Mathematische Annalen (2020) 378:447–484
https://doi.org/10.1007/s00208-020-02028-y

Mathematische Annalen

A diagrammatic approach to the AJ Conjecture

Renaud Detcherry[1] · Stavros Garoufalidis[2]

Received: 11 July 2019 / Revised: 29 May 2020 / Published online: 25 June 2020

Abstract

The AJ Conjecture relates a quantum invariant, a minimal order recursion for the colored Jones polynomial of a knot (known as the $\hat{A}$ polynomial), with a classical invariant, namely the defining polynomial A of the $\mathrm{PSL}_2(\mathbb{C})$ character variety of a knot. More precisely, the AJ Conjecture asserts that the set of irreducible factors of the $\hat{A}$-polynomial (after we set $q = 1$, and excluding those of L-degree zero) coincides with those of the A-polynomial. In this paper, we introduce a version of the $\hat{A}$-polynomial that depends on a planar diagram of a knot (that conjecturally agrees with the $\hat{A}$-polynomial) and we prove that it satisfies one direction of the AJ Conjecture. Our proof uses the octahedral decomposition of a knot complement obtained from a planar projection of a knot, the R-matrix state sum formula for the colored Jones polynomial, and its certificate.

Mathematics Subject Classification Primary 57N10; Secondary 57M25

Contents

Communicated by Thomas Schick.

✉ Stavros Garoufalidis
stavros@mpim-bonn.mpg.de
http://people.mpim-bonn.mpg.de/stavros

Renaud Detcherry
detcherry@mpim-bonn.mpg.de
http://people.mpim-bonn.mpg.de/detcherry

1 Max Planck Institute for Mathematics, Vivatsgasse 7, 53111 Bonn, Germany

2 International Center for Mathematics, Department of Mathematics, Southern University of Science and Technology, Shenzhen, China

1 Introduction

1.1 The colored Jones polynomial and the AJ Conjecture

The Jones polynomial of a knot [20] is a powerful knot invariant with deep connections with quantum field theory, discovered by Witten [36]. The discoveries of Jones and Witten gave rise to Quantum Topology. An even more powerful invariant is the colored Jones polynomial $J_K(n) \in \mathbb{Z}[q^{\pm 1}]$ of a knot K, a sequence of Laurent polynomials that encodes the Jones polynomial of a knot and its parallels. Since the dependence of the colored Jones polynomial $J_K(n)$ on the variable q plays no role in our paper, we omit it from the notation. The colored Jones polynomial determines the Alexander polynomial [4], is conjectured to determine the volume of a hyperbolic knot [21,28,31], is conjectured to select two out of finitely many slopes of incompressible surfaces of the knot complement [11], and is expected to determine the $\mathrm{SL}(2,\mathbb{C})$ character variety of the knot, viewed from the boundary [9]. The latter is the AJ Conjecture, which is the focus of our paper.

The starting point of the AJ Conjecture [9] is the fact that the colored Jones polynomial $J_K(n)$ of a knot K is q-holonomic [15], that is, it satisfies a nontrivial linear recursion relation

$$\sum_{j=0}^{d} c_j(q, q^n) J_K(n+j) = 0, \qquad \text{for all } \ n \in \mathbb{N}, \tag{1}$$

where $c_j(u, v) \in \mathbb{Z}[u, v]$ for all j. We can write the above equation in operator form as follows $P J_K = 0$ where $P = \sum_j c_j(q, Q) E^j$ is an element of the ring $\mathbb{Z}[q, Q]\langle E \rangle$ where $EQ = qQE$ are the operators that act on sequences of functions $f(n)$ by:

$$(Ef)(n) = f(n+1), \qquad (Qf)(n) = q^n f(n). \tag{2}$$

Observe that the set

$$\mathrm{Ann}(f) = \{P \in \mathbb{Z}[q, Q]\langle E\rangle \mid Pf = 0\} \tag{3}$$

is a left ideal of $\mathbb{Z}[q, Q]\langle E\rangle$, nonzero when f is q-holonomic. Although the latter ring is not a principal left ideal domain, its localization $\mathbb{Q}(q, Q)\langle E\rangle$ is, and cleaning denominators allows one to define a minimal E-order, content-free element $\hat{A}_K(q, Q, E) \in \mathbb{Z}[q, Q]\langle E\rangle$ which annihilates the colored Jones polynomial.

On the other hand, the A-polynomial of a knot [5] $A_K(L, M) \in \mathbb{Z}[L, M]$ is the defining polynomial for the character variety of $\mathrm{SL}(2, \mathbb{C})$ representations of the boundary of the knot complement that extend to representations of the knot complement.

The AJ Conjecture asserts that the irreducible factors of $\hat{A}_K(1, Q, E)$ of positive E-degree coincide with those of $A_K(Q, E^{-2})$. The AJ Conjecture is known for most 2-bridge knots, and some 3-strand pretzel knots; see [25] and [27].

