9} (2\pi i\tau)^3 + \dots \right) \\
&\quad - ie^{-\frac{2\text{Vol}(4_1)}{2\pi\tau}} \frac{1}{3^{\frac{3}{4}} 2^{\frac{1}{2}} \sqrt{\tau}} \left(1 + \frac{19}{24\sqrt{-3}^3} 2\pi i\tau + \frac{1333}{1152\sqrt{-3}^6} (2\pi i\tau)^2 + \frac{1601717}{414720\sqrt{-3}^9} (2\pi i\tau)^3 + \dots \right) \\
&\quad + \kappa_{4_1} \frac{i}{\tau} + \kappa'_{4_1} i\tau + \dots
\end{aligned} \tag{3}$$

The first two series in (3) are exactly those appearing in (2), but the third series is a new one starting with $\kappa_{4_1}/(i\tau)$ for an unknown constant

$$\kappa_{4_1} = 0.4458257949935614977\dots \tag{4}$$

From our previous experience, we tried to recognise this constant as an algebraic number, after perhaps multiplying it with a small power of the square root of π . But all our attempts failed. With considerable effort, we computed the number κ_{4_1} to higher precision, but the recognition failed again. We then thought to use a different version of the asymptotic series, a power series in \hbar with coefficients rational functions of the A -polynomial curve. Doing so, the summation over $n \in \mathbb{Z}$ corresponds to integration of the 1-loop term of the above series with respect to u , as in explained in detail in Section 5.9. Much to our surprise, we found out that $\kappa_{4_1} = (\varpi_{4_1} + \overline{\varpi}_{4_1})/(2\pi)$ where

$$\varpi_{4_1} = \int_{\mathbb{R}} \frac{1}{\sqrt{e^{-2u} - 2e^{-u} - 1 - 2e^u + e^{2u}}} du. \tag{5}$$

This explained our failure to recognise $(\kappa_{4_1}/2\pi)$ as an algebraic number. Instead, it is a period of the A -polynomial curve, where the word period refers to the countable subring of the complex numbers introduced by Kontsevich–Zagier [32]. What is more, the above discovery tells us that the vertical asymptotics of the rotated 3D-index involve, in addition to the asymptotic series with algebraic coefficients, further power series with period coefficients! Continuing our experiment with 4_1 , and using the exact value of κ_{4_1} from before, we numerically computed the coefficient κ'_{4_1} of $i\tau$ in (3) and found it to be

$$\kappa'_{4_1} = 0.10059754907380012789\dots \tag{6}$$

We then theoretically computed the next term in the asymptotic series (which was an explicit rational function of e^u) and checked that $\kappa'_{4_1} = 2\pi(\varpi'_{4_1} + \overline{\varpi}'_{4_1})$ where

$$\varpi'_{4_1} = \int_C \frac{e^{-3u} - e^{-2u} - 2e^{-u} + 5 - 2e^u - e^{2u} + e^{3u}}{(e^{-2u} - 2e^{-u} - 1 - 2e^u + e^{2u})^{7/2}} du, \tag{7}$$

for a suitable contour C . Although the above identity would be impossible to guess from its numerical values, this was hardly a surprise, and confirmed the theory that we will discuss at a later section.

This completed our numerical experiments with the 4_1 knot, which admittedly has a simple trace field $\mathbb{Q}(\sqrt{-3})$ and a rather easy second degree A -polynomial. To test our ideas further, we then looked for a knot with more interesting trace field, i.e., the 5_2 knot with cubic trace field and a cubic A -polynomial. Repeating the analysis for the meromorphic 3D-index of the 5_2 knot we found that,

$$I_{5_2}^{\text{mer}}(0,0)(q) = 1 - 8q + 21q^2 + 92q^3 + 80q^4 - 95q^5 - 546q^6 - 1092q^7 - 1333q^8 - 756q^9 + \dots, \tag{8}$$

and that when $\tau \in i\mathbb{R}_+$ tends to zero, we have a similar expansion to (3) involving two known asymptotic series with algebraic coefficients, as well as a third series that starts with $i\kappa_{5_2}/\tau + O(1)$ where the new constant is numerically given by

$$\kappa_{5_2} = 0.3973476532990492346\dots \quad (9)$$

We then discovered that this constant is also a period of the A -polynomial curve. Let us summarise our findings here. Let $y_j(x)$ for $j = 1, 2, 3$ denote the three roots of the equation [16, Eqn.(233)]

$$y^2 = (1 - y)(1 - xy)(1 - y/x). \quad (10)$$

These are algebraic functions of x that collide when x is a root of the discriminant of (10) $x^8 - 6x^7 + 11x^6 - 12x^5 - 11x^4 - 12x^3 + 11x^2 - 6x + 1$ with respect to y . This discriminant has two real roots at $a_0 = 0.235344\dots$ and $a_1 = 4.249090\dots$ that satisfy $a_0 a_1 = 1$. All three branches $y_j(x)$ are real when $x > a_1$, and they are chosen so that $y_3(x)$ real for all x in \mathbb{R} and $y_1(x)$ and $y_2(x)$ complex-conjugate when x is real with

$$y_1(a_1 + 1) = 1.433146\dots, \quad y_2(a_1 + 1) = 3.823982\dots, \quad y_3(a_1 + 1) = 0.182470\dots \quad (11)$$

Consider the rational function $1/\delta_{5_2}(x, y)$ on the affine curve (10) where [16, Eqn.(235)]

$$\delta_{5_2}(x, y) = -y + (1 + x + 1/x)/y - 2/y^2, \quad (12)$$

and let $f_j(x) = 1/\delta_{5_2}(x, y_j(x))$ denote the three branches for $j = 1, 2, 3$. Since $y_j(x) = y_j(1/x)$, it follows that $f_j(x) = f_j(1/x)$ for all x and $j = 1, 2, 3$. Consider the periods

$$\varpi_{5_2}^{(j)} = \int_{\log(a_1)}^{\infty} f_j(e^u) du, \quad j = 1, 2, 3, \quad \varpi_{5_2}^{(3+j)} = \int_{\log(a_0)}^{\log(a_1)} f_j(e^u) du. \quad (13)$$

These periods are given by absolutely convergent integrals and their numerical value can easily be computed to high precision (e.g., to 500 digits), and given to 20 digits by

$$\begin{aligned} \varpi_{5_2}^{(1)} &= 0.50571950675093952382\dots, & \varpi_{5_2}^{(4)} &= 0.23686545502355828387\dots - i1.6144780233538382224\dots, \\ \varpi_{5_2}^{(2)} &= -0.47549190713818022860\dots, & \varpi_{5_2}^{(5)} &= 0.23686545502355828387\dots + i1.6144780233538382224\dots, \\ \varpi_{5_2}^{(3)} &= -0.030227599612759295211\dots, & \varpi_{5_2}^{(6)} &= -0.47373091004711656775\dots \end{aligned} \quad (14)$$

The above periods satisfy the relations

$$\varpi_{5_2}^{(1)} + \varpi_{5_2}^{(2)} + \varpi_{5_2}^{(3)} = 0, \quad \varpi_{5_2}^{(4)} + \varpi_{5_2}^{(5)} + \varpi_{5_2}^{(6)} = 0, \quad (15)$$

which is a consequence of the fact that $f_1(x) + f_2(x) + f_3(x) = 0$ for all x .

We then found numerically that

$$\kappa_{5_2} = \frac{1}{2\pi} (4\varpi_{5_2}^{(1)} + \varpi_{5_2}^{(4)} + \varpi_{5_2}^{(5)}). \quad (16)$$

Our constant is different from the number $\kappa_{5_2}^{\text{HKS}} = .534186\dots$ of [28, Fig.10.2], given to three digits in the above reference, but three more digits were also given to us by A. Kricker. We found out that the latter constant also agrees numerically with a period of the A -polynomial (for the real branch),

$$\kappa_{5_2}^{\text{HKS}} = -2\varpi_{5_2}^{(3)} - \varpi_{5_2}^{(6)} = - \int_{\mathbb{R}} f_1(e^u) du. \quad (17)$$

This leads to a conjectured identity

$$\frac{1}{(2\pi i)^2} \int_{i\mathbb{R}^2} B^2\left(\frac{1}{2} - x, \frac{1}{3} + x - y\right) B\left(\frac{1}{2} - x, \frac{1}{3} + 2y\right) dx dy = - \int_{\mathbb{R}} f_1(e^u) du, \quad (18)$$

(which we have checked to around 20 digits), where $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x + y)$ is Euler's beta function. The above identity implies that the double integral on left hand-side is a period of an elliptic curve (10) over \mathbb{Q} .

Identities of the above form should be valid for all hyperbolic knots.

1.3. Horizontal asymptotics and the Turaev–Viro invariant. Our observations discussed in the previous section lead to conjectures for the horizontal and the vertical asymptotics of the meromorphic 3D-index, which not only allow the effective computation of the asymptotic series, but also explain their structure.

The ingredients of first conjecture are: the meromorphic 3D-index (defined in [21] and briefly reviewed in Section 2.2 below), the Turaev–Viro invariant of a knot (defined in [5, 9] and recalled in Section 6), the asymptotic series $\widehat{\Phi}_{K,n}^{(\sigma)}(\hbar)$ for $s = \sigma_1$ and $s = \sigma_2$ of the form

$$\widehat{\Phi}_{K,n}^{(\sigma)}(\hbar) = e^{\frac{V_\sigma}{\hbar}} \Phi_{K,n}^{(\sigma)}(\hbar), \quad \Phi_{K,n}^{(\sigma)}(\hbar) \in \frac{1}{\sqrt{\delta_\sigma}} (1 + F_\sigma[[n, \hbar]]) \quad (19)$$

defined in Section 3 below and agreeing when $\sigma = \sigma_1$ and $n = 0$ with the asymptotic series of the Kashaev invariant of a knot. The labels σ_1 and σ_2 refer to the geometric $\mathrm{PSL}_2(\mathbb{C})$ -representation σ_1 of the hyperbolic knot and its complex-conjugate representation. The final ingredient is a summation over the integers.

Conjecture 1. For every hyperbolic knot K in S^3 , there exists a computable series $\widehat{\Phi}_{K,n}^{(\sigma_1)}$ which gives the asymptotic expansion of the meromorphic 3D-index (as $\tau \rightarrow 0$ on a ray sufficiently close to the positive real numbers), of the Turaev–Viro invariant (as $m \rightarrow \infty$)

$$I_K^{\mathrm{mer}}(0, 0)(e^{2\pi i\tau}) \sim \sum_{n \in \mathbb{Z}} \widehat{\Phi}_{K,n}^{(\sigma_1)}(2\pi i\tau) \widehat{\Phi}_{K,n}^{(\sigma_2)}(-2\pi i\tau), \quad (20)$$

$$\mathrm{TV}_{K,m+1/2} \sim \sum_{n \in \mathbb{Z}} \widehat{\Phi}_{K,n}^{(\sigma_1)}(2\pi i/(m+1/2)) \widehat{\Phi}_{K,n}^{(\sigma_2)}(-2\pi i/(m+1/2)), \quad (21)$$

and of the rotated 3D-index (as $\tau \rightarrow 0$ on a ray sufficiently close to \mathbb{R}_+):

$$I_K^{\mathrm{rot}}(n, n')(e^{2\pi i\tau}) \sim \widehat{\Phi}_{K,n}^{(\sigma_1)}(2\pi i\tau) \widehat{\Phi}_{K,n'}^{(\sigma_2)}(-2\pi i\tau). \quad (22)$$

This conjectured was verified numerically for the 4_1 and the 5_2 knots.

We make two remarks. The first is that the meromorphic 3D-index is a sum over the integers of the rotated 3D-index (Equation (34)), the latter being a bilinear combination of colored holomorphic blocks (Equation (42)) whose asymptotics are expressed linearly in terms of a matrix of $\widehat{\Phi}(\hbar)$ -series (Equation (45)). Thus, the asymptotics of the meromorphic 3D-index are bilinear combinations of asymptotic series. But when τ tends to zero nearly horizontally, one term (σ_1, σ_2) of this bilinear combination exponentially dominates all others, thus explaining Equation (20). On the other hand, the Turaev–Viro invariant of a knot is Reshetikhin–Turaev invariant of the double of the knot complement, thus doubling occurs already on the level of topology.

The second is that the above conjecture can be explained by a conjectural identity among analytic functions, one being the Borel resummation of the asymptotic series $\widehat{\Phi}_n^{(\sigma)}(2\pi i\tau)$ and a second being a state-integral; see Section 3.1 below. When τ is nearly horizontal, one can ignore all \tilde{q} -corrections and keep only the dominant bilinear combination of asymptotic series, leading to the asymptotic statements (20) and (22) of Conjecture 1.

1.4. Vertical asymptotics and periods. In this section, we discuss the vertical asymptotics (denoted by $\tau \downarrow 0$) of the 3D-index, where τ tends to 0 on the positive imaginary axis. In this case, we ignore the \tilde{q} -corrections which are theoretically unknown and computationally inaccessible. A new ingredient is a collection of asymptotic series parametrised by a labeling set of boundary parabolic $\mathrm{PSL}_2(\mathbb{C})$ -representation.¹ These series appear as follows. The rotated 3D-index is a bilinear combination of colored holomorphic blocks. The vertical asymptotics of the latter are linear combinations of asymptotic series $\widehat{\Phi}_n^{(\sigma)}(2\pi i\tau)$ where σ denotes a boundary parabolic $\mathrm{PSL}_2(\mathbb{C})$ -representation. It follows that the vertical asymptotics of the rotated 3D-index are *bilinear* combinations of asymptotic series. However, among bilinear combinations of all pairs (σ, σ') , only the pairs $(\sigma, \bar{\sigma})$ contribute, where $\bar{\sigma}$ denotes the complex-conjugate of the representation corresponding to σ .

Conjecture 2. When $\tau \downarrow 0$ vertically, we have

$$I^{\mathrm{rot}}(n, n')(e^{2\pi i\tau}) \sim \sum_{\sigma} \varepsilon_{\sigma} \widehat{\Phi}_n^{(\sigma)}(2\pi i\tau) \widehat{\Phi}_{n'}^{(\bar{\sigma})}(-2\pi i\tau), \quad (23)$$

with $\varepsilon_{\sigma} = -1$ when $\delta_{\sigma} < 0$ and 1 otherwise.

Note that after removing the exponential factors, the constant terms of the power series in τ in (23) will all be positive numbers. Moreover, the asymptotic series are polynomially bounded with respect to τ —perhaps a consequence of a unitarity property of the rotated 3D-index.

We next discuss the vertical asymptotics of the meromorphic 3D-index, ignoring as above any \tilde{q} -corrections. Here, we find a surprise and a new phenomenon: the coefficients of the asymptotic series are no longer algebraic numbers, but periods (in the sense of Kontsevich–Zagier [32]) on a plane curve which is none other than the A -polynomial of the knot [6]. To phrase our conjecture, instead of the power series $\Phi_n^{(\sigma)}(\hbar)$ in \hbar with coefficients polynomials in n , we will use the power series $\Phi(x, y, \hbar)$ in \hbar with coefficients rational functions in the A -polynomial curve. The latter are the x -deformed asymptotic series studied in [16]. The relation between the two types of series is given by $\Phi_n^{(\sigma)}(\hbar) = \Phi^{(\sigma)}(e^{n\hbar}, \hbar)$ where $\Phi^{(\sigma)}(x, \hbar) = \Phi(x, y_{\sigma}(x), \hbar)$ and $y_{\sigma} = y_s(x)$ is a branch of the A -polynomial curve.

Conjecture 3. When $\tau \downarrow 0$ vertically, we have

$$\begin{aligned} I^{\mathrm{mer}}(0, 0)(e^{2\pi i\tau}) \sim & \sum_{\sigma \text{ complex}} \sum_{n \in \mathbb{Z}} \widehat{\Phi}_n^{(\sigma)}(2\pi i\tau) \widehat{\Phi}_n^{(\bar{\sigma})}(-2\pi i\tau) \\ & + \frac{1}{2\pi i\tau} \sum_{\sigma} \int_{C_{\sigma}} \widehat{\Phi}^{(\sigma)}(e^{\tau u}, 2\pi i\tau) \widehat{\Phi}^{(\bar{\sigma})}(e^{\tau u}, -2\pi i\tau) du. \end{aligned} \quad (24)$$

¹There are subtleties involved since the latter space can be positive dimensional, however we (and [24, 28]) will ignore such subtleties.

The above conjecture explains the shape of the vertical asymptotics of the meromorphic 3D-index found by [28]. A bit of mystery remains: our vertical asymptotics include more terms than those of [28]. For example, for the 4_1 knot, the series starting with the term κ_{4_1} in Equation (3) is missing from [28, Sec.10.1.1], and for the 5_2 knot, our constant κ_{5_2} differs from the one of [28, Fig.10.2]. We do not understand this discrepancy which involves some global differences in the contours appearing in Equation (24).

The paper is organised as follows.

In Section 2, we introduce a rotated form of the 3D-index (see Equation (26) below) which conveniently decouples two commuting actions of the q -Weyl algebra, and also expresses the meromorphic 3D-index as a trace of the rotated 3D-index (see Equation (34) below). We next express the rotated 3D-index bilinearly in terms of colored holomorphic blocks $h_n^{(\alpha)}(q)$. These blocks (which are q -hypergeometric series) can be defined and computed either from a factorisation of a state-integral, or from a linear q -difference equation and their initial values at $n = 0$ given in [25]. A side bonus of the colored holomorphic blocks is an effective computation of the q -series expansion of the rotated and of the meromorphic 3D-index.

We then compute the asymptotics of the holomorphic blocks in terms of (completed) power series $\Phi_n^{(\sigma)}(h)$. These power series can be defined and computed using either formal Gaussian integration [12] (which amounts to a stationary phase expansion of a state-integral), or from a linear q -difference equation and their initial values at $n = 0$ given in [24]. In particular, the dominant series $\Phi^{(\sigma_1)}(h)$ is exactly the asymptotic expansion of the Kashaev invariant to all orders in $h = 2\pi i/N$ [24].

Finally, we compute the radial asymptotics of the rotated 3D-index in terms of bilinear combinations of the completed power series $\Phi_n^{(\sigma)}(h)$, which, after a formal Gaussian integration with respect to n , give the horizontal asymptotics of the rotated 3D-index.

Analysing further the possible bilinear asymptotics that can appear, combined with the fact that the meromorphic 3D-index is bounded when τ tends to zero vertically, allows us to compute the horizontal asymptotics of the meromorphic 3D-index.