Let us briefly now discuss the q-holonomicity of the colored Jones polynomial [15]: this follows naturally from the fact that the latter can be expressed as a state-sum formula using a labeled, oriented diagram D of the knot, placing an R-matrix at each crossing and contracting indices as described for instance in Turaev's book [34]. For a diagram D with $c(D)$ crossings, this leads to a formula of the form

$$J_K(n) = \sum_{\mathbb{Z}^{c(D)+1}} w_D(n, k) \tag{4}$$

where the summand $w_D(n, k)$ is a q-proper hypergeometric function and for fixed n, the support of the summand is a finite set. The fundamental theoreom of q-holonomic functions of Wilf–Zeilberger [37] concludes that $J_K(n)$ is q-holonomic. Usually this ends the benefits of (4), aside from its sometimes use as a means of computing some values of the colored Jones polynomial for knots with small (eg 12 or less) number of crossings and small color (eg, $n < 10$).

Aside from quantum topology, and key to the results of our paper, is the fact that a planar projection D of a knot K gives rise to an ideal octahedral decomposition of its complement minus two spheres, and thus to a gluing equations variety $\mathcal{G}_D$ and to an A-polynomial A_D reviewed in Sect. 2 below. In [22], Kim–Kim–Yoon prove that A_D coincides with the A-polynomial of K, and in [24] Kim–Park prove that $\mathcal{G}_D$ is, up to birational equivalence, invariant under Reidemeister moves, and forms a diagrammatic model for the decorated $\mathrm{PSL}(2, \mathbb{C})$ character variety of the knot.

The aim of the paper is to highlight the fact that formulas of the form (4) lead to further knot invariants which are natural from the point of view of holonomic modules and form a rephrasing of the AJ Conjecture that connects well with the results of [22] and [24].

1.2 *q*-holonomic sums

To motivate our results, consider a sum of the form

$$f(n) = \sum_{k \in \mathbb{Z}^r} F(n,k) \tag{5}$$

where $n \in \mathbb{Z}$ and $k = (k_1, \dots, k_r) \in \mathbb{Z}^r$ and $F(n,k)$ is a proper q-hypergeometric function with compact support for fixed n. Then f is q-holonomic but more is true. The annihilator

$$\mathrm{Ann}(F) \subset \mathbb{Q}[q, Q, Q_k]\langle E, E_k\rangle$$

of the summand is a q-holonomic left ideal where $E_k = (E_{k_1}, \dots, E_{k_r})$ and $Q_k = (Q_{k_1}, \dots, Q_{k_r})$ are operators, each acting in one of the $r+1$ variables (n,k) with the obvious commutation relations (operators acting on different variables commute and the ones acting on the same variable q-commute). Consider the map

$$\varphi : \mathbb{Q}[q, Q]\langle E, E_k\rangle \to \mathbb{Q}[q, Q]\langle E\rangle, \qquad \varphi(E_{k_i}) = 1, \ i = 1, \dots, r. \tag{6}$$

It is a fact (see Proposition 3.2 below) that

$$\varphi(\mathrm{Ann}(F) \cap \mathbb{Q}[q, Q]\langle E, E_k\rangle) \subset \mathrm{Ann}(f) \tag{7}$$

and that the left hand side is nonzero. Elements of the left hand side are usually called "good certificates", and in practice one uses the above inclusion to compute a recursion for the sum [30,39]. If $\hat{A}^c_F(q, Q, E)$ and $\hat{A}_f(q, Q, E)$ denotes generators of the left and the right hand side of (7), it follows that $\hat{A}_f(q, Q, E)$ is a right divisor of $\hat{A}^c_F(q, Q, E)$. We will call the latter the certificate recursion of f obtained from (5).

In a sense, the certificate recursion of f is more natural than the minimal order recursion and that is the case for holonomic D-modules and their push-forward, discussed for instance by Lairez [26].

What is more important for us is that if one allows presentations of f of the form (5) where F is allowed to change by for instance, consequences of the q-binomial identity, then one can obtain an operator $\hat{A}^c_f(q, Q, E)$ which is independent of the chosen presentation.

1.3 Our results

Applying the above discussion to (4) with $F = w_D$, allows us to introduce the certificate recursion $\hat{A}^c_D(q, Q, E) \in \mathbb{Z}[q, Q]\langle E\rangle$ of the colored Jones polynomial, which depends on a labeled, oriented planar diagram D of a knot. We can also define $\hat{A}^c_K(q, Q, E) \in \mathbb{Z}[q, Q]\langle E\rangle$ to be the left gcd of the elements $\hat{A}^c_D$ in the local ring $\mathbb{Q}(q, Q)\langle E\rangle$, lifted back to $\mathbb{Z}[q, Q]\langle E\rangle$.

We now have all the ingredients to formulate one direction of a refined AJ Conjecture. Our proof uses the octahedral decomposition of a knot complement obtained from a planar projection of a knot, the R-matrix state sum formula for the colored Jones polynomial, and its certificate.

Theorem 1.1 *For every knot K,*

(a) $\hat{A}_K$ *divides* $\hat{A}^c_K$.
(b) *Every irreducible factor of* $A_K(Q, E^{-2})$ *of positive* E*-degree is a factor of* $\hat{A}^c_K(1, Q, E)$.

Remark 1.2 The $\hat{A}_K$-polynomial has only been computed in a handful of cases, see [13,18,19] and [14]. In all cases where $\hat{A}_K$ is known, it is actually obtained from certificates and in that case $\hat{A}^c_K = \hat{A}_K$.

Question 1.3 *Is it true that for any knot* K*, one has* $\hat{A}^c_K = \hat{A}_K$*?*

Question 1.4 *Is it true that the certificate recursion* $\hat{A}^c_D$ *of a planar projection of a knot is invariant under Reidemester moves on* D*?*

A positive answer to the latter question is a quantum analogue of the fact that the gluing equation variety $\mathcal{G}_D$ associated to a diagram D is independent of D, a result that was announced by Kim and Park [24]. We believe that the above question has a positive answer, coming from the fact that the Yang–Baxter equation for the R-matrix follows from a q-binomial identity, but we will postpone this investigation to a future publication.

1.4 Sketch of the proof

To prove Theorem 1.1, we fix a planar projection D of an oriented knot K. On the one hand, the planar projection gives rise to an ideal decomposition of the knot complement (minus two points) using one ideal octahedron per crossing, subdividing further each octahedron to five ideal tetrahedra. This ideal decomposition gives rise to a gluing equations variety, discussed in Sect. 2. On the other hand, the planar projection gives a state-sum for the colored Jones polynomial, by placing one R-matrix per crossing and contracting indices. The summand of this state-sum is q-proper hypergeometric and its annihilator defines an ideal in a quantum Weyl algebra, discussed in Sect. 4. The annihilator ideal is matched when $q = 1$ with the gluing equations ideal in the key Proposition 5.1. This matching, implicit in the Grenoble notes of Thurston [33], combined with a certificate (which is a quantum version of the projection map from gluing equations variety to $\mathbb{C}^* \times \mathbb{C}^*$), and with the fact that the gluing equation variety sees all components of the $\mathrm{PSL}(2, \mathbb{C})$ character variety [22], conclude the proof of our main theorem.

Our method of proof for Theorem 1.1 using certificates to show one direction of the AJ Conjecture is general and flexible and can be applied in numerous other situations, in particular to a proof of one direction of the AJ Conjecture for state-integrals, and to one direction of the AJ Conjecture for the 3Dindex [1,8]. This will be studied in detail in a later publication. For a discussion of the AJ Conjecture for state-integrals and for a proof in the case of the simplest hyperbolic knot, see [1].

Finally, our proof of Theorem 1.1 does not imply any relation between the Newton polygon of the $\hat{A}_K(q, Q, E)$ polynomial and that of $A_K(1, Q, E)$. If the two Newton polygons coincided, the Slope Conjecture of [11] would follow, as was explained in [10]. Nonetheless, the Slope Conjecture is an open problem.

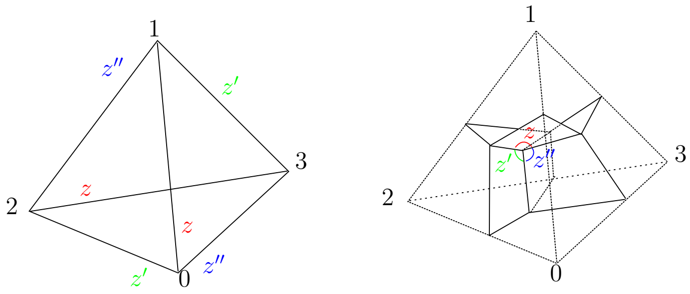


Fig. 1 The dual spine to the triangulation and the shape parameters associated to corners of the spine

2 Knot diagrams, their octahedral decomposition and their gluing equations

2.1 Ideal triangulations and their gluing equations

Given an ideal triangulation $\mathcal{T}$ of a 3-manifold M with cusps, Thurston's gluing equations (one for each edge of $\mathcal{T}$) give a way to describe the hyperbolic structure on M and its deformation if M is hyperbolic [29,32]. The gluing equations define an affine variety $\mathcal{G}_{\mathcal{T}}$, the so-called gluing equations variety, whose definition we now recall. The edges of each combinatorial ideal tetrahedron get assigned variables, with opposite edges having the same variable as in the left hand side of Fig. 1. The triple of variables (often called a triple of shapes of the tetrahedron)

$$(z, z', z'') = \left(z, \frac{1}{1-z}, 1 - \frac{1}{z}\right)$$

satisfies the equations

$$zz'z'' = -1, \quad zz'' - z + 1 = 0 \tag{8}$$

and every solution of (8) uniquely defines a triple of shapes of a tetrahedron. Note that the shapes of the tetrahedron z, z', or z'' lie in $\mathbb{C}^{**} = \mathbb{C}\setminus\{0, 1\}$, and that they are uniquely determined by $z \in \mathbb{C}^{**}$. When we talk about assigning a shape z to a tetrahedron below, it determines shapes z' and z'' as in Fig. 1.