We end this introduction with a remark on rigour. Although the 3D-index (in its original, or rotated form) is a well-defined collection of q -series [10, 11] which was shown to be a topological invariant of hyperbolic knots in two different ways [19, 21], its factorisation in terms of holomorphic blocks or colored holomorphic blocks discussed in Sections 2.3 and 2.4 are conjectural, and so are our asymptotic statements which form the main body of this paper. Likewise, the Turaev–Viro invariant of a knot is a well-defined topological invariant, however the conjecture on its all-orders asymptotic expansion is new, and unproven. However, the stated conjectures for the 3D-index and for the Turaev–Viro invariant are numerically testable (and have been tested for the case of the 4_1 and the 5_2 knots), and lead to surprising predictions as well as an explanation of the structure and computation of the asymptotics of these quantum invariants.

2. A REVIEW OF THE 3D-INDEX

In this section, we review the 3D-index in various forms: the original one of [10, 11], the rotated one, the one coming from colored holomorphic blocks, and the meromorphic version [21].

In their seminal papers, Dimofte–Gaiotto–Gukov introduced the 3D-index [10, 11], which comes from the low energy limit of an $N = 2$ supersymmetric conformal theory in 6 dimensions, compactified in 3-dimensions. More concretely, the 3D-index is a collection of Laurent q -series with integer coefficients attached to every integer homology class of the boundary of an ideally triangulated 3-manifold with torus boundary components. It is known that this collection is a topological invariant of hyperbolic 3-manifolds [19], and that it can be assembled to a meromorphic function on a complex torus [21] which, too, is a topological invariant.

In a further direction, Beem–Dimofte–Pasquetti studied the relationship between the 3D-index and a vector space of holomorphic blocks [2].

We next discuss several equivalent formulations of the 3D-index, namely the fugacity version, the rotated version, the meromorphic version, and the relation with holomorphic blocks.

2.1. The rotated 3D-index. Recall that the 3D-index of a 3-manifold with a torus boundary component (marked by a meridian and a longitude) is a collection of q -series parametrised by a pair of integers (m, e) . It turns out that the 3D-index is annihilated by two actions of the q -Weyl algebra, which are acting in the variables (m, e) in a coupled way. The rotated 3D-index is also an equivalent collection of q -series where the two actions are decoupled. To define the rotated 3D-index, we suppress the ambient 3-manifold M writing $I(m, e)(q)$ instead of $I_M(m, e)(q) \in \mathbb{Z}((q^{\frac{1}{2}}))$. Recall two commuting actions of the q -Weyl algebra [10, Eqn.2.29a]:

$$\begin{aligned} L_+ I(m, e)(q) &= q^{\frac{e}{2}} I(m+1, e)(q), & M_+ I(m, e)(q) &= q^{\frac{m}{2}} I(m, e-1)(q), \\ L_- I(m, e)(q) &= q^{\frac{e}{2}} I(m-1, e)(q), & M_- I(m, e)(q) &= q^{\frac{m}{2}} I(m, e+1)(q). \end{aligned} \quad (25)$$

It is easy to see that the four operators L_{\pm} and M_{\pm} commute except in the following instance:

$$L_+ M_+ = q M_+ L_+, \quad L_- M_- = q^{-1} M_- L_-.$$

The two commuting actions of the q -Weyl algebra mix the coordinates of the 3D-index. A separation of the two actions is possible and this motivates the *rotated form* of the 3D-index given by

$$I^{\text{rot}}(n, n')(q) = \sum_{e \in \mathbb{Z}} I(n - n', e)(q) q^{\frac{e(n+n')}{2}}. \quad (26)$$

When the ideal triangulation has a strict angle structure, it can be shown that the above series is well-defined. In the above equation, and the ones that follow, convergence is ensured if the minimum q -degree of $I(m, e)(q)$ is a positive quadratic function of (m, e) bounded below by $O(m^2 + e^2)$ except at finitely many directions of linear growth $O(|m|, |e|)$ (parallel to the normal vectors of the Newton polygon of the A -polynomial). This convergence is deduced from the results of [19, 18].

Lemma 4. We have:

$$\begin{aligned} L_+ I^{\text{rot}}(n, n')(q) &= I^{\text{rot}}(n+1, n')(q), & M_+ I^{\text{rot}}(n, n')(q) &= q^n I^{\text{rot}}(n, n')(q), \\ L_- I^{\text{rot}}(n, n')(q) &= I^{\text{rot}}(n, n'+1)(q), & M_- I^{\text{rot}}(n, n')(q) &= q^{-n'} I^{\text{rot}}(n, n')(q). \end{aligned} \quad (27)$$

Proof. We have:

$$\begin{aligned} L_+ I^{\text{rot}}(n, n')(q) &= \sum_e q^{\frac{e}{2}} I(n+1-n', e)(q) q^{\frac{e(n+n')}{2}} \\ &= \sum_e I(n+1-n', e)(q) q^{\frac{e(n+1+n')}{2}} = I^{\text{rot}}(n+1, n')(q), \end{aligned}$$

and

$$\begin{aligned} M_+ I^{\text{rot}}(n, n')(q) &= \sum_e q^{\frac{n-n'}{2}} I(n-n', e-1)(q) q^{\frac{e(n+n')}{2}} = \sum_e I(n-n', e-1)(q) q^{\frac{e(n+n')+n-n'}{2}} \\ &= \sum_e I(n-n', e)(q) q^{\frac{(e+1)(n+n')+n-n'}{2}} = \sum_e I(n-n', e)(q) q^{\frac{e(n+n')}{2}} q^n \\ &= q^n I^{\text{rot}}(n, n')(q). \end{aligned}$$

The case of (L_-, M_-) is similar. \square

Note that $I_K^{\text{rot}}(0, 0)(q)$ coincides with the invariant denoted by $I_K^{\text{tot}}(q)$ in Equation (2) of [19], and with the invariant studied in [25].

We now discuss some symmetries of the 3D-index and its rotated version. The symmetry,

$$I(m, e)(q) = I(-m, -e)(q), \quad (28)$$

of the 3D-index implies one for the rotated version $I^{\text{rot}}(n, n')(q) = I^{\text{rot}}(-n, -n')(q)$. The rotated 3D-index has two more symmetries

$$I^{\text{rot}}(n, n')(q) = I^{\text{rot}}(n, -n')(q) = I^{\text{rot}}(-n, n')(q) = I^{\text{rot}}(-n, -n')(q), \quad (29)$$

and

$$I^{\text{rot}}(n, n')(q^{-1}) = I^{\text{rot}}(n', n)(q), \quad (30)$$

which are manifest from the expression of the rotated 3D-index in terms of colored holomorphic blocks discussed below.

We next discuss how to express the original 3D-index from its rotated version, reversing Equation (26).

Lemma 5. For all integers m and e we have:

$$I(m, e)(q) = q^{-\frac{me}{2}} \sum_{k \in \mathbb{Z}} I^{\text{rot}}(m+k, k)(q) q^{-ke}. \quad (31)$$

Proof. We have:

$$\begin{aligned} \sum_k I^{\text{rot}}(m+k, k)(q) q^{-ke} &= \sum_{k, e'} I(m, e')(q) q^{\frac{e'm}{2} + k(e'-e)} = \sum_{e'} q^{\frac{e'm}{2}} I(m, e')(q) \sum_k q^{k(e'-e)} \\ &= \delta_{e, e'} \sum_{e'} q^{\frac{e'm}{2}} I(m, e')(q) = q^{\frac{em}{2}} I(m, e)(q). \end{aligned}$$

For completeness, we can also compute the action of the operators L_{\pm} and M_{\pm} on the right hand side $J(m, e)(q)$ of Equation (31). We have:

$$\begin{aligned} L_+ J(m, e)(q) &= q^{-\frac{me}{2}} \sum_{k \in \mathbb{Z}} L_+ I^{\text{rot}}(m+k, k)(q) q^{-ke} = q^{-\frac{me}{2}} \sum_{k \in \mathbb{Z}} I^{\text{rot}}(m+k+1, k)(q) q^{-ke} \\ &= q^{\frac{e}{2}} J(m+1, e)(q), \end{aligned}$$

and

$$\begin{aligned} M_+ J(m, e)(q) &= q^{-\frac{me}{2}} \sum_{k \in \mathbb{Z}} M_+ I^{\text{rot}}(m+k, k)(q) q^{-ke} = q^{-\frac{me}{2}} \sum_{k \in \mathbb{Z}} q^{m+k} I^{\text{rot}}(m+k, k)(q) q^{-ke} \\ &= q^{\frac{m}{2}} J(m, e-1)(q). \end{aligned}$$

The case of L_- and M_- is similar. \square

2.2. The meromorphic 3D-index. We now discuss another realisation of the 3D-index as a meromorphic function. In [21], it was shown that if a triangulation has a strict angle structure, then the following function

$$I^{\text{mer}}(z, w)(q) = \sum_{m, e \in \mathbb{Z}} I(m, e)(q) e^{2\pi i \tau(mz + ew)} \quad (32)$$

is meromorphic and regular at $(z, w) = (0, 0)$. The relation between the meromorphic and the rotated 3D-index is given by the next lemma.

Lemma 6. For all integers ℓ and ℓ' , we have:

$$I^{\text{mer}}(\ell, \ell')(q) = q^{-2\ell\ell'} \sum_{n \in \mathbb{Z}} q^{2\ell n} I^{\text{rot}}(n, n - 2\ell')(q). \quad (33)$$

Proof. Using Equations (32) and (31), we have:

$$\begin{aligned} I^{\text{mer}}(\ell, \ell')(q) &= \sum_{e, m} I(m, e)(q) q^{m\ell + e\ell'} = \sum_{e, m} q^{-\frac{me}{2}} \sum_k I^{\text{rot}}(m+k, k)(q) q^{m\ell + e\ell' - ke} \\ &= \sum_{m, k} q^{m\ell} I^{\text{rot}}(m+k, k)(q) \sum_e q^{-(\frac{m}{2} + k - \ell')e} = \sum_{m, k} q^{m\ell} I^{\text{rot}}(m+k, k)(q) \delta_{\frac{m}{2} + k - \ell', 0}, \end{aligned}$$

where the last identity follows from the fact that $\sum_e q^{ae} = \delta_a$ (interpreted distributionally). When m is odd the delta function vanishes, and for $m = 2m'$, we obtain that

$$\begin{aligned} I^{\text{mer}}(\ell, \ell')(q) &= \sum_{m', k} q^{2m'\ell} I^{\text{rot}}(2m'+k, k)(q) \delta_{m'+k-\ell', 0} = \sum_{m'} q^{2m'\ell} I^{\text{rot}}(m'+\ell', \ell' - m')(q) \\ &= \sum_{m'} q^{2m'\ell} I^{\text{rot}}(m'+\ell', m' - \ell')(q) = \sum_n q^{2n\ell} I^{\text{rot}}(n + \ell', n - \ell')(q), \end{aligned}$$

where the next to last equality follows from Equation (29). The result follows. \square

In [28], the asymptotics of $I^{\text{mer}}(0, 0)(q)$ when $q \in (0, 1)$ tends to 1 were studied. The relation of this q -series and the rotated 3D-index follows from the lemma above, and is given by

$$I^{\text{mer}}(0, 0)(q) = \sum_{n \in \mathbb{Z}} I^{\text{rot}}(n, n)(q). \quad (34)$$

2.3. Holomorphic blocks. Consider the 3D-index in the so-called fugacity basis [10, Eqn.2.5]:

$$I^\sharp(\zeta, m)(q) = \sum_e I(m, e)(q)\zeta^e. \quad (35)$$

Its expression in terms of holomorphic blocks is given below Eqn.6.46 of [2]. Comparing Equations (26) and (35), and keeping in mind that $I^\sharp(\zeta, m)$ is a meromorphic function of ζ with potential singularities at $\zeta \in q^{\mathbb{Z}}$, it follows that

$$I^{\text{rot}}(n, n')(q) = \lim_{x \rightarrow 1} I^\sharp(q^{\frac{n+n'}{2}}x, n - n')(q). \quad (36)$$

Now, according to [2, Eqn.1.3] (see also [2, Eqn.6.46] for a similar statement for state-integrals), we have

$$I^\sharp(\zeta, m)(q) = \sum_\alpha B^{(\alpha)}(x; q)B^{(\alpha)}(\tilde{x}; \tilde{q}), \quad x = q^{\frac{m}{2}}\zeta, \quad \tilde{x} = q^{\frac{m}{2}}\zeta^{-1}, \quad \tilde{q} = q^{-1}, \quad (37)$$

where $B^{(\alpha)}(x, q)$ are holomorphic blocks, meromorphic functions of x which are defined both inside and outside the unit q -disk, and are annihilated by the homogeneous \hat{A} -polynomial. Equations (36) and (37) imply that

$$I^{\text{rot}}(n, n')(q) = \lim_{x \rightarrow 1} \sum_\alpha B^{(\alpha)}(q^{-n'}x^{-1}; q^{-1})B^{(\alpha)}(q^n x; q). \quad (38)$$

This is our starting point for the asymptotics of the 3D-index, which in particular predicts that the rotated 3D-index satisfies the equation

$$\hat{A}(M_+, L_+)I^{\text{rot}} = \hat{A}(M_-, L_-)I^{\text{rot}} = 0. \quad (39)$$

2.4. Colored holomorphic blocks. The holomorphic blocks $B^{(\alpha)}(x; q)$ are meromorphic functions of x with poles at $x \in q^{\mathbb{Z}}$, which makes the limit in equation (38) difficult to compute. Instead, we will introduce colored holomorphic blocks $h^{(\alpha)}(q)$ and express the rotated 3D-index as follows:

$$I^{\text{rot}}(n, n')(q) = \sum_\alpha B_{\alpha, \bar{\alpha}} h_n^{(\alpha)}(q) h_{n'}^{(\bar{\alpha})}(q^{-1}), \quad (40)$$

where $B = B^t \in \text{GL}_r(\mathbb{Q})$ and $\alpha \mapsto \bar{\alpha}$ is an involution corresponding to the complex conjugation of the set of σ . The colored holomorphic blocks, defined for $|q| \neq 1$ form a fundamental solution to the linear q -difference equation

$$\hat{A}(M_+, L_+)H(q) = \hat{A}(M_-, L_-)H(q^{-1}) = 0, \quad (41)$$

which can be defined and computed by applying the Frobenius method. In particular, this implies that the labeling set of α 's consists of the pairs of edges of the lower Newton polygon of the A -polynomial, together with with a root of the edge polynomial (the latter known to be product of cyclotomic polynomials).

Going one step further, we can define a $\mathbb{Z} \times \mathbb{Z}$ matrix $I^{\text{rot}}(q)$ whose (n, n') entry is $I^{\text{rot}}(n, n')(q)$, and a $\mathbb{Z} \times r$ matrix $H(q)$ whose (n, α) entry is $h_n^{(\alpha)}(q)$, and then write the above equation in the matrix form

$$I^{\text{rot}}(q) = H(q)BH(q^{-1})^t. \quad (42)$$

Equations (42) and (34) give that

$$I^{\text{mer}}(0,0)(q) = \text{tr}(H(q)BH(q^{-1})^t). \quad (43)$$

The symmetry,

$$h_n^{(\alpha)}(q) = h_{-n}^{(\alpha)}(q), \quad (44)$$

of the colored holomorphic blocks (for all α and all integers n) and Equation (42) implies the symmetries (29) and (30) for the rotated 3D-index.

To use equation (42) effectively, we need a method to compute the colored holomorphic blocks. We can do so either from the polar decomposition of the holomorphic blocks when the latter are available (this is the case of the 4_1 and 5_2 knots), or by applying the Frobenius method (see, [14]) to the linear q -difference equation that they satisfy when the latter is available (this is also the case of the 4_1 and 5_2 knots). Either way, there is an ambiguity in the normalisation of the colored holomorphic blocks.

3. ASYMPTOTICS

The previous section expressed the rotated and the meromorphic 3D-index of a knot bilinearly in terms of a matrix of colored holomorphic blocks—see equations (42) and (43). In this section, we discuss the structure of the horizontal asymptotics of the colored holomorphic blocks, and consequently of the rotated and of the meromorphic 3D-index.

3.1. A matrix of asymptotic series. We begin by defining a matrix of asymptotic series which ultimately expresses the asymptotics of the colored holomorphic blocks and of the 3D-index, in all of its forms. Let $H(q)$ and $\widehat{\Phi}(h)$ denote the $\mathbb{Z} \times r$ matrix with $H(q)_{n,\alpha} = h_n^{(\alpha)}(q)$ and $\widehat{\Phi}(h)_{n,\alpha} = \widehat{\Phi}_n^{(\alpha)}(h)$. For an integer n , let $H_n(q)$ and $\widehat{\Phi}_n(h)$ denote the $r \times r$ matrices whose rows are the n th, $n+1$, \dots , $n+r-1$ rows of $H(q)$ and $\widehat{\Phi}(h)$, respectively. The entries of the matrix $\widehat{\Phi}(h)$ are completed formal power series of the form given in Equation (19) where σ labels a boundary parabolic $\text{PSL}_2(\mathbb{C})$ -representation of the knot complement, and $V_\sigma = i\text{Vol}(\sigma) + \text{CS}(\sigma)$ denotes its complexified volume whose imaginary part is the volume and the real part is the Chern–Simons invariant [35]. The Borel resummation of the completed formal power series $\widehat{\Phi}$, whose definition we omit, will be denoted by $(s\widehat{\Phi})(h)$ (for a detailed definition, see [34]). The Borel resummation depends on a ray in a sector that does not contain Stokes lines. In our case, the sector is in the upper or lower half-plane and close to the positive real axis, in which case it is conjectured that there are no Stokes lines in such a sector [15], and that Borel resummation is well-defined.

The relation between the colored holomorphic blocks and the asymptotic series is the following conjectural identity:

$$H_n(q) = (s\widehat{\Phi}_n)(2\pi i\tau)M(\tilde{q})H_0(\tilde{q})\text{diag}(\tau^{\kappa_\alpha}), \quad (0 < |\arg(\tau)| < \varepsilon) \quad (45)$$

valid for some $\varepsilon > 0$ (which depends on the knot), where $M(q) \in \text{GL}_r(\mathbb{Q}(q))$. This equality was conjectured in [15, 16] together with [25, 24]. The first papers conjectured that state integrals give rise to Borel re-summation and the second essentially conjecture that state integrals come as cocycles of associated quantum modular forms in this case H_n . Both of these statements are known to hold in examples [7, 20]. The above equality of exact functions allows us to compute the matrix $\widehat{\Phi}_n(h)$ of asymptotic series in several ways:

- (a) via numerical asymptotics of the colored holomorphic blocks as τ tends to zero nearly horizontally.
- (b) via WKB, using that fact that both matrices $H_n(q)$ and $\widehat{\Phi}_n(h)$ are fundamental solutions of the same linear q -difference equation, namely the \widehat{A} -equation. In fact, it appears that $\widehat{\Phi}_n(h)$ is the unique discrete WKB solution to the \widehat{A} equation with initial condition $\widehat{\Phi}_0(h)$.
- (c) via the formulas of [12], where a power series was defined by formal Gaussian integration using the Neumann–Zagier matrices of an ideal triangulation of a knot complement and a solution to the Neumann–Zagier equations, whose developing map is α . In that paper, if we let the solution of the Neumann–Zagier equation vary infinitesimally around α , we obtain a power series in h whose coefficients are polynomials in n .
- (d) via the asymptotics of state-integrals that equal to a suitable matrix-valued cocycle.