Given an ideal triangulation $\mathcal{T}$ with N tetrahedra, assign shapes z_i for $i = 1, \dots, N$ to each tetrahedron. If e is an edge of $\mathcal{T}$ the corresponding gluing equation is given by

$$\prod_{\Delta \in N(e)} z_\Delta = 1,$$

where $N(e)$ is the set of all tetrahedra that meet along the edge e, and z_Δ is the shape parameter corresponding to the edge e of Δ. The gluing equation variety $\mathcal{G}_{\mathcal{T}}$ is the affine variety in the variables $(z_1, \dots, z_N) \in (\mathbb{C}^{**})^N$ defined by the edge gluing equations, for all edges of $\mathcal{T}$. Equivalently, it is the affine variety in the variables $(z_1, z_1', z_1'', \ldots, z_N, z_N', z_N'') \in \mathbb{C}^{3N}$ defined by the edge equations and the Eq. (8), one for each tetrahedron.

We next discuss the relation between a solution to the gluing equations and decorated (or sometimes called, augmented) $\mathrm{PSL}(2, \mathbb{C})$ representations of the fundamental group of the underlying 3-manifold M. The construction of decorated representations from solutions to the gluing equations appears for instance in Zickert's thesis [40] and also in [12]. Below, we follow the detailed exposition by Dunfield given in [2, Sec.10.2–10.3].

A solution of the gluing equations gives rise to a developing map $\widetilde{M} \to \mathbb{H}^3$ from the universal cover $\widetilde{M}$ to the 3-dimensional hyperbolic space $\mathbb{H}^3$. Since the orientation preserving isometries of $\mathbb{H}^3$ are in $\mathrm{PSL}(2, \mathbb{C})$, this in turn gives rise to a $\mathrm{PSL}(2, \mathbb{C})$ representation of the fundamental group $\pi_1(M)$, well-defined up to conjugation. What's more, we get a decorated representation (those were called augmented representations in Dunfield's terminology). Following the notation of [2, Sec.10.2–10.3], let $\overline{X}(M, \mathrm{PSL}(2, \mathbb{C}))$ denote the augmented character variety of M. Thus, we get a map:

$$\mathcal{G}_{\mathcal{T}} \to \overline{X}(M, \mathrm{PSL}(2, \mathbb{C})). \tag{9}$$

So far, M can have boundary components of arbitrary genus. When the boundary ∂M consists of a single torus boundary component, and γ is a simple closed curve on ∂M, the holonomy of an augmented representation gives a regular function $h_\gamma : \overline{X}(M, \mathrm{PSL}(2, \mathbb{C})) \to \mathbb{C}^*$. Note that for a decorated representation ρ, the set of squares of the eigenvalues of $\rho(\gamma) \in \mathrm{PSL}(2, \mathbb{C})$ is given by $\{h_\gamma(\rho), h_\gamma(\rho)^{-1}\}$. Once we fix a pair of meridian and longitude (μ, λ) of the boundary torus, then we get a map

$$(h_\mu, h_\lambda) : \overline{X}(M, \mathrm{PSL}(2, \mathbb{C})) \to \mathbb{C}^* \times \mathbb{C}^*. \tag{10}$$

The defining polynomial of the 1-dimensional components of the above map is the A-polynomial of the 3-manifold M. Technically, this is the $\mathrm{PSL}(2, \mathbb{C})$-version of the A-polynomial and its precise relation with the $\mathrm{SL}(2, \mathbb{C})$-version of the A-polynomial (as defined by [5]) is discussed in detail in Champanerkar's thesis [7]; see also [2, Sec.10.2–10.3].

We should point out that although (9) is a map of affine varieties, its image may miss components of $\overline{X}(M, \mathrm{PSL}(2, \mathbb{C}))$, and hence the gluing equations of the triangulation may not detect some factors of the A-polynomial. In fact, when the boundary of M consists of tori, the image of (9) always misses the components of abelian $\mathrm{SL}(2, \mathbb{C})$ representations (and every knot complement has a canonical such component), but it may also miss others. For instance, there is a 5-tetrahedron ideal triangulation of the 4_1 knot with an edge of valency one, and for that triangulation, $\mathcal{G}_{\mathcal{T}}$ is empty.