In the present paper, we will mostly use the WKB ansatz since the linear q -difference equation (i.e., the \widehat{A} -polynomial) of the 4_1 and the 5_2 knots are known. Recall that the WKB ansatz

$$\Phi_n(\hbar) = \exp\left(\sum_{\ell=-1}^{\infty} S_\ell(n\hbar)\right) = \Phi_0(\hbar) \exp\left(\sum_{\ell=-1}^{\infty} \sum_{k=1}^{\infty} a_{k,\ell}(n\hbar)^k \hbar^\ell\right). \quad (46)$$

substitutes the above power series in the linear q -difference equation, and solves for the coefficients, one degree of \hbar at a time. The Weyl symmetry $n \leftrightarrow -n$ of the linear q -difference equation implies that we get polynomials in n^2 , thus $a_{k,\ell} = 0$ for all odd k . The first unknown to be determined is $a_{2,-1}$, which satisfies a polynomial equation obtained by setting $q = 1$ in the linear q -difference equation, $q^n = 1 + u + O(u^2)$ and replacing the shift operator by $1 + a_{2,-1}u + O(u^2)$. This polynomial equation is identical to the polynomial equation for the cusp-shape c of the hyperbolic knot, obtained by replacing $M = 1 + u$, $L = 1 + cu$ in the A -polynomial of a knot and equating the lower power of u arising to zero. The coincidence of the polynomial equations for $a_{2,-1}$ and of the cusp shape is consistent with the AJ-Conjecture which states that the specialisation of $q = 1$ to the \widehat{A} -polynomial is the A -polynomial. Once $a_{2,-1}$ is known, the remaining coefficients $a_{k,\ell}$ for $k \geq 2$ are obtained from the WKB ansatz as a solution to a block triangular linear system.

3.2. Horizontal asymptotics of the rotated 3D-index. Equation (45), together with Equation (42) implies that

$$I^{\text{rot}}(q) = (s\widehat{\Phi})(2\pi i\tau)M(\tilde{q})I_0^{\text{rot}}(\tilde{q})M(\tilde{q}^{-1})(s\widehat{\Phi})^*(-2\pi i\tau), \quad (47)$$

where $*$ means the conjugate transpose. The above is an exact identity that gives the $q \rightarrow 1$ asymptotics all the q -series $I^{\text{rot}}(n, n')(q)$ to all orders in τ and with all exponential terms \tilde{q} included. Ignoring the \tilde{q} terms, and assuming that $\tau \rightarrow 0$ in a fixed ray with $0 < |\arg(\tau)| < \varepsilon$, we obtain that

$$I^{\text{rot}}(q) \sim \widehat{\Phi}(2\pi i\tau)\text{diag}(M_0)(\widehat{\Phi})^*(-2\pi i\tau), \quad (48)$$

for a matrix $M_0 \in \text{GL}_r(\mathbb{Q})$. Even further, when $\tau \rightarrow 0$ in a ray $\arg(\tau) = \theta_0$ with θ_0 positive and sufficiently small, this implies the asymptotic expansion of Equation (22).

3.3. Horizontal asymptotics and Gaussian integration. Equations (43) and (22) imply that when $\tau \rightarrow 0$ in a ray $\arg(\tau) = \theta_0$ with θ_0 positive and sufficiently small, Equation (20) holds. In this section, we explain how to expand the sums over the integers of the bilinear asymptotic series in (20), as power series in \hbar .

We will use the form (46) for the two asymptotic series $\Phi_n^{(\sigma)}(\hbar)$ and $\Phi_n^{(\sigma')}(\hbar)$, and abbreviate $\beta_{k,\ell} = a_{k,\ell}^{(\sigma)} + (-1)^{k+\ell} a_{k,\ell}^{(\sigma')}$ and $\beta = 2\beta_{2,-1}$. Of course, β and $\beta_{k,\ell}$ depend on (σ, σ') . Then, we have

$$\begin{aligned} \Phi_n^{(\sigma)}(\hbar)\Phi_n^{(\sigma')}(-\hbar) &= \exp\left(\sum_{\ell=-1}^{\infty}\sum_{k=1}^{\infty}(a_{k,\ell}^{(\sigma)} + (-1)^{k+\ell}a_{k,\ell}^{(\sigma')})(n\hbar)^k\hbar^\ell\right) \\ &= \Phi_0^{(\sigma)}(\hbar)\Phi_0^{(\sigma')}(-\hbar)e^{\frac{1}{2}\beta n^2\hbar}\exp\left(\sum_{\ell=-1}^{\infty}\sum_{k=1}^{\infty}\beta_{k,\ell}(n\hbar)^k\hbar^\ell\right) \\ &= \Phi_0^{(\sigma)}(\hbar)\Phi_0^{(\sigma')}(-\hbar)e^{\frac{1}{2}\beta n^2\hbar}I(n, \hbar), \end{aligned} \quad (49)$$

where $I(n, \hbar) = \exp\left(\sum_{\ell=-1}^{\infty}\sum_{k=1}^{\infty}\beta_{k,\ell}(n\hbar)^k\hbar^\ell\right)$. The coefficient of \hbar^k in $I(n, \hbar)$ is a polynomial in n of degree $4k/3$ (resp., $4k/3 - 4/3$, $4k/3 - 2/3$) if $k = 0 \pmod 3$ (resp., $k = 1 \pmod 3$, $k = 2 \pmod 3$), with the first few terms given by

$$\begin{aligned} I(n, \hbar) &= 1 + n^2\beta_{2,0}\hbar^2 + (n^2\beta_{2,1} + n^4\beta_{4,-1})\hbar^3 + \left(\frac{1}{2}n^4\beta_{2,0}^2 + n^2\beta_{2,2} + n^4\beta_{4,0}\right)\hbar^4 \\ &\quad + (n^4\beta_{2,0}\beta_{2,1} + n^2\beta_{2,3} + n^6\beta_{2,0}\beta_{4,-1} + n^4\beta_{4,1} + n^6\beta_{6,-1})\hbar^5 \\ &\quad + \left(\frac{1}{6}n^6\beta_{2,0}^3 + \frac{1}{2}n^4\beta_{2,1}^2 + n^4\beta_{2,0}\beta_{2,2} + n^2\beta_{2,4} + n^6\beta_{2,1}\beta_{4,-1}\right. \\ &\quad \left. + \frac{1}{2}n^8\beta_{4,-1}^2 + n^6\beta_{2,0}\beta_{4,0} + n^4\beta_{4,2} + n^6\beta_{6,0}\right)\hbar^6 + O(\hbar^7). \end{aligned} \quad (50)$$

To sum over $n \in \mathbb{Z}$, we can use either Poisson summation, or the modular transformation property

$$\Theta(\mathbf{e}(\tau)) = \zeta_8\tau^{-1/2}\Theta(\mathbf{e}(-1/\tau)), \quad (51)$$

of the theta function

$$\Theta(q) = \sum_{n \in \mathbb{Z}} q^{n^2/2}, \quad (52)$$

where $\zeta_8 = \mathbf{e}(1/8)$ so that $(\zeta_8)^8 = 1$. Differentiating Equation (51), we find that

$$\sum_{n \in \mathbb{Z}} n^{2k} \mathbf{e}\left(\frac{1}{2}n^2\tau\right) \in \zeta_8 \sum_{j=0}^{k-1} \frac{1}{\tau^{k+j+1/2}} \mathbb{Q} \left[\left[\mathbf{e}\left(-\frac{1}{2\tau}\right) \right] \right] \quad (53)$$

is polynomial in $\tau^{-1/2}$ with coefficients a power series of $\mathbf{e}(-1/\tau)$. When $\tau \sim 0$, ignoring all the powers $\mathbf{e}(-1/\tau)$, we obtain that $\sum_{n \in \mathbb{Z}} n^k \mathbf{e}\left(\frac{1}{2}n^2\tau\right) \sim \langle n^k \rangle_{2\pi i\tau}$ to all orders in τ , where

$$\langle n^{2k} \rangle_{2\pi i\tau} = \frac{(-1)^k \zeta_8 (2k-1)!!}{(2\pi i)^k \tau^{(2k+1)/2}}, \quad \langle n^{2k+1} \rangle_{2\pi i\tau} = 0 \quad (54)$$

and $(2k - 1)!! = (2k)!/(k!2^k)$ for $k \geq 0$. The above equation can also be written in terms of a formal Gaussian integration as follows:

$$\langle n^{2k} \rangle_{2\pi i\tau} = \int_{\mathbb{R}} u^{2k} \mathbf{e} \left(\frac{1}{2} u^2 \tau \right) du. \quad (55)$$

Using the above asymptotics with $2\pi i\tau = \beta\hbar$ and Equation (50) we find that

$$\begin{aligned} \sum_{n \in \mathbb{Z}} \Phi_n^{(\sigma)}(\hbar) \Phi_n^{(\sigma')}(-\hbar) &= \Phi_0^{(\sigma)}(\hbar) \Phi_0^{(\sigma')}(-\hbar) \langle 1 \rangle_{\beta\hbar} \times \left(1 + \left(-\frac{\beta_{2,0}}{\beta} + \frac{3\beta_{4,-1}}{\beta^2} \right) \hbar \right. \\ &\quad \left. + \left(3\frac{\beta_{2,0}^2}{2\beta^2} - \frac{\beta_{2,1}}{\beta} - 15\frac{\beta_{2,0}\beta_{4,-1}}{\beta^3} + 105\frac{\beta_{4,-1}^2}{2\beta^4} + 3\frac{\beta_{4,0}}{\beta^2} - 15\frac{\beta_{6,-1}}{\beta^3} \right) \hbar^2 + O(\hbar^3) \right). \end{aligned} \quad (56)$$

From the bounds of the n -degree of each coefficient of \hbar^k in $I(n, \hbar)$, it follows that the coefficient of \hbar^k in the sum over n depends only on $I(n, \hbar) + O(\hbar^{3k+1})$.

4. FURTHER DISCUSSION

In this section, we discuss briefly three further aspects of the 3D-index: its relation with Stokes matrices, following [15, 16], its relation to stability, following [22], and its possible relation with topological recursion [4].

4.1. The 3D-index and Stokes matrices. A discovery of [15, 16] was that the rotated 3D-index essentially determines the Stokes constants of the Borel resummation of the asymptotic series that appear in complex Chern–Simons theory. For a precise statement, see Conjecture 4 of [16]. A further extension that includes the so-called trivial flat connection was proposed in [17]. An explanation of this relation between the BPS counts of the 3D-index and the Stokes constants of perturbative power series from first principles is currently missing. Without doubt, the Stokes constants of the Borel resummation of the matrix $\Phi_n(\hbar)$ of factorially divergent \hbar -power series can be expressed in terms of the rotated 3D-index, in a manner analogous to the results of [16], only that the descendant variable m there is replaced by the variable n here. We will not pursue this numerical investigation further.

4.2. Stability. In this short subsection, which is independent of our asymptotic results, we make some comments regarding the $q = 0$ expansion of the colored holomorphic blocks and their relation with the 3D-index. Whereas the 3D-index is a well-defined topological invariant, computable from the Neumann–Zagier matrices of an ideal triangulation of a 3-manifold, the $\mathbb{Z} \times r$ matrix $H(q)$ of the colored holomorphic blocks has not been defined in terms of an ideal triangulation, nor is a priori a topological invariant. Yet, as we will see in examples, the matrix of colored holomorphic blocks satisfies some stability properties and this makes possible to recover it from the matrix of the rotated 3D-index.

Perhaps stability properties of $I^{\text{rot}}(n, n')(q)$ as n or n' are large allow one to determine the colored holomorphic blocks $H(q)$ (in a suitable normalisation) from the rotated 3D-index. If so, it would give a definition of the colored holomorphic blocks in terms of Neumann–Zagier matrices as well as conclude their topological invariance, something which is currently missing. This works for 4_1 (whose details we give) and 5_2 (whose details we omit). Unfortunately, this hope has not been realised so far for all knots.

4.3. Periods from Topological Recursion? We have seen in the introduction that periods of the A -polynomial curve appear in the vertical asymptotics of the meromorphic 3D-index (but not of the rotated, or the original 3D-index). On the other hand, the rotated 3D-index satisfies (in two variables) a linear q -difference equation whose $q = 1$ limit is conjectured to be the A -polynomial of [6].

Since periods are nonperturbative (and classical) information of the A -polynomial curve, is there an explanation of their appearance in the context of topological recursion? Note that the latter can explain the asymptotics of the Kashaev invariant of a knot [4].

5. THE 3D-INDEX OF THE 4_1 KNOT

In this section, we give a detailed and computable description of the 3D-index of the 4_1 knot and of its asymptotics, illustrating the conjectures of Section 1.3. All quantities (functions and numbers) in this section refer to the 4_1 knot, which will be omitted from the notation.

5.1. Holomorphic blocks. The 4_1 knots has two colored holomorphic blocks $B^{(1)}(x; q)$ and $B^{(2)}(x; q)$ are given by (see [2, Eqn.6.45] and [16, Sec.5.2])

$$\begin{aligned} B^{(1)}(x; q) &= \frac{(qx^2; q)_\infty}{\theta(x; q)\theta(-q^{1/2}x; q)} \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(k+1)/2} x^k}{(q; q)_k (qx^2; q)_k}, \\ B^{(2)}(x; q) &= B^{(1)}(x^{-1}; q), \\ B^{(1)}(x; q^{-1}) &= \frac{\theta(x; q)\theta(-q^{-1/2}x; q)}{(x^2; q)_\infty} \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(k+1)/2} x^{-k}}{(q; q)_k (qx^{-2}; q)_k}, \\ B^{(2)}(x; q^{-1}) &= B^{(1)}(x^{-1}; q^{-1}), \end{aligned} \tag{57}$$

where

$$\theta(x; q) = (-q^{1/2}x; q)_\infty (-q^{1/2}x^{-1}; q)_\infty. \tag{58}$$

Since the two holomorphic blocks are given by 1-dimensional proper q -hypergeometric sums, it follows by Wilf-Zeilberger theory [41], implemented by Koutschan in [33], that they both satisfy the second order linear q -difference equation

$$P_0(x, q)B^{(\alpha)}(x; q) + P_1(x, q)B^{(\alpha)}(qx; q) + P_2(x, q)B^{(\alpha)}(q^2x; q) = 0, \tag{59}$$

for $\alpha = 1, 2$, where

$$\begin{aligned} P_0(x, q) &= q^2 x^2 (q^3 x^2 - 1), \\ P_1(x, q) &= -q^{1/2} (1 - q^2 x^2) (1 - qx - qx^2 - q^3 x^2 - q^3 x^3 + q^4 x^4), \\ P_2(x, q) &= q^3 x^2 (-1 + qx^2). \end{aligned} \tag{60}$$

Our normalisation of the above q -difference equation differs slightly from [16, Eqn.(140)]. Our choice was dictated by the facts that it annihilates in two ways the rotated 3D-index, the latter being canonically normalised, and by the fact that it respects the $\mathbb{Z}/2\mathbb{Z}$ Weyl symmetry, (i.e., it is invariant under the map $x \mapsto 1/x$), which when $x = q^n$, (n being the weight of an $\mathfrak{sl}_2(\mathbb{C})$ representation), means invariance under the map $n \mapsto -n$.

5.2. Colored holomorphic blocks. The holomorphic blocks are meromorphic functions of (x, q) . We now define the colored holomorphic blocks by looking at their expansions at $x = q^n e^u$ at $u = 0$. Explicitly, we define the colored holomorphic blocks $h_n^{(0)}(q)$ and $h_n^{(1)}(q)$ for $|q| < 1$ by

$$\begin{aligned} B^{(1)}(q^n e^u; q) &= \frac{1}{(q; q)_\infty (-q^{1/2}; q)_\infty^2} h_n^{(0)}(q) u^{-1} + O(u^0), \\ B^{(1)}(q^n e^u; q) + B^{(2)}(q^n e^u; q) &= \frac{1}{(q; q)_\infty (-q^{1/2}; q)_\infty^2} h_n^{(1)}(q) + O(u^1), \end{aligned} \quad (61)$$

and for $|q| < 1$ by

$$\begin{aligned} B^{(1)}(q^{-n} e^{-u}; q^{-1}) &= \frac{(q; q)_\infty (-q^{1/2}; q)_\infty^2}{2} h_n^{(0)}(q^{-1}) + O(u^1), \\ B^{(1)}(q^{-n} e^{-u}; q^{-1}) - B^{(2)}(q^{-n} e^{-u}; q^{-1}) &= \frac{(q; q)_\infty (-q^{1/2}; q)_\infty^2}{2} h_n^{(1)}(q^{-1}) u + O(u^2). \end{aligned} \quad (62)$$

Since the holomorphic blocks $B^{(\alpha)}(x; q)$ satisfy the q -difference equation (59), it follows that the colored holomorphic blocks, defined in (61) and (62) satisfy the linear q -difference equation

$$P_0(q^n, q) h_n^{(\alpha)}(q) + P_1(q^n, q) h_{n+1}^{(\alpha)}(q) + P_2(q^n, q) h_{n+2}^{(\alpha)}(q) = 0 \quad (\alpha = 0, 1, n \in \mathbb{Z}). \quad (63)$$

Next, we give q -hypergeometric formulas for the colored holomorphic blocks for $|q| \neq 1$. To do so, we use the next lemma which can be deduced from [20, Prop.2.2] or from [25]. For a positive integer ℓ , consider the function

$$E_\ell(q) = \frac{\zeta(1-\ell)}{2} + \sum_{s=1}^{\infty} s^{\ell-1} \frac{q^s}{1-q^s}, \quad (64)$$

(where $\zeta(s)$ is the Riemann zeta function), analytic for $|q| < 1$ and extended to $|q| > 1$ by the symmetry $E_\ell(q^{-1}) = -E_\ell(q)$.

Lemma 7 ([20, 25]). We have:

$$\begin{aligned} \frac{(qe^\varepsilon; q)_\infty}{(q; q)_\infty} \frac{1}{(qe^\varepsilon; q)_m} &= \frac{1}{(q; q)_m} \sqrt{\frac{-\varepsilon}{1-\exp(\varepsilon)}} \exp\left(-\sum_{\ell=1}^{\infty} \left(E_\ell(q) - \sum_{n=1}^m \text{Li}_{1-\ell}(q^n)\right) \frac{\varepsilon^\ell}{\ell!}\right), \\ \frac{(q^{-1}e^\varepsilon; q^{-1})_\infty}{(q^{-1}; q^{-1})_\infty} \frac{1}{(q^{-1}e^\varepsilon; q^{-1})_m} &= \frac{1}{(q^{-1}; q^{-1})_m} \frac{-1}{\varepsilon} \sqrt{\frac{-\varepsilon}{1-\exp(\varepsilon)}} \\ &\quad \exp\left(-\sum_{\ell=1}^{\infty} \left(E_\ell(q^{-1}) - \sum_{n=1}^m \text{Li}_{1-\ell}(q^{-n})\right) \frac{\varepsilon^\ell}{\ell!}\right). \end{aligned} \quad (65)$$

Proposition 8. We have:

$$h_n^{(0)}(q) = (-1)^n q^{|n|(2|n|+1)/2} \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(k+1)/2+|n|k}}{(q; q)_k (q; q)_{k+2|n|}}, \quad (66)$$

and

$$h_n^{(1)}(q) = (-1)^n q^{|n|(2|n|+1)/2} \sum_{k=0}^{\infty} \left(-4E_1(q) + \sum_{\ell=1}^{k+2|n|} \frac{1+q^\ell}{1-q^\ell} + \sum_{\ell=1}^k \frac{1+q^\ell}{1-q^\ell} \right) (-1)^k \frac{q^{k(k+1)/2+|n|k}}{(q; q)_k (q, q)_{k+2|n|}} \\ - 2(-1)^n q^{|n|(2|n|-1)/2} \sum_{k=0}^{2|n|-1} (-1)^k \frac{q^{k(k+1)/2-|n|k} (q^{-1}, q^{-1})_{2|n|-1-k}}{(q; q)_k}, \quad (67)$$

for $|q| \neq 1$. The colored holomorphic blocks satisfy the symmetries

$$h_n^{(0)}(q^{-1}) = h_n^{(0)}(q), \quad h_n^{(1)}(q^{-1}) = -h_n^{(1)}(q), \quad (68)$$

and

$$h_{-n}^{(\alpha)}(q) = h_n^{(\alpha)}(q), \quad \alpha = 0, 1. \quad (69)$$

Proof. The proof of the above formulas (66) and (67) follows from the definitions of the colored holomorphic blocks combined with the expansions of Lemma 7 above. One simply collects terms associated to each piece of the sum to deduce the proposition. For example,

$$\frac{1}{\theta(q^n e^u; q)} = \frac{q^{n^2/2}}{(-q^{1/2}; q)_\infty^2} (1 + nu + O(u^2)), \quad (70)$$

or when $n \geq 0$,

$$\frac{(q^{2n+1}x^2; q)_\infty}{(q^{2n+1}x^2; q)_k} = \frac{(q; q)_\infty}{(q; q)_{k+2n}} \left(1 - \left(\frac{1}{2} + 2E_1(q) + 2 \sum_{\ell=1}^{k+2n} \frac{q^\ell}{1-q^\ell} \right) u + O(u^2) \right), \quad (71)$$

or when $k - 2n < 0$,

$$\frac{(q^{-2n+1}x^{-2}; q)_\infty}{(q^{-2n+1}x^{-2}; q)_k} = -2(q^{-1}; q^{-1})_{2n-k-1} u + O(u^2). \quad (72)$$

The $q \leftrightarrow q^{-1}$ symmetry of $h_n^{(0)}(q)$ follows from the definition and the symmetry of its summand itself and similarly for $h^{(1)}(q)$ using the first few coefficients in the u expansion, where we also use the definition of $E_1(q^{-1})$. \square

The colored holomorphic blocks when $n = 0$ agree with the pair of q -series in [25]:

$$h_0^{(0)}(q) = g(q), \quad h_0^{(1)}(q) = G(q). \quad (73)$$

This is manifest from the definitions of these q -series. The relation to the descendant q -series of [15, Eqn.(13)] is given by

$$g_0^{(i)}(q) = h_0^{(i)}(q), \\ (1-q)g_1^{(i)}(q) = h_0^{(i)}(q) - q^{1/2}h_1^{(i)}(q). \quad (74)$$

This follows from the relation for the two Wronskians [16, Eqn.(156)] noting the $-q^{1/2}$ difference as seen in Equation (59). Note that the colored Jones polynomial of the 4_1 knot

is given

$$\begin{aligned}
J_n(q) &= \sum_{k=0}^{n-1} q^{-kn} (q^{n-1}; q^{-1})_k (q^{n+1}; q)_k \\
&= \frac{1}{1-q^n} \sum_{k=-n}^{-1} q^{nk+n} \frac{(q; q)_{n-k-1}}{(q; q)_{k+n}} \\
&= \frac{1}{1-q^n} \sum_{k=1}^n q^{n-nk} \frac{(q; q)_{n+k-1}}{(q; q)_{n-k}}.
\end{aligned} \tag{75}$$

The last formula appears up to some simple factor in the expression for $h_n^{(1)}(q)$. The first few values of the colored holomorphic blocks are given by

$$\begin{aligned}
h_0^{(0)}(q) &= 1 - 2 - 2q^2 - 2q^3 - 2q^4 + q^6 + 5q^7 + 7q^8 + 11q^9 + \dots, \\
h_1^{(0)}(q) &= q^{1/2}(-q - q^2 - q^3 + q^5 + 3q^6 + 5q^7 + 7q^8 + 9q^9 + \dots), \\
h_2^{(0)}(q) &= q^5 + q^6 + 2q^7 + 2q^8 + 3q^9 + \dots,
\end{aligned} \tag{76}$$

and

$$\begin{aligned}
h_0^{(1)}(q) &= 1 - 7q - 14q^2 - 8q^3 - 2q^4 + 30q^5 + 43q^6 + 95q^7 + 109q^8 + 137q^9 + \dots, \\
h_1^{(1)}(q) &= q^{1/2}(-2q^{-1} - 5q - 3q^2 + 3q^3 + 16q^4 + 33q^5 + 51q^6 + 73q^7 + 77q^8 + 81q^9 + \dots), \\
h_2^{(1)}(q) &= 2q^{-3} - 2q^{-2} + 2q + 2q^2 + 2q^3 + 2q^4 + 11q^5 + 9q^6 + 14q^7 + 6q^8 + 3q^9 + \dots.
\end{aligned} \tag{77}$$

The colored holomorphic blocks have the q -degree (i.e., minimum power of q)

$$\deg_q h_n^{(0)}(q) = n^2 + \frac{1}{2}|n|, \quad h_n^{(1)}(q) = -n^2 + \frac{1}{2}|n|. \tag{78}$$

The above statement for the degree follows easily from Equations (66) and (67). Alternatively, observe that the two colored holomorphic blocks are fundamental solutions of the linear q -difference equation (63) obtained by the Frobenius (or the WKB) method (one power of q^n at a time, as explained in [14]), and their degrees are quadratic polynomials whose slopes are 4 and -4 , those of the Newton polygon of the A -polynomial.

5.3. The rotated and the meromorphic 3D-index. We next express the rotated 3D-index in terms of the colored holomorphic blocks.

Proposition 9. For all integers n and n' we have:

$$I^{\text{rot}}(n, n')(q) = -\frac{1}{2}h_{n'}^{(1)}(q^{-1})h_n^{(0)}(q) + \frac{1}{2}h_{n'}^{(0)}(q^{-1})h_n^{(1)}(q). \tag{79}$$

Proof. Equation (38), and the definition of the colored holomorphic blocks imply that

$$\begin{aligned}
I^{\text{rot}}(n, n')(q) &= \lim_{u \rightarrow 0} (B^{(1)}(q^{-n'} e^{-u}; q^{-1}) B^{(1)}(q^n e^u; q) + B^{(2)}(q^{-n'} e^{-u}; q^{-1}) B^{(2)}(q^n e^u; q)) \\
&= \lim_{u \rightarrow 0} \left(-\frac{1}{2} h_{n'}^{(1)}(q^{-1}) h_n^{(0)}(q) + \frac{1}{2} h_{n'}^{(0)}(q^{-1}) h_n^{(1)}(q) + O(u^1) \right) \\
&= -\frac{1}{2} h_{n'}^{(1)}(q^{-1}) h_n^{(0)}(q) + \frac{1}{2} h_{n'}^{(0)}(q^{-1}) h_n^{(1)}(q).
\end{aligned} \tag{80}$$

This and Equation (34) imply (1). Note that the sum over the integers in (1) is a convergent q -series since $\deg_q h_n^{(1)}(q) h_n^{(0)}(q) = |n|$ (as follows from (78)). \square

Using Equations (66) and (67) and the above proposition, it follows that the first few values of $I^{\text{rot}}(n, n)(q) = h_n^{(0)}(q) h_n^{(1)}(q)$ are given by

$$\begin{aligned}
I^{\text{rot}}(0, 0)(q) &= 1 - 8q - 9q^2 + 18q^3 + 46q^4 + 90q^5 + 62q^6 + 10q^7 - 170q^8 - 424q^9 + \dots, \\
I^{\text{rot}}(1, 1)(q) &= 2q + 2q^2 + 7q^3 + 8q^4 + 3q^5 - 22q^6 - 67q^7 - 132q^8 - 206q^9 + \dots, \\
I^{\text{rot}}(2, 2)(q) &= 2q^2 + 2q^4 + 4q^6 + 2q^7 + 8q^8 + 8q^9 + \dots,
\end{aligned} \tag{81}$$

and their degree is given by $\deg_q I^{\text{rot}}(n, n)(q) = |n|$. This, together with Equation (34), gives the first few terms of the meromorphic 3D-index as stated in Equation (1). Incidentally, the diagonal $I^{\text{rot}}(n, n)(q)$ of the rotated 3D-index satisfies the following third order recursion

$$\begin{aligned}
0 &= -q^{4+4n}(-1+q^{2+n})(1+q^{2+n})(-1+q^{5+2n})(1-q^{2+n}-q^{3+2n}-q^{5+2n}-q^{6+3n}+q^{8+4n})I^{\text{rot}}(n, n)(q) \\
&\quad + q(-1+q^{1+n})(1+q^{1+n})(-1+q^{5+2n})(1-q^{1+n}-q^{1+2n}-q^{3+2n}-q^{3+3n}+q^{4+4n}) \\
&\quad \quad \times (1-q^{1+n}-q^{2+n}-q^{1+2n}-q^{2+2n}+q^{3+2n}-q^{4+2n}-q^{5+2n}+q^{3+3n}+q^{6+3n}+q^{3+4n}+q^{5+4n} \\
&\quad \quad \quad + 3q^{6+4n}+q^{7+4n}+q^{9+4n}+q^{6+5n}+q^{9+5n}-q^{7+6n}-q^{8+6n}+q^{9+6n} \\
&\quad \quad \quad - q^{10+6n}-q^{11+6n}-q^{10+7n}-q^{11+7n}+q^{12+8n})I^{\text{rot}}(n+1, n+1)(q) \\
&\quad - (-1+q^{2+n})(1+q^{2+n})(-1+q^{1+2n})(1-q^{2+n}-q^{3+2n}-q^{5+2n}-q^{6+3n}+q^{8+4n}) \\
&\quad \quad \times (1-q^{1+n}-q^{2+n}-q^{1+2n}-q^{2+2n}+q^{3+2n}-q^{4+2n}-q^{5+2n}+q^{3+3n}+q^{6+3n}+q^{3+4n}+q^{5+4n} \\
&\quad \quad \quad + 3q^{6+4n}+q^{7+4n}+q^{9+4n}+q^{6+5n}+q^{9+5n}-q^{7+6n}-q^{8+6n}+q^{9+6n}-q^{10+6n}-q^{11+6n}-q^{10+7n} \\
&\quad \quad \quad - q^{11+7n}+q^{12+8n})I^{\text{rot}}(n+2, n+2)(q) \\
&\quad + q^{9+4n}(-1+q^{1+n})(1+q^{1+n})(-1+q^{1+2n})(1-q^{1+n}-q^{1+2n}-q^{3+2n}-q^{3+3n}+q^{4+4n})I^{\text{rot}}(n+3, n+3)(q),
\end{aligned} \tag{82}$$

for all integers n .

5.4. Stability. In this section, we discuss briefly the stability properties of the colored holomorphic blocks of the 4_1 knot. By stability, we mean the limit as $n \rightarrow \infty$ of a sequence of q -series, as explained in detail in [22]. Equations (66) and (67) imply easily that

$$\begin{aligned}
\lim_{n \rightarrow \infty} h_n^{(0)}(q) q^{-n(2n+1)/2} &= \frac{1}{(q; q)_\infty}, \\
\lim_{n \rightarrow \infty} h_n^{(1)}(q) q^{n(2n-1)/2} &= 2(q; q)_\infty.
\end{aligned} \tag{83}$$

It follows that

$$\begin{aligned} \lim_{n' \rightarrow \infty} q^{n'(2n'+1)/2} I^{\text{rot}}(n, n')(q) &= \lim_{n' \rightarrow \infty} \frac{q^{n'(2n'+1)/2}}{2} h_{n'}^{(1)}(q) h_n^{(0)}(q) + \frac{q^{n'(2n'+1)/2}}{2} h_{n'}^{(0)}(q) h_n^{(1)}(q) \\ &= (q; q)_\infty h_n^{(0)}(q), \end{aligned} \tag{84}$$

which implies that

$$\frac{I^{\text{rot}}(n, n)(q)}{(q; q)_\infty h_n^{(0)}(q)} = \frac{h_n^{(1)}(q)}{(q; q)_\infty}. \tag{85}$$

In other words, the degree (78) of the colored holomorphic blocks $h_n^{(\alpha)}(q)$ for $\alpha = 0, 1$ and the rotated 3D-index determines them up to multiplication by a power of $(q; q)_\infty$.

5.5. Numerical asymptotics of the meromorphic 3D-index. In this section, we discuss the numerical computation of the asymptotics of the meromorphic 3D-index at $q \rightarrow 1$ following closely the ideas and the numerical methods of [24, 25]. We first fix a ray and use τ in that ray with absolute value $|\tau| = 1/N$ for $N = 200, \dots, 300$. For these values of τ , we compute numerically the values of the colored holomorphic blocks $h_n^{(\alpha)}(q)$ for $n = 0, 1$ and $\alpha = 0, 1$ with precision about 20000 digits using their q -hypergeometric definition (66) and (67). We then use the q -difference equation (63) to compute the values of the colored holomorphic blocks $h_n^{(\alpha)}(q)$ for $n = 0, \dots, 25N$ and $\alpha = 0, 1$, and sum up these terms using (1), keeping in mind that the error is $O(q^N)$. Having done so, we obtain the values of the meromorphic 3D-index at the above range of τ to about 30 digits. We then numerically extrapolate the coefficients of each power of τ in the asymptotic series, and finally recognise them exactly. After doing so, we find numerically the following asymptotic expansion given in Equation (2), as $\arg(\tau)$ is fixed, positive and small, and τ tends to zero.

5.6. Asymptotic series. In this section, we will explain how to compute the asymptotic series $\Phi^{(\alpha)}(\hbar)$ using the discrete WKB applied to the \hat{A} -equation of the figure-eight knot.

Proposition 10. If the ansatz (46) satisfies

$$P_0(e^{nh}, e^h) \Phi_n^{(\sigma)}(\hbar) + P_1(e^{nh}, e^h) \Phi_{n+1}^{(\sigma)}(\hbar) + P_2(e^{nh}, e^h) \Phi_{n+2}^{(\sigma)}(\hbar) = 0, \tag{86}$$

it follows that $a_{2,-1}$ is a root of the polynomial equation $4a_{2,-1}^2 + 3 = 0$. A choice of such a root and the initial condition at $n = 0$ uniquely determines the power series $\Phi_n^{(\sigma)}(h)$.

The first few terms of the asymptotic series are given by

$$\begin{aligned}
\widehat{\Phi}_n^{(\sigma_1)}(\hbar) &= (-3)^{-1/4} e^{\frac{i\text{Vol}}{\hbar}} \exp\left(\sqrt{-3}\hbar\frac{n^2}{2}\right) \left(1 - \frac{11}{24\sqrt{-3}^3}\hbar \right. \\
&+ \left(-\frac{9}{2\sqrt{-3}^6}n^2 + \frac{697}{1152\sqrt{-3}^6}\right)\hbar^2 \\
&+ \left(-\frac{81}{4\sqrt{-3}^9}n^4 + \frac{285}{16\sqrt{-3}^9}n^2 - \frac{724351}{414720\sqrt{-3}^9}\right)\hbar^3 \\
&+ \left(\frac{4185}{32\sqrt{-3}^{12}}n^4 - \frac{19825}{256\sqrt{-3}^{12}}n^2 + \frac{278392949}{39813120\sqrt{-3}^{12}}\right)\hbar^4 \\
&+ \left(\frac{12879}{20\sqrt{-3}^{15}}n^6 - \frac{129933}{64\sqrt{-3}^{15}}n^4 + \frac{5481733}{5760\sqrt{-3}^{15}}n^2 - \frac{17049329117}{209018880\sqrt{-3}^{15}}\right)\hbar^5 \\
&+ \left(\frac{6561}{8\sqrt{-3}^{18}}n^8 - \frac{646461}{80\sqrt{-3}^{18}}n^6 + \frac{18984649}{1280\sqrt{-3}^{18}}n^4 - \frac{1718762189}{276480\sqrt{-3}^{18}}n^2 + \frac{39021801798779}{75246796800\sqrt{-3}^{18}}\right)\hbar^6 + O(\hbar^7).
\end{aligned} \tag{87}$$

We have computed 300 terms of the series $\widehat{\Phi}_0^{(\sigma_1)}(\hbar)$, obtained either by extrapolation, and matching the stationary phase of the corresponding state-integral. Using 10 terms of the above series, one can compute the series $\widehat{\Phi}_n^{(\sigma_1)}(\hbar)$ up to $O(\hbar^{11})$.