For later use, let us record how to compute the holonomy of a peripheral curve on the gluing equations variety. Given a path γ in a component of ∂M that is normal with respect to this triangulation, it intersects the triangles of ∂M in segment joining different sides. Each segment may go from one side of the triangle to either the adjacent left side or right side. Also it separates one corner of the triangle from the other two; this corner correspond to a shape parameter which we name z_{left} or z_{right} depending whether the segment goes left or right. The holonomy of γ is then:

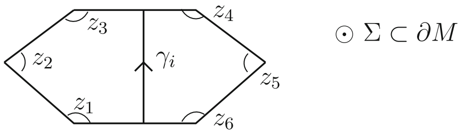


Fig. 2 A segment γ_i of a peripheral loop γ intersecting a region of the spine. The boundary component Σ to which γ belongs lies above the region. In this example, $h_{\gamma_i} = -z_1 z_2 z_3 = -\frac{1}{z_4 z_5 z_6}$

$$h_\gamma = \prod_{left\ segments} z_{left} \prod_{right\ segments} z_{right}^{-1}.$$

2.2 Spines and gluing equations

The ideal triangulations that we will discuss in the next section come from a planar projection of a knot, and it will be easier to work with their spines, that is the the dual 2-skeleton. Because of this reason, we discuss the gluing equations of an ideal triangulation $\mathcal{T}$ in terms of its spine. In that case, edges of $\mathcal{T}$ are dual to 2-cells of the spine, and give rise to gluing equations. Recall that a spine S of M is a CW-complex embedded in M, such that each point of S has a neighborhood homeomorphic to either D^2, $Y \times [0, 1]$ where Y is the Y-shaped graph or to the cone over the edges of a tetrahedron, and such that $M \backslash S$ is homeomorphic to $\partial M \times [0, 1)$. Points of the third type are vertices of the spine, points of the second type form the edges of the spines and points of the first type form the regions of the spine.

For any ideal triangulation of M, the dual spine is obtained as shown in Fig. 1. Shape parameters that were assigned to tetrahedra are now assigned to vertices of the spine. At each vertex, two opposite corners bear the same shape parameter z, and the other bear the parameters z', z'' according to the cyclic ordering (see Fig. 1). Edge equations translate into region equations, the region equation associated to the region R being:

$$\prod_{c \in \mathrm{corners}(R)} z_c = 1.$$

For a path γ on the spine S that is in normal position with respects to S, it intersects each region in a collection of segments $(\gamma_i)_{i \in I}$. The holonomy of the segment γ_i is

$$h_{\gamma_i} = -\prod_{c\ \text{left corner}} z_c = -\prod_{c\ \text{right corner}} z_c^{-1},$$

where left and right corners are defined as in Fig. 2, and the holonomy of γ is

$$h_\gamma = \prod_{i \in I} h_{\gamma_i}.$$

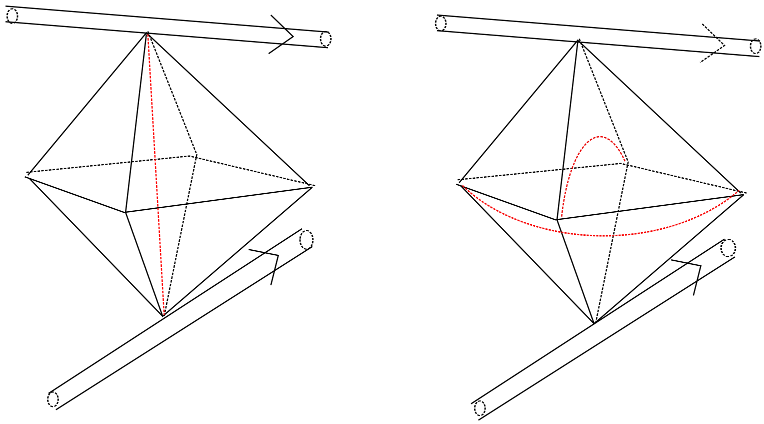


Fig. 3 Any octahedron can be split into 4 or 5 tetrahedra by adding the red dashed edges to it (color figure online)

2.3 The octahedral decomposition of a knot diagram

In this section we fix a diagram D in S^2 of an oriented knot K. By diagram, we mean an embedded 4-valent graph in the plane, with an overcrossing/undercrossing choice at each vertex. Let $X(D)$ and $c(D)$ denote the set and the number of crossings of D. In this section as well as the remainder of the paper, an arc of D will be the segment of the diagram joining two successive crossings of D. An *overpass* (resp. *underpass*) will be a small portion of the upper strand (resp. lower strand) of a crossing. We will denote the set of overpasses by $O(D)$ and the set of underpasses by $U(D)$. An *overarc* (resp. *underarc*) will be the portion of the knot joining two successive underpasses (resp. overpasses). An overarc of K may pass through some number of crossings of K, doing so as the upper strand each time.

Given a diagram D of the knot K with $c(D)$ crossings, let B_1 be some ball lying above the projection plane and B_2 another ball lying under the projection plane. A classical construction, first introduced by Weeks in his thesis, and implemented in `SnapPy` as a method of constructing ideal triangulations of planar projections of knots [6,35], yields a decomposition of $S^3\backslash(K \cup B_1 \cup B_2)$ into $c(D)$ ideal octahedra. The decomposition works as follows: at each crossing of K, put an octahedron whose top vertex is on the overpass and bottom vertex is on the underpass. Pull the two middle vertices lying on the two sides of the overpass up towards B_1 and the two other middle vertices down towards B_2. One can then patch all these octahedra together to get a decomposition of $S^3\backslash(K \cup B_1 \cup B_2)$. We refer to [22] as well as [33] for figures and more details on this construction.

From the octahedral decomposition of $S^3\backslash(K \cup B_1 \cup B_2)$, one can get an ideal triangulation of $S^3\backslash(K \cup B_1 \cup B_2)$ simply by splitting the octahedra further into tetrahedra. There are two natural possibilities for this splitting, as one can cut each octahedra into either 4 or 5 tetrahedra as shown in Fig. 3. We will be interested in the decomposition where we split each octahedra into 5 tetrahedra, obtaining thus

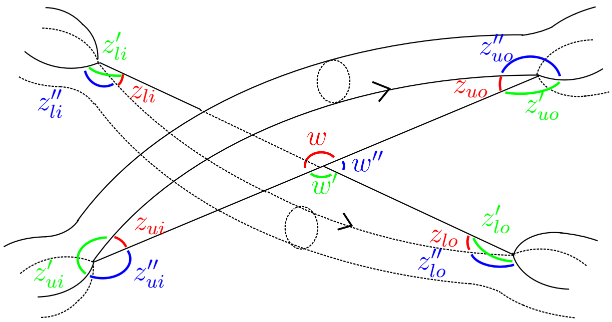


Fig. 4 The $5T$-spine near a crossing of D, and the shape parameters of each corner of the spine. The arrows specify the orientation of strands

a decomposition of $S^3\backslash(K \cup B_1 \cup B_2)$ into $5c(D)$ tetrahedra. We denote this ideal triangulation by $\mathcal{T}_D^{5T}$, and we call it the "$5T$-triangulation of D".

Since the inclusion map $S^3\backslash(K \cup B_1 \cup B_2) \to S^3\backslash K$ is an isomorphism on fundamental groups, a solution to the gluing equations of $\mathcal{T}_D^{5T}$ gives rise to a decorated $\mathrm{PSL}(2, \mathbb{C})$ representation of the knot complement.

2.4 The spine of the 5*T*-triangulation of a knot diagram and its gluing equations

Let $\mathcal{G}_D$ denote the gluing equation variety of $\mathcal{T}_D^{5T}$. To write down the equations of $\mathcal{G}_D$, we will work with the dual spine, and use the spine formulation of the gluing equations introduced in Sect. 2.1. We describe this spine just below. This well-known spine is studied in detail by several authors including [22].

Figure 4 shows a picture of the spine near a crossing of D. The spine contains 5 vertices near each crossing of D and can be described as follows:

First we embed K in S^3 as a solid torus sitting in the middle of the projection plane; except for overpasses which go above the projection plane and underpasses which go below. We let the boundary of a tubular neighborhood of K to be a subset of the spine. At each crossing we connect the overpass and the underpass using two triangles that intersects transversally in one point. Finally we glue the regions of the projection plane that lie outside D to the rest of the spine. The regions of the spine are then of 3 types:

- An upper/lower triangle region for each crossing, and $2c(D)$ in total.
- For each region of D one gets an horizontal region in the spine; we call these *big regions*, $c(D) + 2$ in total.
- The boundary of a neighborhood of K is cut by the triangle regions and the big regions into regions lying over the projection plane (*upper shingle region*) and some lying under the projection plane (*lower shingle regions*). Note that upper shingle regions start and end at underpasses; they are in correspondance with the overarcs of the diagram, $c(D)$ in total. Similarly, the lower shingle regions are in correspondance with underarcs, and there is also $c(D)$ of them.

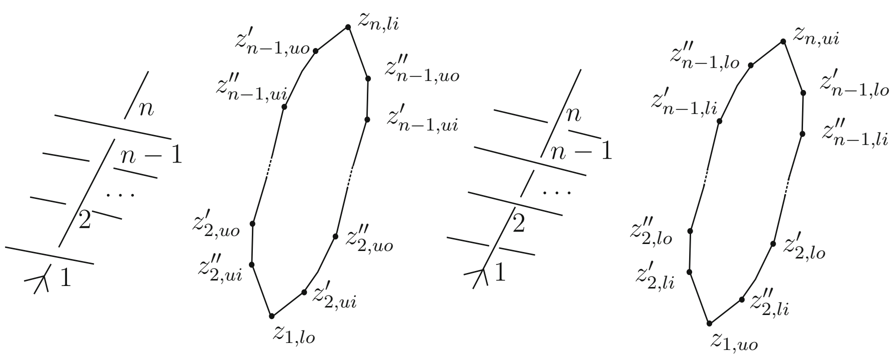


Fig. 5 An overarc (resp. underarc) and the corresponding upper (resp. lower) shingle region of the spine, with shape parameters

We now assign shape parameters to each vertex of the spine as shown in Fig. 4. There are 5 shape parameters for each crossing c: a central one which we call w_c and 4 others: $z_{c,li}, z_{c,lo}, z_{c,ui}, z_{c,uo}$ standing for lower-in, lower-out, upper-in and upper-out. When the crossing c we consider is clear, we will sometimes write $w, z_{li}, z_{lo} \dots$ dropping the index c.