Proof. To determine $a_{2,-1}$ (which satisfies the same polynomial equation as the cusp shape) note that

$$\begin{aligned}
P_0(1+u, 1) &= u^4 + 4u^3 + 5u^2 + 2u, \\
P_1(1+u, 1) &= u^6 + 5u^5 + 7u^4 - 2u^3 - 10u^2 - 4u, \\
P_2(1+u, 1) &= u^4 + 4u^3 + 5u^2 + 2u,
\end{aligned} \tag{88}$$

and so

$$P_0(1+u, 1) + P_1(1+u, 1)(1+2a_{2,-1}u) + P_2(1+u, 1)(1+2a_{2,-1}u)^2 = (8a_{2,-1}^2 + 6)u^3 + O(u^4). \tag{89}$$

Therefore, $a_{2,-1} = \frac{1}{2}\sqrt{-3}$. Choosing a root, substituting into (46), and then expanding (86) to $O(\hbar^6)$ we find

$$\begin{aligned}
&\left(\left(16a_{4,-1} + 8\sqrt{-3}a_{2,0} - \frac{8}{3}\sqrt{-3}\right)n^3 + \dots\right)\hbar^4 \\
&+ \left(\left(24\sqrt{-3}a_{4,-1} - 12a_{2,0} + 8\right)n^5 + \dots\right)\hbar^5 + O(\hbar^6) = 0,
\end{aligned} \tag{90}$$

which give two independent equations for $a_{4,-1}$ and $a_{2,0}$ which have solution

$$a_{4,-1} = \frac{\sqrt{-3}}{12}, \quad a_{2,0} = \frac{1}{6}. \tag{91}$$

This shows that

$$\Phi_0(\hbar)^{-1}\Phi_n(\hbar) + O(\hbar^3) = 1 + \frac{\sqrt{-3}}{2}n^2\hbar + \left(-\frac{3}{8}n^4 + \frac{1}{6}n^2\right)\hbar^2 + O(\hbar^3). \tag{92}$$

We can continue inductively in N computing $a_{k,\ell}$ where $k + 2\ell = 2N$. The coefficients of $n^3\hbar^{3+N}, \dots, n^{3+2N}\hbar$ form a linear system in $a_{k,\ell}$ with $k + 2\ell = N$ which depends on $a_{k,\ell}$ with

$k + 2\ell < N$. Using the initial condition

$$\begin{aligned} \widehat{\Phi}_0^{(\sigma_1)}(\hbar) = & (-3)^{-1/4} e^{\frac{i\text{Vol}}{\hbar}} \left(1 - \frac{11}{24\sqrt{-3}}\hbar + \frac{697}{1152\sqrt{-3}^6}\hbar^2 - \frac{724351}{414720\sqrt{-3}^9}\hbar^3 \right. \\ & \left. + \frac{278392949}{39813120\sqrt{-3}^{12}}\hbar^4 - \frac{17049329117}{209018880\sqrt{-3}^{15}}\hbar^5 + \frac{39021801798779}{75246796800\sqrt{-3}^{18}}\hbar^6 + O(\hbar^7) \right), \end{aligned} \quad (93)$$

taken from [24], together with the above discussion gives (87). \square

Finally, we define the second asymptotic series $\widehat{\Phi}_n^{(\sigma_2)}(\hbar)$ by using the other choice of $a_{2,-1}$ in the WKB. It is easy to see that the two asymptotic series are related by

$$\widehat{\Phi}_n^{(\sigma_2)}(\hbar) = i\widehat{\Phi}_n^{(\sigma_1)}(-\hbar), \quad (94)$$

from which it follows that they trivially satisfy the quadratic relation

$$\widehat{\Phi}_n^{(\sigma_1)}(\hbar)\widehat{\Phi}_n^{(\sigma_1)}(-\hbar) + \widehat{\Phi}_n^{(\sigma_2)}(\hbar)\widehat{\Phi}_n^{(\sigma_2)}(-\hbar) = 0. \quad (95)$$

5.7. Asymptotics of the colored holomorphic blocks. Having defined the asymptotic series $\widehat{\Phi}_n^{(\sigma_1)}(\hbar)$ and $\widehat{\Phi}_n^{(\sigma_2)}(\hbar)$, we now use them to give the asymptotics of the colored holomorphic blocks in suitable rays, with exponentially small terms included. To Borel resum the above two asymptotic series, we use about 300 coefficients of \hbar when $n = 0$, and about 170 coefficients (each a polynomial of n) for general n . Using for example $\tau = \mathbf{e}(0.001)/50$, one can check the following identities (96) to 10 to 20 coefficients of \tilde{q} within the accuracy of the numerical Borel resummation.

For τ in a cone $C = \arg(\theta) \in (\theta_0, \theta_1)$ with $0 < \theta_0 < \theta_1$ sufficiently small, we find numerically that

$$h_n^{(0)}(q) = \zeta_8 \tau^{1/2} \sum_{k=1}^2 s\widehat{\Phi}_n^{(\sigma_k)}(2\pi i\tau) H_{\sigma_k}^{(0)}(\tilde{q}), \quad h_n^{(1)}(q) = \zeta_8 \tau^{-1/2} \sum_{k=1}^2 s\widehat{\Phi}_n^{(\sigma_k)}(2\pi i\tau) H_{\sigma_k}^{(1)}(\tilde{q}), \quad (96)$$

where

$$H_{\sigma_1}^{(\alpha)}(\tilde{q}) = h_0^{(\alpha)}(\tilde{q}), \quad H_{\sigma_2}^{(\alpha)}(\tilde{q}) = \frac{2\tilde{q} - 1}{1 - \tilde{q}} h_0^{(\alpha)}(\tilde{q}) - \frac{\tilde{q}^{1/2}}{1 - \tilde{q}} h_1^{(\alpha)}(\tilde{q}), \quad (97)$$

for $\alpha = 0, 1$. The above identities were found in [16, Sec.5] when $n = 0$ and were verified for general n here. We stress that Equation (96) is an identity of analytic functions on the cone C . Writing the above identity in matrix form, we obtain that

$$\begin{pmatrix} h_n^{(0)}(q) & h_n^{(1)}(q) \\ h_{n+1}^{(0)}(q) & h_{n+1}^{(1)}(q) \end{pmatrix} = \begin{pmatrix} s\widehat{\Phi}_n^{(\sigma_1)}(2\pi i\tau) & s\widehat{\Phi}_n^{(\sigma_2)}(2\pi i\tau) \\ s\widehat{\Phi}_{n+1}^{(\sigma_1)}(2\pi i\tau) & s\widehat{\Phi}_{n+1}^{(\sigma_2)}(2\pi i\tau) \end{pmatrix} \begin{pmatrix} H_{\sigma_1}^{(0)}(\tilde{q}) & H_{\sigma_1}^{(1)}(\tilde{q}) \\ H_{\sigma_2}^{(0)}(\tilde{q}) & H_{\sigma_2}^{(1)}(\tilde{q}) \end{pmatrix} \begin{pmatrix} \tau^{1/2} & 0 \\ 0 & \tau^{-1/2} \end{pmatrix}, \quad (98)$$

where

$$\begin{aligned} \begin{pmatrix} H_{\sigma_1}^{(0)}(\tilde{q}) & H_{\sigma_1}^{(1)}(\tilde{q}) \\ H_{\sigma_2}^{(0)}(\tilde{q}) & H_{\sigma_2}^{(1)}(\tilde{q}) \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ \frac{2-\tilde{q}}{1-\tilde{q}} & -\frac{\tilde{q}^{1/2}}{1-\tilde{q}} \end{pmatrix} \begin{pmatrix} h_0^{(0)}(\tilde{q}) & h_0^{(1)}(\tilde{q}) \\ h_1^{(0)}(\tilde{q}) & h_1^{(1)}(\tilde{q}) \end{pmatrix} \\ &= \begin{pmatrix} 1 - \tilde{q} - 2\tilde{q}^2 - 2\tilde{q}^3 - 2\tilde{q}^4 + \dots & 1 - 7\tilde{q} - 14\tilde{q}^2 - 8\tilde{q}^3 - 2\tilde{q}^4 + \dots \\ -1 + 2\tilde{q} + 3\tilde{q}^2 + 2\tilde{q}^3 + \tilde{q}^4 + \dots & 1 + 10\tilde{q} + 15\tilde{q}^2 + 2\tilde{q}^3 + 19\tilde{q}^4 + \dots \end{pmatrix}. \end{aligned} \quad (99)$$

Although we will not use this in our current paper, we point out that the entries of

$$\begin{pmatrix} h_{n'}^{(0)}(\tilde{q}) & h_{n'}^{(1)}(\tilde{q}) \\ h_{n'+1}^{(0)}(\tilde{q}) & h_{n'+1}^{(1)}(\tilde{q}) \end{pmatrix}^{-1} \begin{pmatrix} \tau^{-1/2} & 0 \\ 0 & \tau^{1/2} \end{pmatrix} \begin{pmatrix} h_n^{(0)}(q) & h_n^{(1)}(q) \\ h_{n+1}^{(0)}(q) & h_{n+1}^{(1)}(q) \end{pmatrix} \quad (100)$$

can be expressed as combinations of the elementary functions holomorphic for $\tau \in \mathbb{C}'$ times state integrals (see [20, 16] with the notation as in those papers)

$$\int \Phi_{\mathbf{b}}(x + i\mathbf{b}^{-1}(m\mathbf{b} + m'\mathbf{b}^{-1}))\Phi_{\mathbf{b}}(x - i\mathbf{b}^{-1}(m\mathbf{b} + m'\mathbf{b}^{-1})) \exp(-\pi i x^2) dx, \quad (101)$$

and therefore extend to analytic functions for $\tau \in \mathbb{C}'$.

Now for $\tau \in C$, the above discussion and Equation (79) imply that the rotated 3D-index is given by

$$\begin{aligned} I^{\text{rot}}(n, n')(q) &= -\frac{1}{2}h_{n'}^{(1)}(q^{-1})h_n^{(0)}(q) + \frac{1}{2}h_{n'}^{(0)}(q^{-1})h_n^{(1)}(q) \\ &= \frac{1}{2}h_{n'}^{(1)}(q)h_n^{(0)}(q) + \frac{1}{2}h_{n'}^{(0)}(q)h_n^{(1)}(q) \\ &= \frac{i}{2}(s\widehat{\Phi}_{n'}^{(\sigma_1)}(2\pi i\tau)H_{\sigma_1}^{(1)}(\tilde{q}) + s\widehat{\Phi}_{n'}^{(\sigma_2)}(2\pi i\tau)H_{\sigma_2}^{(1)}(\tilde{q}))(s\widehat{\Phi}_n^{(\sigma_1)}(2\pi i\tau)H_{\sigma_1}^{(0)}(\tilde{q}) + s\widehat{\Phi}_n^{(\sigma_2)}(2\pi i\tau)H_{\sigma_2}^{(0)}(\tilde{q})) \\ &\quad + \frac{i}{2}(s\widehat{\Phi}_{n'}^{(\sigma_1)}(2\pi i\tau)H_{\sigma_1}^{(0)}(\tilde{q}) + s\widehat{\Phi}_{n'}^{(\sigma_2)}(2\pi i\tau)H_{\sigma_2}^{(0)}(\tilde{q}))(s\widehat{\Phi}_n^{(\sigma_1)}(2\pi i\tau)H_{\sigma_1}^{(1)}(\tilde{q}) + \tau^{1/2}s\widehat{\Phi}_n^{(\sigma_2)}(2\pi i\tau)H_{\sigma_2}^{(1)}(\tilde{q})). \end{aligned} \quad (102)$$

Now, let $\tau \rightarrow 0$ with $\arg(\tau)$ fixed, positive and sufficiently small. Then, \tilde{q} is exponentially small compared to τ , and ignoring the exponential small corrections in the above identity, and using the constant terms of the series in (99) and the quadratic relations (95) we obtain that

$$I^{\text{rot}}(n, n)(q) \sim \widehat{\Phi}_n^{(\sigma_1)}(2\pi i\tau)\widehat{\Phi}_{n'}^{(\sigma_2)}(-2\pi i\tau). \quad (103)$$

5.8. Asymptotics of the meromorphic 3D-index. In the previous sections we gave “horizontal” asymptotic expansion of the colored holomorphic blocks and of the rotated 3D-index. We now discuss the horizontal asymptotics of the meromorphic 3D-index

$$I^{\text{mer}}(0, 0)(q) = \sum_{n \in \mathbb{Z}} I^{\text{rot}}(n, n)(q) \sim \sum_{n \in \mathbb{Z}} \widehat{\Phi}_n^{(\sigma_1)}(2\pi i\tau)\widehat{\Phi}_n^{(\sigma_2)}(-2\pi i\tau).$$

Equation (87) implies that $\widehat{\Phi}_n^{(\sigma_1)}(\hbar)\widehat{\Phi}_n^{(\sigma_2)}(-\hbar)$ up to $O(\hbar^7)$ is given by

$$\begin{aligned} &3^{-1/2}e^{\frac{2i\text{Vol}}{\hbar}} \exp\left(2\sqrt{-3}\hbar\right)^{n^2/2} \left(1 - \frac{11}{12\sqrt{-3}}\hbar\right. \\ &\quad + \left(\frac{-9}{\sqrt{-3}^6}n^2 + \frac{409}{288\sqrt{-3}^6}\right)\hbar^2 \\ &\quad + \left(\frac{-81}{2\sqrt{-3}^9}n^4 + \frac{159}{4\sqrt{-3}^9}n^2 - \frac{209839}{51840\sqrt{-3}^9}\right)\hbar^3 \\ &\quad + \left(\frac{2403}{8\sqrt{-3}^{12}}n^4 - \frac{5653}{32\sqrt{-3}^{12}}n^2 + \frac{39693941}{2488320\sqrt{-3}^{12}}\right)\hbar^4 \\ &\quad + \left(\frac{12879}{20\sqrt{-3}^{15}}n^6 - \frac{129933}{64\sqrt{-3}^{15}}n^4 + \frac{5481733}{5760\sqrt{-3}^{15}}n^2 - \frac{17049329117}{209018880\sqrt{-3}^{15}}\right)\hbar^5 \\ &\quad \left. + \left(\frac{6561}{8\sqrt{-3}^{18}}n^8 - \frac{646461}{80\sqrt{-3}^{18}}n^6 + \frac{18984649}{1280\sqrt{-3}^{18}}n^4 - \frac{1718762189}{276480\sqrt{-3}^{18}}n^2 + \frac{39021801798779}{75246796800\sqrt{-3}^{18}}\right)\hbar^6 + O(\hbar^7)\right). \end{aligned} \quad (104)$$

As explained in Section 3.3, the coefficient of \hbar^k in $A(\hbar) := \sum_{n \in \mathbb{Z}} \widehat{\Phi}_n^{(\sigma_1)}(2\pi i \hbar) \widehat{\Phi}_n^{(\sigma_2)}(-2\pi i \hbar)$ depends on the coefficients of the summand up to order \hbar^{3k} , and is computed by formal Gaussian integration. Then as $\langle 1 \rangle_{2\sqrt{-3}\hbar} = \zeta_8 \sqrt{\frac{2\pi i}{2\sqrt{-3}\hbar}}$ we have

$$A(\hbar) = e^{\frac{2i \text{Vol}}{\hbar}} \sqrt{\frac{2\pi i}{\hbar}} 3^{-\frac{3}{4}} 2^{-\frac{1}{2}} B(\hbar), \quad (105)$$

where $B(\hbar) = 1 + O(\hbar)$. Explicitly, the terms of (104) that contribute to the coefficient of \hbar in $B(\hbar)$ are

$$-\frac{11}{12\sqrt{-3}^3} \hbar - \frac{9}{\sqrt{-3}^6} n^2 \hbar^2 - \frac{81}{2\sqrt{-3}^9} n^4 \hbar^3. \quad (106)$$

It follows that

$$\begin{aligned} \text{coeff}(B(\hbar), \hbar) &= -\frac{11}{12\sqrt{-3}^3} \hbar - \frac{9}{\sqrt{-3}^6} \frac{\langle n^2 \rangle_{2\sqrt{-3}\hbar}}{\langle 1 \rangle_{2\sqrt{-3}\hbar}} \hbar^2 - \frac{81}{2\sqrt{-3}^9} \frac{\langle n^4 \rangle_{2\sqrt{-3}\hbar}}{\langle 1 \rangle_{2\sqrt{-3}\hbar}} \hbar^3 \\ &= -\frac{11}{12\sqrt{-3}^3} \hbar - \frac{9}{\sqrt{-3}^6} (-1) \left(\frac{1}{2\sqrt{-3}\hbar} \right) \hbar^2 - \frac{81}{2\sqrt{-3}^9} 3 \left(\frac{1}{2\sqrt{-3}\hbar} \right)^2 \hbar^3 \\ &= \frac{-19}{24\sqrt{-3}^3} \hbar. \end{aligned} \quad (107)$$

Similarly, we obtain that

$$\begin{aligned} \text{coeff}(B(\hbar), \hbar^2) &= \frac{409}{288\sqrt{-3}^6} \hbar^2 + \frac{159}{4\sqrt{-3}^9} \frac{\langle n^2 \rangle}{\langle 1 \rangle} \hbar^3 + \frac{2403}{8\sqrt{-3}^{12}} \frac{\langle n^4 \rangle}{\langle 1 \rangle} \hbar^4 + \frac{12879}{20\sqrt{-3}^{15}} \frac{\langle n^6 \rangle}{\langle 1 \rangle} \hbar^5 + \frac{6561}{8\sqrt{-3}^{18}} \frac{\langle n^8 \rangle}{\langle 1 \rangle} \hbar^6 \\ &= \frac{409}{288\sqrt{-3}^6} \hbar^2 + \frac{159}{4\sqrt{-3}^9} (-1) \left(\frac{1}{2\sqrt{-3}\hbar} \right) \hbar^3 + \frac{2403}{8\sqrt{-3}^{12}} 3 \left(\frac{1}{2\sqrt{-3}\hbar} \right)^2 \hbar^4 \\ &\quad + \frac{12879}{20\sqrt{-3}^{15}} (-15) \left(\frac{1}{2\sqrt{-3}\hbar} \right)^3 \hbar^5 + \frac{6561}{8\sqrt{-3}^{18}} 105 \left(\frac{1}{2\sqrt{-3}\hbar} \right)^4 \hbar^6 \\ &= \frac{1333}{1152\sqrt{-3}^6} \hbar^2. \end{aligned} \quad (108)$$

Equations (107) and (108) are in complete agreement with the numerical extrapolation (2). As we will see in Section 6, the same rational numbers with numerator 19 and 1333 will appear in the asymptotics of the Turaev–Viro invariant of the 4_1 knot.

5.9. Vertical asymptotics. In this section, we discuss the vertical asymptotics of the colored holomorphic blocks and of the rotated 3D-index. As was mentioned in the introduction, we have no theoretical or computational control in the \tilde{q} -corrections, nor to exponentially small terms, which we will therefore ignore.