Note that the assignment of shape parameters is such that the main version of the parameter $w, z_{li}, \dots$ lies on a corner of a triangle region, while the auxiliary $w', w'', z'_{li}, z''_{li} \dots$ are prescribed by the cyclic ordering induced by the boundary of $S^3\backslash(K \cup B_1 \cup B_2)$.

We can now write down the gluing equations coming from the $5T$-spine:

• The upper/lower triangle equations are (in the notation of Fig. 4)

$$w z_{ui} z_{uo} = 1, \quad w z_{li} z_{lo} = 1. \tag{11}$$

• The upper/lower shingle equations. Consider an upper shingle region corresponding to an overarc going from some crossing labeled 1 to the crossing n, going through crossings $1, 2, \dots, n-1$ as overpasses. Then the shingle region has one corner for each of its ends, and 4 corners for each overpasses, as explained in Fig. 5. We get:

$$z_{1,lo} z'_{2,ui} z''_{2,uo} \cdots z'_{n-1,ui} z''_{n-1,uo} z_{n,li} z'_{n-1,uo} z''_{n-1,ui} \cdots z'_{2,uo} z''_{2,ui} = 1.$$

Lemma 2.1 *The upper/lower shingle equations have the equivalent forms, respectively:*

$$z_{n,lo} = z_{1,lo} w_n^{-1} \prod_{j=2}^{n-1} w_j, \quad z_{n,li} = z_{1,li} w_1 \prod_{j=2}^{n-1} w_j^{-1}. \tag{12}$$

$$z_{n,ui} = z_{1,ui} w_1 \prod_{j=2}^{n-1} w_j^{-1}, \quad z_{n,uo} = z_{1,uo} w_n^{-1} \prod_{j=2}^{n-1} w_j. \tag{13}$$

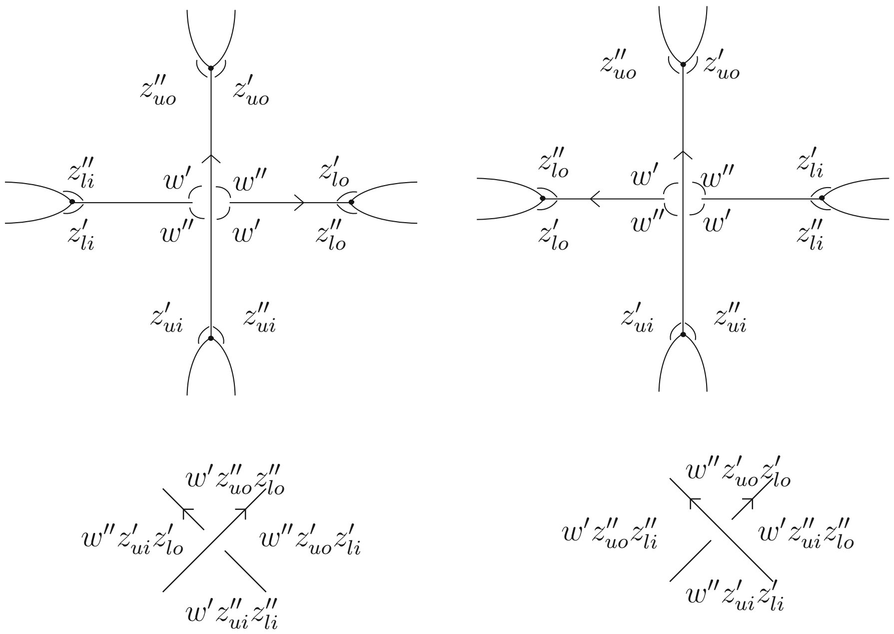


Fig. 6 On the top, a top view of the $5t$-spine near a positive and a negative crossing. On the bottom, the rule describing the corner factors

Proof Grouping together shape parameters coming from the same vertex and using $zz'z'' = -1$, we get:

$$z_{n,li} z_{1,lo} = \prod_{j=2}^{n-1} z_{j,ui} z_{j,uo}$$

and then, using Eq. (11):

$$z_{n,li} z_{1,lo} = \prod_{j=2}^{n-1} w_j^{-1}$$

Finally, using Eq. (11), we can rewrite this as Eq. (12) between only z'_{lo}s (or only z'_{li}s) parameters.