Recall that the two colored holomorphic blocks of 4_1 coincide, when $n = 0$ with the pair of q -series of the 4_1 studied in [25] (see Equation (73)). In [25] it was found that their vertical asymptotics $\tau \downarrow 0$ were given by

$$h_0^{(0)}(q) \sim \zeta_8 \sqrt{\tau} \left(\widehat{\Phi}_0^{(\sigma_1)}(2\pi i \tau) - \widehat{\Phi}_0^{(\sigma_2)}(2\pi i \tau) \right), \quad h_0^{(1)}(q) \sim \frac{\zeta_8}{\sqrt{\tau}} \left(\widehat{\Phi}_0^{(\sigma_1)}(2\pi i \tau) + \widehat{\Phi}_0^{(\sigma_2)}(2\pi i \tau) \right). \quad (109)$$

The factor ζ_8 in the above formula is due to a different normalisation of the constant term of the asymptotic series in [25], a famous eighth root of unity ambiguity arising from the half integral weight. Note also that the asymptotic series in the above equation involve an

oscillatory term, namely $e^{\pm \text{Vol}/(2\pi\tau)}$, times a power series in τ . We can complement the above with the asymptotics $\tau \downarrow 0$

$$h_0^{(0)}(q^{-1}) \sim \zeta_8 \sqrt{-\tau} \left(\widehat{\Phi}_0^{(\sigma_1)}(-2\pi i\tau) + \widehat{\Phi}_0^{(\sigma_2)}(-2\pi i\tau) \right), \quad h_0^{(1)}(q^{-1}) \sim \frac{\zeta_8}{\sqrt{-\tau}} \left(-\widehat{\Phi}_0^{(\sigma_1)}(-2\pi i\tau) + \widehat{\Phi}_0^{(\sigma_2)}(-2\pi i\tau) \right). \quad (110)$$

It comes with no surprise that for all integers n , we have

$$\begin{aligned} h_n^{(0)}(q) &\sim \zeta_8 \sqrt{\tau} \left(\widehat{\Phi}_n^{(\sigma_1)}(2\pi i\tau) - \widehat{\Phi}_n^{(\sigma_2)}(2\pi i\tau) \right), & h_n^{(0)}(q^{-1}) &\sim \zeta_8 \sqrt{-\tau} \left(\widehat{\Phi}_n^{(\sigma_1)}(-2\pi i\tau) + \widehat{\Phi}_n^{(\sigma_2)}(-2\pi i\tau) \right), \\ h_n^{(1)}(q) &\sim \frac{\zeta_8}{\sqrt{\tau}} \left(\widehat{\Phi}_n^{(\sigma_1)}(2\pi i\tau) + \widehat{\Phi}_n^{(\sigma_2)}(2\pi i\tau) \right), & h_n^{(1)}(q^{-1}) &\sim \frac{\zeta_8}{\sqrt{-\tau}} \left(-\widehat{\Phi}_n^{(\sigma_1)}(-2\pi i\tau) + \widehat{\Phi}_n^{(\sigma_2)}(-2\pi i\tau) \right) \end{aligned} \quad (111)$$

as $\tau \downarrow 0$. Note that these vertical asymptotics are bounded and oscillatory. This, together with Equations (79) and (94), implies that for all integers n and n' we have

$$\begin{aligned} I^{\text{rot}}(n, n')(q) &\sim -i \left(\widehat{\Phi}_n^{(\sigma_1)}(2\pi i\tau) \widehat{\Phi}_{n'}^{(\sigma_1)}(2\pi i\tau) - \widehat{\Phi}_n^{(\sigma_2)}(2\pi i\tau) \widehat{\Phi}_{n'}^{(\sigma_2)}(2\pi i\tau) \right) \\ &= \widehat{\Phi}_n^{(\sigma_1)}(2\pi i\tau) \widehat{\Phi}_{n'}^{(\sigma_2)}(-2\pi i\tau) + \widehat{\Phi}_n^{(\sigma_2)}(2\pi i\tau) \widehat{\Phi}_{n'}^{(\sigma_1)}(-2\pi i\tau), \end{aligned} \quad (112)$$

confirming Conjecture 2 for the 4_1 knot. We now discuss the vertical asymptotics of the meromorphic 3D-index. There are two key problems: the first is to make sense of the asymptotic series $\sum_{n \in \mathbb{Z}} \widehat{\Phi}_n^{(\sigma_1)}(\hbar) \widehat{\Phi}_n^{(\sigma_1)}(-\hbar)$ which has no exponential term and it is simply a power series in $n\hbar$ and \hbar . The second is that the sum of the above series for (σ_1, σ_1) and for (σ_2, σ_2) simply vanishes, if we interchange the summation over n with the terms of the above series. On the other hand, the numerical vertical asymptotics of the meromorphic 3D-index suggest a non-vanishing contribution. We now explain a solution to this apparent contradiction. The key idea is to use a version of the asymptotic series whose coefficients of the powers of \hbar are not polynomials in $n\hbar$ but rather branches of algebraic functions on an affine curve (the A -polynomial curve). These algebraic functions were discussed in detail and computed in the paper [16] whose notation we follow here.

Recall the asymptotic series $\Phi_{4_1}^{(\sigma)}(x, \hbar)$ from [16, Sec.5.1]: it comes from a power series in \hbar of the form $\sqrt{\delta_{4_1}(x, y)} \Phi_{4_1}(x, y, \hbar) \in \mathbb{Q}(X_{4_1})[[\hbar]]$ with first few terms given by [16, Eqn.(123)]

$$\begin{aligned} \Phi_{4_1}(x, y, \hbar) &= \frac{1}{\sqrt{\delta_{4_1}}} \left(1 - \frac{1}{24 \delta_{4_1}^3} (x^{-3} - x^{-2} - 2x^{-1} + 15 - 2x - x^2 + x^3) \hbar \right. \\ &\quad + \frac{1}{1152 \delta_{4_1}^6} (x^{-6} - 2x^{-5} - 3x^{-4} + 610x^{-3} - 606x^{-2} - 1210x^{-1} \\ &\quad \left. + 3117 - 1210x - 606x^2 + 610x^3 - 3x^4 - 2x^5 + x^6) \hbar^2 + \dots \right). \end{aligned} \quad (113)$$

Here X_{4_1} is the affine curve given by the equation [16, Eqn.(116)]

$$-x^2 y = (1 - y)(1 - xy), \quad (114)$$

$\mathbb{Q}(X_{4_1}) = \mathbb{Q}(x)[y]/(x^2 y + (1 - y)(1 - xy))$ is the field of rational functions of X_{4_1} , and [16, Eqn.(118)]

$$\delta_{4_1}(x, y) = (xy^2 - 1)/(xy). \quad (115)$$

Let $y_j(x)$ for $j = 1, 2$ denote the roots of Equation (114); they are algebraic functions that collide when x is a root of the discriminant $(1 - 3x + x^2)(1 + x + x^2)$ of (114) with respect to

y . The discriminant has two real roots at $a_0 = (3 - \sqrt{5})/2$ and $a_1 = (3 + \sqrt{5})/2$ satisfying $a_0 a_1 = 1$. The two branches are real when $x > a_1$, and they are chosen so that

$$y_1(a_1 + 1) = 0.16866\dots, \quad y_2(a_1 + 1) = -0.16866\dots \quad (116)$$

Given that (114) is a quadratic equation in y , an easy calculation shows that

$$\delta_{4_1}(x, y_j(x)) = (-1)^{j-1} \sqrt{x^{-2} - 2x^{-1} - 1 - 2x + x^2} \quad (117)$$

for $j = 1, 2$. We now define $\Phi^{(\sigma_j)}(x, \hbar) = \Phi(x, y_j(x), \hbar)$ for $j = 1, 2$. In the vertical asymptotics of the meromorphic 3D-index, we use the bilinear expression

$$\widehat{\Phi}_{4_1}(x, y, \hbar) \widehat{\Phi}_{4_1}(x, y, -\hbar) = \frac{1}{\delta_{4_1}} \left(1 + \frac{1}{\delta_{4_1}^6} (x^{-3} - x^{-2} - 2x^{-1} + 5 - 2x - x^2 + x^3) \hbar^2 + \dots \right), \quad (118)$$

evaluated at $y = y_j(x)$ for $j = 1, 2$. The branches $y_j(x)$ satisfy the symmetry $y_j(x) = y_j(1/x)$ for $j = 1, 2$ and the linear relation

$$1/\delta_{4_1}(x, y_1(x)) + 1/\delta_{4_1}(x, y_2(x)) = 0. \quad (119)$$

The above was called a quadratic relation of asymptotic series in [24].

The functions $y_j(x)$ are analytic functions on $\mathbb{R} \setminus \{a_0, a_1\}$ and satisfy the bounds $y_j(x) = O(1/x)$ for $|x| \gg 0$ and $y_j(x) = (x - a_0)^{-1/2}$ for x near a_0 and $y_j(x) = (x - a_1)^{-1/2}$ for x near a_1 . It follows that the period integrals $\int_{C_k} f_j(u) du$ of $f_j(u) = 1/\delta_j(e^u)$ for $j = 1, 2$ and for $C_1 = (-\infty, \log(a_0))$, $C_2 = (\log(a_0), \log(a_1))$ and $C_3 = (\log(a_1), \infty)$ are well-defined (i.e., absolutely-convergent) and in fact they are integer linear combinations of

$$\begin{aligned} \int_{C_3} f_1(e^u) du &= 0.70030152116630101159\dots, \\ \int_{C_2} f_1(e^u) du &= -i1.5962422221317835101\dots \end{aligned} \quad (120)$$

6. THE TURAEV–VIRO INVARIANT OF A KNOT

6.1. Definition. The Turaev–Viro invariant of a closed 3-manifold was defined in [40] using a triangulation of the manifold where its extension to triangulated 3-manifolds with nonempty boundary (in particular to link complements), was also discussed. It was noticed by Walker, Turaev and Roberts [36] independently that the Turaev–Viro invariant of a closed 3-manifold M equals to the $SU(2)$ Reshetikhin–Turaev invariant of its double $D(M) = M\sharp - M$. This relation was extended to 3-manifolds with boundary (where the double $D(M) = M \cup_{\partial M} -M$) by Benedetti and Petronio [3]. A careful extension of the above relation from $SU(2)$ to $SO(3)$ was given by Chen–Yang in [5] and by Detcherry–Kalfagianni–Yang [9, Thm.3.1], and a Volume Conjecture for them was formulated in [5]. A comparison between the Turaev–Viro invariants of a knot complement with the Turaev–Viro invariant of the closed 3-manifold obtained by 0-surgery on the knot was given by Detcherry–Kalfagianni in [8, Cor.5.3]. The comparison takes the form of an inequality whose proof follows easily from the Cauchy–Schwarz inequality of an inner product.

In the present paper, we will focus on the Turaev–Viro invariant of a knot, which is defined by a state-sum formula in terms of colorings of edges of an ideal triangulation of the knot

complement. Theorem 3.1 of [9] expresses this state-sum in terms of the values of the colored Jones polynomial of the knot at roots of unity as follows:

$$\mathrm{TV}_{K,m+1/2} = \mathrm{ev}_{1/(m+1/2)} \left(\eta_{2m+1}^2 \sum_{k=1}^m [k]^2 |J_{K,k}(q)|^2 \right). \quad (121)$$

Here, $J_{K,n}(q) \in \mathbb{Z}[q^{\pm 1}]$ denotes the n -th colored Jones polynomial of a knot K in 3-space [31, 39], normalised by $J_{\mathrm{unknot},n}(q) = 1$, and such that $J_{K,2}(q)$ is the Jones polynomial [29] of K , and $\eta_{2m+1} = \frac{2}{\sqrt{2m+1}} \sin\left(\frac{2\pi}{2m+1}\right)$, $\mathbf{e}(x) = e^{2\pi i x}$, $[k] = (q^{k/2} - q^{-k/2})/(q^{1/2} - q^{-1/2})$ and $\mathrm{ev}_{a/c}(f(q)) = f(\mathbf{e}(a/c))$ for $a/c \in \mathbb{Q}$.

We stress that in the definition of the Turaev–Viro invariants of a knot, we choose as our variable $m + 1/2$ rather than $2m + 1$ for reasons that will become clear later.

It is obvious from (121) that the Turaev–Viro invariant of a knot at $m + 1/2$ is a finite sum of positive m terms. Experimentally it appears that the dominant term is near the end (i.e., $J_{K,m}(\mathbf{e}(1/(m + 1/2)))$) and grows exponentially with respect to m and has a full asymptotic expansion into power series in $1/(m + 1/2)$. Moreover, each term for $k = m - n$ with $n > 0$ fixed and n large also grows exponentially with respect to m , at the same rate as the one with $n = 0$, and with subleading corrections which are power series in $1/(m + 1/2)$ with coefficients polynomials in n . Moreover, a symmetry of the colored Jones polynomial allows one to double the sum over m terms to one over $2m$ terms, and thus consider n to be a fixed integer. One can replace the sum over n by a formal Gaussian integral and one can further experimentally observe the expansion of the asymptotics of the Turaev–Viro invariant of a knot to all orders in $m + 1/2$. The resulting series have interesting arithmetic and analytic properties analogous to those in [24] with one key difference: they are the double of the series of [24].

6.2. Numerical asymptotics. As an illustration of the above discussion, we will give the asymptotics of the Turaev–Viro invariant of the simplest hyperbolic (4_1) knot. Let $v = \mathrm{Vol}(4_1)/(2\pi) = 2 \mathrm{ImLi}_2(\mathbf{e}(1/6))/(2\pi) = 0.3230659472\dots$ denote the volume of the 4_1 knot [38], divided by 2π .

Observation 11. *We have:*

$$\mathrm{TV}_{4_1,m+1/2} \sim \frac{e^{2v(m+1/2)}}{2^{\frac{1}{2}} \cdot 3^{\frac{3}{4}}} (m + 1/2)^{\frac{1}{2}} B \left(\frac{\pi}{2^2 \cdot 3^2 \cdot \sqrt{3} \cdot (m + 1/2)} \right), \quad (122)$$

where $B(h) = \sum_{k=0}^{\infty} \frac{b_k}{k!} h^k$ with $b_k \in \mathbb{Q}$ and the first 10 coefficients are given by

k	0	1	2	3	4	5	6
b_k	1	19	1333	$\frac{1601717}{5}$	$\frac{704696117}{5}$	$\frac{683123156521}{7}$	$\frac{3461441912579591}{5 \cdot 7}$

and

k	7	8	9
b_k	$\frac{691282346978984873}{5}$	$\frac{1274507463563873288357}{5}$	$\frac{164770744067453335344413873}{5^2 \cdot 11}$

This is a numerical observation, obtained by numerical computation of the values of the colored Jones N -th colored Jones polynomials at roots of unity (using either the Le-Habiro

formula for the colored Jones polynomial, or its recursion, known to exist for all knots, and which implies that computing the N -th colored Jones polynomial at a root of unity can be done so in $O(N)$ -steps) for $N = 1000, \dots, 1100$, to high precision, then numerically extrapolating the answer and finally recognizing the corresponding elements of the trace field (which in our case is $\mathbb{Q}(\sqrt{-3})$ hence the numbers in question are rational). A detailed description of this method with numerous examples of asymptotics of quantum invariants is given in [24] and in [25].

Note that the growth rate of the Turaev–Viro invariant of the 4_1 knot is not the volume, but twice the volume. This doubling has to do with the fact that the Turaev–Viro invariant of a knot complement is the Reshetikhin–Turaev invariant of the doubled manifold, and matches the fact that the rotated 3D-index is a bilinear function of the colored holomorphic blocks (see Equation (43)), and hence its asymptotics is a bilinear (and not linear) combination of asymptotic series of colored holomorphic blocks.

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APPENDIX A. THE 3D-INDEX OF THE 5_2 KNOT

In this appendix, we discuss in detail the 3D-index of the 5_2 knot and its holomorphic blocks and colored holomorphic blocks. All functions and number in this section depend on the knot 5_2 which we omit from the notation.

A.1. Holomorphic blocks. The 5_2 knot has three nontrivial boundary parabolic $\mathrm{PSL}_2(\mathbb{C})$ representations, all Galois conjugate to the geometric one, the latter having a cubic trace field of discriminant -23 . Hence, this knot has three holomorphic blocks $B^{(\alpha)}(x; q)$ for $\alpha = 1, 2, 3$ (see [2, Eqn.6.48] and [16, Sec.6.2])

$$\begin{aligned} B^{(1)}(x; q) &= \theta(x; q)G(x, x^{-1}, 1, q), \\ B^{(2)}(x; q) &= \frac{\theta(x; q)}{\theta(-q^{\frac{1}{2}}x; q)}G(x, x^2, x, q), \\ B^{(3)}(x; q) &= \frac{\theta(x^{-1}; q)}{\theta(-q^{\frac{1}{2}}x^{-1}; q)}G(x^{-1}, x^{-2}, x^{-1}, q), \end{aligned} \tag{123}$$

expressed in terms of the function

$$G(x, y, z, q) = \begin{cases} (qx; q)_\infty (qy; q)_\infty \sum_{k=0}^{\infty} \frac{z^k}{(q^{-1}; q^{-1})_k (qx; q)_k (qy; q)_k} & \text{for } |q| < 1 \\ \frac{1}{(x; q^{-1})_\infty (y; q^{-1})_\infty} \sum_{k=0}^{\infty} \frac{z^k}{(q^{-1}; q^{-1})_k (qx; q)_k (qy; q)_k} & \text{for } |q| > 1. \end{cases} \tag{124}$$

As in the case of the 4_1 knot, Wilf-Zeilberger theory [41] implemented by Koutschan [33] implies that the holomorphic blocks satisfy the same q -difference equation

$$P_0(x, q)B^{(\alpha)}(x, q) + P_1(x, q)B^{(\alpha)}(q^{-1}x, q) + P_2(x, q)B^{(\alpha)}(q^{-2}x, q) + P_3(x, q)B^{(\alpha)}(q^{-3}x, q) = 0, \quad (125)$$

for $\alpha = 1, 2, 3$, where

$$\begin{aligned} P_0(x, q) &= -q^{-2}x^2(1 - q^{-2}x)(1 + q^{-2}x)(1 - q^{-5}x^2), \\ P_1(x, q) &= q^{3/2}x^{-3}(1 - q^{-1}x)(1 + q^{-1}x)(1 - q^{-5}x^2) \\ &\quad \cdot (1 - q^{-1}x - q^{-1}x^2 - q^{-4}x^2 + q^{-2}x^2 + q^{-3}x^2 + q^{-2}x^3 + q^{-5}x^3 + q^{-5}x^4 + q^{-5}x^4 - q^{-6}x^5), \\ P_2(x, q) &= q^5x^{-5}(1 - q^{-2}x)(1 + q^{-2}x)(1 - q^{-1}x^2) \\ &\quad \cdot (1 - q^{-2}x - q^{-2}x - q^{-2}x^2 - q^{-5}x^2 + q^{-4}x^3 + q^{-7}x^3 - q^{-5}x^3 - q^{-6}x^3 + q^{-7}x^4 - q^{-9}x^5), \\ P_3(x, q) &= q^{\frac{11}{2}}x^{-5}(1 - q^{-1}x)(1 + q^{-1}x)(1 - q^{-1}x^2). \end{aligned} \quad (126)$$

Compare with [16, Eqn.(259)] (with $m = 0$), the slight difference in the coefficients being due to the fact that we insist our q -difference equation to be satisfied by the rotated 3D-index, as we already discussed for the case of the 4_1 knot.

A.2. Colored holomorphic blocks. The holomorphic blocks are meromorphic functions of (x, q) . Now we define the colored holomorphic blocks from the expansions of the expansions of the holomorphic blocks around $x = q^n$, or equivalently when $x = q^n e^u$, around $u = 0$. To begin with, we define $h_n^{(0)}(q)$ for $|q| < 1$ by

$$B^{(1)}(q^n e^u; q) = (-q^{\frac{1}{2}}; q)_\infty^2 (q; q)_\infty^2 h_n^{(0)}(q) + O(u^1). \quad (127)$$

One can then show that

$$\begin{aligned} B^{(2)}(q^n e^u; q) &= (-q^{\frac{1}{2}}; q)_\infty^2 h_n^{(0)}(q)u^{-1} + O(u^0), \\ B^{(3)}(q^n e^u; q) &= -(-q^{\frac{1}{2}}; q)_\infty^2 h_n^{(0)}(q)u^{-1} + O(u^0). \end{aligned} \quad (128)$$

Next, we define $h_n^{(1)}(q)$ for $|q| < 1$ by

$$B^{(2)}(q^n e^u; q) + B^{(3)}(q^n e^u; q) = -2(-q^{\frac{1}{2}}; q)_\infty^2 h_n^{(1)}(q) + O(u^1). \quad (129)$$

Then, one can show that

$$-B^{(1)}(q^n e^u; q) \frac{\wp'(e^u, q)}{2\wp(e^u, q)(q; q)_\infty^2} - B^{(2)}(q^n e^u; q) = (-q^{\frac{1}{2}}; q)_\infty^2 h_n^{(1)}(q) + O(u^1), \quad (130)$$

where \wp is the Weierstrass \wp -function. Last, we define $h_n^{(2)}(q)$ for $|q| < 1$ by

$$-B^{(1)}(q^n e^u; q) \frac{\wp'(e^u, q)}{\wp(e^u, q)(q; q)_\infty^2} - B^{(2)}(q^n e^u; q) + B^{(3)}(q^n e^u; q) = (-q^{\frac{1}{2}}; q)_\infty^2 h_n^{(2)}(q)u + O(u^2). \quad (131)$$

Next we define the colored holomorphic blocks when $|q| > 1$. We begin by defining $h_n^{(0)}(q^{-1})$ for $|q| < 1$ by

$$-B^{(1)}(q^{-n} e^{-u}; q^{-1}) = \frac{1}{(-q^{\frac{1}{2}}; q)_\infty^2 (q; q)_\infty^2} h_n^{(0)}(q)u^{-2} + O(u^{-1}). \quad (132)$$

One can show that

$$\begin{aligned} -B^{(2)}(q^{-n}e^{-u}; q^{-1}) &= \frac{1}{(-q^{\frac{1}{2}}; q)_{\infty}^2} h_n^{(0)}(q) u^{-1} + O(u^0), \\ -B^{(3)}(q^{-n}e^{-u}; q^{-1}) &= -\frac{1}{(-q^{\frac{1}{2}}; q)_{\infty}^2} h_n^{(0)}(q) u^{-1} + O(u^0). \end{aligned} \quad (133)$$

Then, we define $h_n^{(1)}(q^{-1})$ for $|q| < 1$ by

$$B^{(2)}(q^{-n}e^{-u}; q) + B^{(3)}(q^{-n}e^{-u}; q) = \frac{1}{(-q^{\frac{1}{2}}; q)_{\infty}^2} h_n^{(1)}(q) + O(u^1). \quad (134)$$

One can show

$$-2B^{(1)}(q^{-n}e^{-u}; q) \frac{\wp(e^u; q)(q; q)_{\infty}^2}{\wp'(e^u; q)} - 2B^{(2)}(q^{-n}e^{-u}; q) = \frac{1}{(-q^{\frac{1}{2}}; q)_{\infty}^2} h_n^{(1)}(q) + O(u^1). \quad (135)$$

Finally, we define $h_n^{(2)}(q^{-1})$ for $|q| < 1$ by

$$2B^{(1)}(q^{-n}e^{-u}; q) \frac{\wp(e^u; q)(q; q)_{\infty}^2}{\wp'(e^u; q)} + B^{(2)}(q^{-n}e^{-u}; q) - B^{(3)}(q^{-n}e^{-u}; q) = \frac{h_n^{(2)}(q)}{(-q^{\frac{1}{2}}; q)_{\infty}^2} u + O(u^2). \quad (136)$$

Since the holomorphic blocks $B^{(\alpha)}(x; q)$ satisfy the q -difference equation (125), it follows that the colored holomorphic blocks, given in (139), (140), and (141), satisfy the linear q -difference equation

$$P_0(q^n, q) h_n^{(\alpha)}(q) + P_1(q^n, q) h_{n-1}^{(\alpha)}(q) + P_2(q^n, q) h_{n-2}^{(\alpha)}(q) + P_3(q^n, q) h_{n-3}^{(\alpha)}(q) = 0, \quad (137)$$

for all $\alpha = 1, 2, 4$ and all integers n .

Our next lemma gives explicit q -hypergeometric sums for the colored holomorphic blocks of the 5_2 knot. Although the derivation of these sums is rigorous, albeit routine, the formulas are lengthy. To avoid clutter, we will introduce some notation for q -harmonic sums. The standard notation $H_n = \sum_{j=1}^n 1/j$ for harmonic numbers, and its extension $H_n^{(k)} = \sum_{j=1}^n 1/j^k$ leads to several recent q -generalisations whose notation is not standard, see e.g., Singer [37]. In our formulas for the colored holomorphic blocks of the 5_2 knot, we will use

$$H_n(q) = \sum_{j=1}^n \frac{q^j}{1-q^j}, \quad H_n^{(2)}(q) = \sum_{j=1}^n \frac{q^j}{(1-q^j)^2}. \quad (138)$$

Lemma 12. We have:

$$h_n^{(0)}(q) = (-1)^n q^{|n|/2} \sum_{k=0}^{\infty} \frac{q^{|n|k}}{(q^{-1}; q^{-1})_k (q; q)_{k+2|n|} (q; q)_{k+|n|}}, \quad (139)$$

$$\begin{aligned}
h_n^{(1)}(q) &= -(-1)^n q^{|n|/2} \sum_{k=0}^{\infty} \frac{q^{|n|k}}{(q; q)_{k+2|n|} (q^{-1}; q^{-1})_k (q; q)_{k+|n|}} \\
&\quad \times \left(k + |n| - \frac{1}{4} - 3E_1(q) + H_k(q) + H_{k+|n|}(q) + H_{k+2|n|}(q) \right) \\
&\quad + q^{-n^2/2} \sum_{k=0}^{|n|-1} \frac{(q^{-1}, q^{-1})_{|n|-1-k}}{(q^{-1}, q^{-1})_k (q; q)_{k+|n|}},
\end{aligned} \tag{140}$$

and

$$\begin{aligned}
h_n^{(2)}(q) &= (-1)^n q^{|n|/2} \sum_{k=0}^{\infty} \frac{q^{|n|k}}{(q^{-1}; q^{-1})_k (q; q)_{k+|n|} (q; q)_{k+2|n|}} \\
&\quad \times \left(E_2(q) + \frac{1}{8} - H_k^{(2)}(q) - H_{k+|n|}^{(2)}(q) - H_{k+2|n|}^{(2)}(q) \right. \\
&\quad \left. - \left(k + |n| - \frac{1}{4} - 3E_1(q) + H_k(q) + H_{k+|n|}(q) + H_{k+2|n|}(q) \right)^2 \right) \\
&\quad + 2q^{-n^2/2} \sum_{k=0}^{|n|-1} \frac{(q^{-1}, q^{-1})_{|n|-1-k}}{(q^{-1}, q^{-1})_k (q; q)_{k+|n|}} \\
&\quad \times \left(|n| - \frac{3}{4} - 3E_1(q) + H_k(q) + H_{k+|n|}(q) + H_{|n|-k-1}(q) \right) \\
&\quad - 2(-1)^n q^{-|n|/2} \sum_{k=0}^{|n|-1} q^{-|n|k} \frac{(q^{-1}; q^{-1})_{2|n|-k-1} (q^{-1}; q^{-1})_{|n|-k-1}}{(q^{-1}; q^{-1})_k},
\end{aligned} \tag{141}$$

for $|q| \neq 1$. The colored holomorphic blocks satisfy the symmetries

$$h_{-n}^{(\alpha)}(q) = h_n^{(\alpha)}(q), \quad \alpha = 0, 1, 2. \tag{142}$$

The first few values of the colored holomorphic blocks are given by

$$\begin{aligned}
h_0^{(0)}(q) &= 1 - q - 3q^2 - 5q^3 - 7q^4 - 6q^5 - 3q^6 + 8q^7 + 24q^8 + 52q^9 + \dots, \\
h_1^{(0)}(q) &= -q^{\frac{1}{2}} - 2q^{\frac{3}{2}} - 3q^{\frac{5}{2}} - 3q^{\frac{7}{2}} - q^{\frac{9}{2}} + 4q^{\frac{11}{2}} + 14q^{\frac{13}{2}} + 29q^{\frac{15}{2}} + 51q^{\frac{17}{2}} + \dots, \\
h_2^{(0)}(q) &= q + 2q^2 + 5q^3 + 8q^4 + 14q^5 + 19q^6 + 26q^7 + 29q^8 + 30q^9 + \dots,
\end{aligned} \tag{143}$$

$$\begin{aligned}
h_0^{(1)}(q) &= -\frac{1}{2} + \frac{9}{2}q + \frac{21}{2}q^2 + \frac{19}{2}q^3 + \frac{9}{2}q^4 - 27q^5 - \frac{133}{2}q^6 - 156q^7 - 252q^8 - 384q^9 + \dots, \\
h_1^{(1)}(q) &= q^{-\frac{1}{2}} + \frac{5}{2}q^{\frac{1}{2}} + 3q^{\frac{3}{2}} - \frac{1}{2}q^{\frac{5}{2}} - \frac{23}{2}q^{\frac{7}{2}} - \frac{73}{2}q^{\frac{9}{2}} - 74q^{\frac{11}{2}} - 133q^{\frac{13}{2}} - \frac{393}{2}q^{\frac{15}{2}} - \frac{529}{2}q^{\frac{17}{2}} + \dots, \\
h_2^{(1)}(q) &= -q^{-3} - 2q^{-1} - 2 - \frac{15}{2}q - 11q^2 - \frac{41}{2}q^3 - 23q^4 - 27q^5 - \frac{19}{2}q^6 + 30q^7 + \frac{237}{2}q^8 + 272q^9 + \dots,
\end{aligned} \tag{144}$$

$$\begin{aligned}
h_0^{(2)}(q) &= -\frac{1}{6} + \frac{37}{6}q + \frac{17}{2}q^2 - \frac{115}{6}q^3 - \frac{389}{6}q^4 - 181q^5 - \frac{579}{2}q^6 - \frac{1414}{3}q^7 - 548q^8 - \frac{1418}{3}q^9 + \dots, \\
h_1^{(2)}(q) &= -2q^{-\frac{3}{2}} + 4q^{-\frac{1}{2}} + \frac{1}{6}q^{\frac{1}{2}} - \frac{29}{3}q^{\frac{3}{2}} - \frac{59}{2}q^{\frac{5}{2}} - \frac{139}{2}q^{\frac{7}{2}} - \frac{755}{6}q^{\frac{9}{2}} - \frac{560}{3}q^{\frac{11}{2}} - \frac{673}{3}q^{\frac{13}{2}} - \frac{941}{6}q^{\frac{15}{2}} + \dots, \\
h_2^{(2)}(q) &= -2q^{-8} + 4q^{-7} - 2q^{-4} - 6q^{-3} + 6q^{-2} - 4q^{-1} + 4 + \frac{11}{6}q + \dots,
\end{aligned} \tag{145}$$

$$\begin{aligned}
h_0^{(0)}(q^{-1}) &= 1 + q^2 + 3q^3 + 6q^4 + 10q^5 + 16q^6 + 24q^7 + 37q^8 + 55q^9 + \dots, \\
h_1^{(0)}(q^{-1}) &= +q^{\frac{7}{2}} + 2q^{\frac{9}{2}} + 4q^{\frac{11}{2}} + 6q^{\frac{13}{2}} + 10q^{\frac{15}{2}} + 15q^{\frac{17}{2}} + 24q^{\frac{19}{2}} + 37q^{\frac{21}{2}} + 58q^{\frac{23}{2}} + 88q^{\frac{25}{2}} + \dots, \\
h_2^{(0)}(q^{-1}) &= q^{12} + 2q^{13} + 5q^{14} + 9q^{15} + 17q^{16} + 27q^{17} + 45q^{18} + 68q^{19} + 105q^{20} + 154q^{21} + \dots,
\end{aligned} \tag{146}$$

$$\begin{aligned}
h_0^{(1)}(q^{-1}) &= 1 - 3q - 3q^2 + 3q^3 + 6q^4 + 12q^5 + 5q^6 + 3q^7 - 12q^8 - 25q^9 + \dots, \\
h_1^{(1)}(q^{-1}) &= -q^{\frac{3}{2}} - q^{\frac{5}{2}} + 2q^{\frac{7}{2}} + 4q^{\frac{9}{2}} + 6q^{\frac{11}{2}} + 3q^{\frac{13}{2}} - q^{\frac{15}{2}} - 9q^{\frac{17}{2}} - 18q^{\frac{19}{2}} - 23q^{\frac{21}{2}} + \dots, \\
h_2^{(1)}(q^{-1}) &= q^5 + q^7 - q^8 - q^9 - 4q^{10} - 6q^{11} - 6q^{12} - 6q^{13} - 2q^{14} + \dots
\end{aligned} \tag{147}$$

$$\begin{aligned}
h_0^{(2)}(q^{-1}) &= -\frac{5}{6} + 5q - \frac{53}{6}q^2 - \frac{117}{2}q^3 - 117q^4 - \frac{601}{3}q^5 - \frac{865}{3}q^6 - 449q^7 - \frac{4523}{6}q^8 - \frac{7547}{6}q^9 + \dots, \\
h_1^{(2)}(q^{-1}) &= 2q^{\frac{1}{2}} + q^{\frac{3}{2}} - q^{\frac{5}{2}} - \frac{119}{6}q^{\frac{7}{2}} - \frac{107}{3}q^{\frac{9}{2}} - \frac{172}{3}q^{\frac{11}{2}} - 62q^{\frac{13}{2}} - \frac{304}{3}q^{\frac{15}{2}} - \frac{349}{2}q^{\frac{17}{2}} - 370q^{\frac{19}{2}} + \dots, \\
h_2^{(2)}(q^{-1}) &= -2q + 4q^2 - 2q^3 - 2q^4 - 5q^5 + 2q^6 + 5q^7 + 15q^8 + 25q^9 + \dots
\end{aligned} \tag{148}$$

The colored holomorphic blocks have q and q^{-1} -degree

$$\begin{aligned}
\deg_q h_n^{(0)}(q) &= |n|/2, & \deg_q h_n^{(0)}(q^{-1}) &= |n|(5|n| + 2)/2, \\
\deg_q h_n^{(1)}(q) &= -|n|(2|n| - 1)/2, & \deg_q h_n^{(1)}(q^{-1}) &= |n|(2|n| + 1), \\
\deg_q h_n^{(2)}(q) &= -|n|(5|n| - 2)/2, & \deg_q h_n^{(2)}(q^{-1}) &= |n|/2.
\end{aligned} \tag{149}$$

This can easily be deduced from Equations (139), (140), and (141).

A.3. The rotated and the meromorphic 3D-index. We next express the rotated 3D-index in terms of the colored holomorphic blocks.

Proposition 13. For all integers n and n' we have:

$$I^{\text{rot}}(n, n')(q) = -\frac{1}{2}h_{n'}^{(0)}(q^{-1})h_n^{(2)}(q) - h_{n'}^{(1)}(q^{-1})h_n^{(1)}(q) - \frac{1}{2}h_{n'}^{(2)}(q^{-1})h_n^{(0)}(q). \tag{150}$$

Proof. The rotated 3D-index is given by the limit $x \rightarrow 1$ in Equation (38). On the other hand, the expansions of the holomorphic blocks in terms of coloured holomorphic blocks, imply that

$$\begin{aligned}
&B^{(1)}(q^{-n'}e^{-u}; q^{-1})B^{(1)}(q^n e^u; q) + B^{(2)}(q^{-n'}e^{-u}; q^{-1})B^{(2)}(q^n e^u; q) + B^{(3)}(q^{-n'}e^{-u}; q^{-1})B^{(3)}(q^n e^u; q) \\
&= -\frac{1}{2}h_{n'}^{(0)}(q^{-1})h_n^{(2)}(q) - h_{n'}^{(1)}(q^{-1})h_n^{(1)}(q) - \frac{1}{2}h_{n'}^{(2)}(q^{-1})h_n^{(0)}(q) + O(u).
\end{aligned} \tag{151}$$

The result follows. \square

Using Equations (139), (140), (141) and the above proposition, it follows that the first few values of the rotated 3D-index $I^{\text{rot}}(n, n)(q)$ are given by

$$\begin{aligned}
I^{\text{rot}}(0, 0)(q) &= 1 - 12q + 3q^2 + 74q^3 + 90q^4 + 33q^5 - 288q^6 - 684q^7 - 1095q^8 - 1140q^9 + \dots, \\
I^{\text{rot}}(1, 1)(q) &= 2q + 7q^2 + 7q^3 - 13q^4 - 68q^5 - 154q^6 - 220q^7 - 165q^8 + 157q^9 + 898q^{10} + \dots, \\
I^{\text{rot}}(2, 2)(q) &= 2q^2 + 6q^4 + 2q^5 + 17q^6 + 14q^7 + 36q^8 + 21q^9 + 6q^{10} - 110q^{11} + \dots,
\end{aligned} \tag{152}$$

and their degree is given by $\deg_q I^{\text{rot}}(n, n)(q) = |n|$. This, together with Equation (34) gives the first few terms of the meromorphic 3D-index as stated in Equation (8).

A.4. Quadratic relations. We now discuss a new phenomenon of the coloured holomorphic blocks of the 5_2 knot, which was trivial for the case of the amphichiral knot 4_1 , namely quadratic relations among the colored holomorphic blocks. These relations were originally discovered in [25], and interpreted in terms of a duality statement of a q -holonomic module by the authors in [23]. These quadratic relations for 5_2 are

$$h_{n'}^{(0)}(q^{-1})h_n^{(2)}(q) - 2h_{n'}^{(1)}(q^{-1})h_n^{(1)}(q) + h_{n'}^{(2)}(q^{-1})h_n^{(0)}(q) = 0 \quad (\text{if } n = n'), \quad (153)$$

else the left hand-side is in $\mathbb{Z}[q^{\pm 1/2}]$ if $n \neq n'$. For instance, if $H_3(q) = \begin{pmatrix} h_0^{(0)}(q) & h_0^{(1)}(q) & h_0^{(2)}(q) \\ h_1^{(0)}(q) & h_1^{(1)}(q) & h_1^{(2)}(q) \\ h_2^{(0)}(q) & h_2^{(1)}(q) & h_2^{(2)}(q) \end{pmatrix}$,

then we have

$$H_3(q) \begin{pmatrix} 0 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 0 \end{pmatrix} H_3(q^{-1})^t = \begin{pmatrix} 0 & -2q^{\frac{1}{2}} + 2q^{\frac{3}{2}} & 2q - 6q^2 + 2q^4 + 2q^5 + 2q^6 - 4q^7 + 2q^8 \\ 2q^{-\frac{3}{2}} - 2q^{-\frac{1}{2}} & 0 & -2q^{\frac{3}{2}} + 2q^{\frac{5}{2}} \\ 2q^{-8} - 4q^{-7} + 2q^{-6} + 2q^{-5} + 2q^{-4} - 6q^{-2} + 2q^{-1} & 2q^{-\frac{9}{2}} - 2q^{-\frac{7}{2}} & 0 \end{pmatrix}. \quad (154)$$

The above relations together with Proposition 13 imply that

$$I^{\text{rot}}(n, n)(q) = -2h_n^{(1)}(q^{-1})h_n^{(1)}(q). \quad (155)$$

A.5. Asymptotic series. Applying the discrete WKB ansatz (46) to the linear q -difference equation (137), we find that $a_{2,-1}$ satisfies the polynomial equation

$$-32a_{2,-1}^3 - 112a_{2,-1}^2 - 120a_{2,-1} - 68 = 0. \quad (156)$$

It follows that $a_{2,-1} = -\frac{3}{2} + \xi$, where

$$\xi^3 - \xi^2 + 1 = 0 \quad (157)$$

generates the trace field of 5_2 , the cubic field of discriminant -23. Using the initial condition $\Phi_0(\hbar)$ from [24] (see also [15]), we find that for

$$\delta = 3\xi - 2, \quad (158)$$

we have,

$$\begin{aligned} \widehat{\Phi}_n^{(\sigma_1)}(\hbar) &= \frac{1}{\sqrt{\delta}} e^{\frac{V(\sigma_1)}{\hbar}} \exp\left(\left(-\frac{3}{2} + \xi\right)n^2\hbar\right) \\ &\times \left(1 + \left(-\frac{33}{2116}\xi^2 - \frac{121}{1058}\xi + \frac{245}{2116}\right)\hbar\right. \\ &+ \left(\left(-\frac{7}{46}\xi^2 + \frac{7}{46}\xi - \frac{1}{23}\right)n^2 + \left(\frac{10025}{389344}\xi^2 - \frac{12643}{389344}\xi + \frac{683}{97336}\right)\right)\hbar^2 \\ &+ \left(\left(-\frac{5}{276}\xi^2 + \frac{17}{276}\xi - \frac{1}{69}\right)n^4 + \left(-\frac{5557}{292008}\xi^2 + \frac{2609}{146004}\xi - \frac{1984}{36501}\right)n^2\right. \\ &+ \left.\left(\frac{50198891}{12357778560}\xi^2 - \frac{3544387}{1235777856}\xi + \frac{6584729}{1029814880}\right)\hbar^3 \right. \\ &+ \left(\left(-\frac{251}{8464}\xi^2 + \frac{67}{1587}\xi - \frac{103}{3174}\right)n^4 + \left(\frac{175913}{26864736}\xi^2 - \frac{71191}{17909824}\xi + \frac{141915}{17909824}\right)n^2\right. \\ &+ \left.\left(-\frac{952485893}{1136915627520}\xi^2 + \frac{1861268771}{4547662510080}\xi + \frac{203137333}{909532502016}\right)\right)\hbar^4 \end{aligned} \quad (159)$$

$$\begin{aligned}
& + \left(\left(\frac{259}{38088} \xi^2 - \frac{773}{47610} \xi + \frac{2209}{190440} \right) n^6 + \left(-\frac{4897763}{80594208} \xi^2 + \frac{30391187}{322376832} \xi - \frac{13140721}{322376832} \right) n^4 \right. \\
& + \left(\frac{24639499193}{852686720640} \xi^2 - \frac{78232585819}{1705373441280} \xi + \frac{3754956391}{213171680160} \right) n^2 \\
& + \left(-\frac{12690681719899}{4393041984737280} \xi^2 + \frac{32954346270143}{8786083969474560} \xi - \frac{3038166526261}{2196520992368640} \right) \Big) \hbar^5 \\
& + \left(\left(\frac{1}{828} \xi^2 - \frac{7}{6624} \xi + \frac{7}{6624} \right) n^8 + \left(\frac{80599}{3504096} \xi^2 - \frac{76921}{3504096} \xi + \frac{4732}{182505} \right) n^6 \right. \\
& + \left(-\frac{5845444967}{74146671360} \xi^2 + \frac{12028059677}{148293342720} \xi - \frac{8174917009}{74146671360} \right) n^4 \\
& + \left(\frac{208659394649}{5457195012096} \xi^2 - \frac{1077794714231}{27285975060480} \xi + \frac{1449480552719}{27285975060480} \right) n^2 \\
& \left. + \left(-\frac{8106425420368231}{2694399083972198400} \xi^2 + \frac{25148935902272269}{8083197251916595200} \xi - \frac{4471134896235023}{898133027990732800} \right) \hbar^6 + O(\hbar^7) \right).
\end{aligned}$$

Equation (159) above implies that for \mathfrak{F}_2 and $(\sigma, \sigma') = (\sigma_1, \sigma_2)$, we have

$$\begin{aligned}
& \widehat{\Phi}_n^{(\sigma_1)}(\hbar) \widehat{\Phi}_n^{(\sigma_2)}(-\hbar) \\
& = \frac{1}{\sqrt{\delta(\sigma_1)\delta(\sigma_2)}} e^{\frac{V(\sigma_1) - V(\sigma_2)}{\hbar}} \exp \left((\xi(\sigma_1) - \xi(\sigma_2)) n^2 \hbar \right) \left(1 \right. \\
& + \left(-\frac{33}{2116} (\xi(\sigma_1))^2 - \frac{121}{1058} \xi(\sigma_1) + \frac{33}{2116} (\xi(\sigma_2))^2 + \frac{121}{1058} \xi(\sigma_2) \right) \hbar \\
& + \left(\left(-\frac{7}{46} (\xi(\sigma_1))^2 + \frac{7}{46} \xi(\sigma_1) - \frac{7}{46} (\xi(\sigma_2))^2 + \frac{7}{46} \xi(\sigma_2) - \frac{2}{23} \right) n^2 \right. \\
& + \left(-\frac{1089}{4477456} (\xi(\sigma_2))^2 - \frac{3993}{2238728} \xi(\sigma_2) + \frac{246745}{8954912} (\xi(\sigma_1))^2 - \frac{3993}{2238728} (\xi(\sigma_2))^2 \right. \\
& \left. \left. - \frac{14641}{1119364} \xi(\sigma_2) - \frac{172209}{8954912} \xi(\sigma_1) + \left(\frac{246745}{8954912} (\xi(\sigma_2))^2 - \frac{172209}{8954912} \xi(\sigma_2) + \frac{2811}{4477456} \right) \right) \right) \hbar^2 \\
& + \left(\left(-\frac{5}{276} (\xi(\sigma_1))^2 + \frac{17}{276} \xi(\sigma_1) + \frac{5}{276} (\xi(\sigma_2))^2 - \frac{17}{276} \xi(\sigma_2) \right) n^4 \right. \\
& + \left(-\frac{1925}{97336} \xi(\sigma_2) - \frac{107}{146004} (\xi(\sigma_1))^2 + \frac{1925}{97336} (\xi(\sigma_2))^2 + \frac{1525}{292008} \xi(\sigma_1) + \left(\frac{107}{146004} (\xi(\sigma_2))^2 - \frac{1525}{292008} \xi(\sigma_2) \right) n^2 \right. \\
& + \left(\frac{2843269}{823851904} \xi(\sigma_2) + \frac{3001169}{3089444640} (\xi(\sigma_1))^2 - \frac{2843269}{823851904} (\xi(\sigma_2))^2 \right. \\
& \left. \left. + \frac{220399}{2471555712} \xi(\sigma_1) + \left(-\frac{3001169}{3089444640} (\xi(\sigma_2))^2 - \frac{220399}{2471555712} \xi(\sigma_2) \right) \right) \right) \hbar^3 + O(\hbar^4) \Big).
\end{aligned} \tag{160}$$

The coefficients of \hbar^k for $k = 4, 5, 6$ in the above expression can be computed from (159), but they are lengthy and will not be given here.

As explained in Section 3.3, the coefficient of \hbar^k in $A(\hbar) := \sum_{n \in \mathbb{Z}} \widehat{\Phi}_n^{(\sigma_1)}(2\pi i \hbar) \widehat{\Phi}_n^{(\sigma_2)}(-2\pi i \hbar)$ depends on the summand up to $O(\hbar^{3k+1})$, and is computed by formal Gaussian integration. Explicitly, we have

$$\begin{aligned}
A(\hbar) & = e^{\frac{2i \text{Vol}}{\hbar}} \sqrt{\frac{2\pi i}{\hbar}} \frac{1}{\sqrt{\delta(\sigma_1)\delta(\sigma_2)(2\xi(\sigma_1) - 2\xi(\sigma_2))}} \\
& \times \left(1 + \left(-\frac{1258}{13225} (\xi(\sigma_1) - \xi(\sigma_2)) + \frac{7}{1840} (\xi(\sigma_1) - \xi(\sigma_2))^3 - \frac{963}{211600} (\xi(\sigma_1) - \xi(\sigma_2))^5 \right) \hbar \right. \\
& \left. + \left(-\frac{226591}{19467200} + \frac{88839}{7786880} (\xi(\sigma_1) - \xi(\sigma_2))^2 + \frac{38187}{155737600} (\xi(\sigma_1) - \xi(\sigma_2))^4 \right) \hbar^2 + O(\hbar^3) \right),
\end{aligned} \tag{161}$$

in complete agreement with the numerical extrapolation of the asymptotics of $I_{\mathfrak{F}_2}^{\text{rot}}(0, 0)(q)$.

A.6. Horizontal asymptotics. We now discuss the horizontal asymptotics of the colored holomorphic blocks when τ tends to zero just above (*resp.* below) the positive (*resp.* negative) reals. The next identity (162) expresses the holomorphic blocks as Borel resummations of the three asymptotic series. This identity was discovered and verified in [15, Sec.6.4] when $n =$ using about 200 coefficients (of powers of \hbar) of the asymptotic series. When n is an arbitrary integer, we use about 90 coefficients of the asymptotic series, each a polynomial of n to numerically evaluate the Borel resummation using Padé approximants. We then find numerically that for τ in a cone $C = \arg(\theta) \in (\theta_0, \theta_1)$ with $0 < \theta_0 < \theta_1$ sufficiently small, we have

$$h_n^{(j)}(q) = \tau^{1-j} \sum_{k=1}^3 s \widehat{\Phi}_n^{(\sigma_k)}(2\pi i \tau) H_{\sigma_k}^{(j)}(\tilde{q}), \quad (162)$$

where for $|\tilde{q}| < 1$, we have

$$\begin{pmatrix} H_{\sigma_1}^{(j)}(\tilde{q}) \\ H_{\sigma_2}^{(j)}(\tilde{q}) \\ H_{\sigma_3}^{(j)}(\tilde{q}) \end{pmatrix} = \begin{pmatrix} \frac{2\tilde{q}^3 - 2\tilde{q}^2 - 2\tilde{q} + 1}{-\tilde{q}^3 + \tilde{q}^2 + \tilde{q} - 1} & \frac{-\tilde{q}^{11/2} + 2\tilde{q}^{9/2} - \tilde{q}^{7/2} + 2\tilde{q}^{5/2} - \tilde{q}^{3/2} + \tilde{q}^{1/2}}{-\tilde{q}^4 + \tilde{q}^3 + \tilde{q} - 1} & -\frac{\tilde{q}^7}{-\tilde{q}^5 + \tilde{q}^3 + \tilde{q}^2 - 1} \\ -1 & 0 & 0 \\ \frac{-\tilde{q}^3 + \tilde{q} - 1}{-\tilde{q}^3 + \tilde{q}^2 + \tilde{q} - 1} & \frac{-\tilde{q}^{11/2} + 3\tilde{q}^{9/2} - \tilde{q}^{7/2} + 2\tilde{q}^{5/2} - 2\tilde{q}^{3/2} + \tilde{q}^{1/2}}{-\tilde{q}^4 + \tilde{q}^3 + \tilde{q} - 1} & -\frac{\tilde{q}^7}{-\tilde{q}^5 + \tilde{q}^3 + \tilde{q}^2 - 1} \end{pmatrix} \begin{pmatrix} h_0^{(j)}(\tilde{q}) \\ h_1^{(j)}(\tilde{q}) \\ h_2^{(j)}(\tilde{q}) \end{pmatrix}, \quad (163)$$

and

$$\begin{pmatrix} H_{\sigma_1}^{(j)}(\tilde{q}^{-1}) \\ H_{\sigma_2}^{(j)}(\tilde{q}^{-1}) \\ H_{\sigma_3}^{(j)}(\tilde{q}^{-1}) \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ \frac{-3\tilde{q}^3 + 2\tilde{q}^2 + 2\tilde{q} - 2}{-\tilde{q}^3 + \tilde{q}^2 + \tilde{q} - 1} & \frac{\tilde{q}^5 - \tilde{q}^4 + 2\tilde{q}^3 - \tilde{q}^2 + 2\tilde{q} - 1}{-\tilde{q}^{11/2} + \tilde{q}^{9/2} + \tilde{q}^{5/2} - \tilde{q}^{3/2}} & -\frac{1}{-\tilde{q}^7 + \tilde{q}^5 + \tilde{q}^4 - \tilde{q}^2} \\ \frac{-\tilde{q}^3 + \tilde{q}^2 - 1}{-\tilde{q}^3 + \tilde{q}^2 + \tilde{q} - 1} & \frac{\tilde{q}^5 - 2\tilde{q}^4 + 2\tilde{q}^3 - \tilde{q}^2 + 3\tilde{q} - 1}{-\tilde{q}^{11/2} + \tilde{q}^{9/2} + \tilde{q}^{5/2} - \tilde{q}^{3/2}} & -\frac{1}{-\tilde{q}^7 + \tilde{q}^5 + \tilde{q}^4 - \tilde{q}^2} \end{pmatrix} \begin{pmatrix} h_0^{(j)}(\tilde{q}^{-1}) \\ h_1^{(j)}(\tilde{q}^{-1}) \\ h_2^{(j)}(\tilde{q}^{-1}) \end{pmatrix}. \quad (164)$$

With around 90 coefficients of $\widehat{\Phi}_n^{(\sigma_k)}$, these identities can be checked to about 10 coefficients of \tilde{q} . Using Equation (150) and the above, we obtain a bilinear expression for $I^{\text{rot}}(n, n')(q)$ in terms of products $\widehat{\Phi}_n^{(\sigma)}(2\pi i \tau) \widehat{\Phi}_n^{(\sigma')}(-2\pi i \tau)$ times \tilde{q} -series. On the fixed ray of τ near the positive real axis, we can ignore all \tilde{q} -terms, and among all bilinear combinations of asymptotic series, there is precisely one with $(\sigma, \sigma') = (\sigma_1, \sigma_2)$ which is exponentially larger than the others. In particular, we find that

$$I^{\text{rot}}(n, n')(q) \sim \widehat{\Phi}_n^{(\sigma_1)}(2\pi i \tau) \widehat{\Phi}_n^{(\sigma_2)}(-2\pi i \tau), \quad (165)$$

concluding the one part of Conjecture 1.

A.7. Vertical asymptotics. Following the method of [25] and using a high precision computation of the colored holomorphic blocks when τ is small on the imaginary axis, we find numerically that the coloured holomorphic blocks have the following vertical asymptotics as $\tau \downarrow 0$

$$\begin{aligned} h_n^{(0)}(q) &\sim \tau \widehat{\Phi}_n^{(\sigma_1)}(2\pi i \tau) - \tau \widehat{\Phi}_n^{(\sigma_2)}(2\pi i \tau), & h_n^{(0)}(q^{-1}) &\sim -\tau \widehat{\Phi}_n^{(\sigma_3)}(-2\pi i \tau), \\ h_n^{(1)}(q) &\sim -\frac{1}{2} \widehat{\Phi}_n^{(\sigma_1)}(2\pi i \tau) - \frac{1}{2} \widehat{\Phi}_n^{(\sigma_2)}(2\pi i \tau), & h_n^{(1)}(q^{-1}) &\sim \widehat{\Phi}_n^{(\sigma_1)}(-2\pi i \tau) + \widehat{\Phi}_n^{(\sigma_2)}(-2\pi i \tau), \\ h_n^{(2)}(q) &\sim -\frac{1}{6\tau} \widehat{\Phi}_n^{(\sigma_1)}(2\pi i \tau) + \frac{1}{6\tau} \widehat{\Phi}_n^{(\sigma_2)}(2\pi i \tau), & h_n^{(2)}(q^{-1}) &\sim -\frac{1}{6\tau} \widehat{\Phi}_n^{(\sigma_3)}(-2\pi i \tau). \end{aligned} \quad (166)$$

As in the case of the 4_1 knot, each asymptotic statement above involves linear combinations of asymptotic series of the *same* exponential growth rate.

Taking the vertical asymptotics of the quadratic relation (153) of the colored holomorphic blocks gives the following quadratic relation of the three asymptotic series (given in [24,

Sec.3.3] for $n = 0$)

$$\widehat{\Phi}_n^{(\sigma_1)}(\hbar)\widehat{\Phi}_n^{(\sigma_1)}(-\hbar) + \widehat{\Phi}_n^{(\sigma_2)}(\hbar)\widehat{\Phi}_n^{(\sigma_2)}(-\hbar) + \widehat{\Phi}_n^{(\sigma_3)}(\hbar)\widehat{\Phi}_n^{(\sigma_3)}(-\hbar) = 0, \quad (167)$$

valid for all integers n .

Equations (150) and (166) imply that

$$I^{\text{rot}}(n, n')(q) \sim -\widehat{\Phi}_n^{(\sigma_1)}(2\pi i\tau)\widehat{\Phi}_{n'}^{(\sigma_1)}(-2\pi i\tau) + \widehat{\Phi}_n^{(\sigma_2)}(2\pi i\tau)\widehat{\Phi}_{n'}^{(\sigma_2)}(-2\pi i\tau) + \widehat{\Phi}_n^{(\sigma_3)}(2\pi i\tau)\widehat{\Phi}_{n'}^{(\sigma_3)}(-2\pi i\tau). \quad (168)$$

verifying Conjecture (2) for the 5_2 knot.

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