Similarly for a lower shingle region corresponding to an underarc running from crossing 1 to crossing n, one gets an equation:

$$z_{1,uo} z''_{2,li} z'_{2,lo} \ldots z''_{n-1,li} z'_{n-1,lo} z_{n,ui} z''_{n-1,lo} z'_{n-1,li} \ldots z''_{2,lo} z'_{2,li} = 1,$$

which simplifies to (13). □

• Figure 6 shows a top-view of the $5T$-spine near a crossing, as well as the shape parameters of horizontal corners of the spine. We see that each vertex of a region of

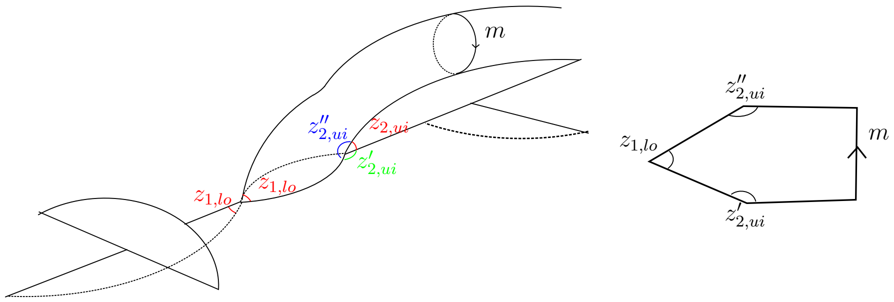


Fig. 7 The meridian positioned on top of overpass 2, and the left part of the region of the $5t$ spine that m intersects

K gives rise to 3 corners in the corresponding big region. For each region R_i of K, we get a big region equation of the form

$$\prod_{v \text{ corner of } R_i} f(v) = 1 \tag{14}$$

where the corner factors $f(v)$ are prescribed by the rule shown in Fig. 6.

Below, we will denote the triangle, region and shingle equations by t_i, r_k and s_j respectively. The above discussion defines the gluing equations variety $\mathcal{G}_D$ as an affine subvariety of $(\mathbb{C}^{**})^{5c(D)}$ defined by

$$\mathcal{G}_D = \{(w_c, z_{c,ui}, z_{c,uo}, z_{c,li}, z_{c,lo})_{c\in c(D)} \in (\mathbb{C}^{**})^{5c(D)} \mid t_i = 1, s_j = 1, r_k = 1\}. \tag{15}$$

We now express the holonomies $w_\mu = h_\mu$ and $w_\lambda = h_\lambda$ of the meridian μ and preferred longitude λ in terms of the above shape parameters. Note that if K is not the unknot, it is always possible to find in the diagram of K an underpass that is followed by an overpass that corresponds to a different crossing of K. We then name those two crossings 1 and 2. Assume that the meridian is positioned as shown in Fig. 7. Then the rule described in Sect. 2.1 gives us the following holonomy:

$$h_\mu = -z_{1,lo} z'_{2,ui} z''_{2,ui}.$$

As $z_{2,ui} z'_{2,ui} z''_{2,ui} = -1$, we get:

$$w_\mu = h_\mu = \frac{z_{1,lo}}{z_{2,ui}}. \tag{16}$$

Finally, we turn to the holonomy of a longitude. We first compute the holonomy of the longitude $\tilde{l}$ corresponding to the blackboard framing of the knot. We can represent this longitude on the diagram D as a right parallel of D. We draw this longitude on the spine in Fig. 8, we can see that it intersects each upper or lower shingle region in one segment.

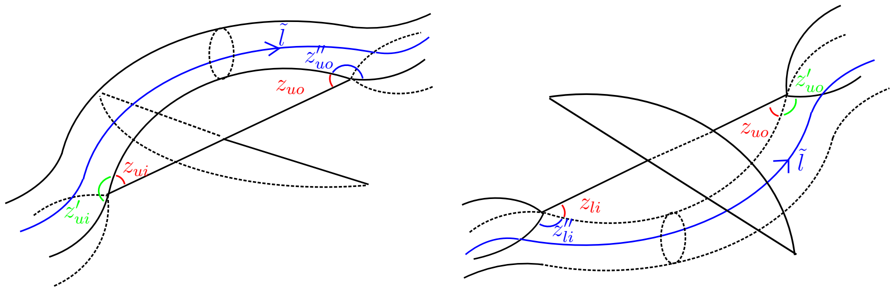


Fig. 8 The longitude $\tilde{l}$ on the $5t$-spine, and the shape parameters to the left (resp. to the right) of it on overpasses (resp. underpasses)

We compute the holonomy of each segment in an upper shingle using the convention

$$h_a = - \prod_{\text{c left corner}} z_c$$

and each lower shingle segment using the convention

$$h_a = - \prod_{\text{c right corner}} z_c^{-1}.$$

We can actually ignore the -1 signs as there are $2c(D)$ segments, an even number.

As Fig. 8 shows, we get:

$$h_{\tilde{\lambda}} = \prod_{\text{overarc } a} \prod_{\text{overpasses} \in a} z''_{uo} z'_{ui} \prod_{\text{underarc } a} \prod_{\text{underpasses} \in a} \frac{1}{z'_{lo} z''_{li}} = \prod_{X(D)} \frac{z''_{uo} z'_{ui}}{z'_{lo} z''_{li}}.$$

The last product is over the set $X(D)$ of crossings of D, and for simplicity we do not indicate the dependence of the variables on the crossing $c \in X(D)$. Let λ be the longitude with zero winding number with K. The winding number of the blackboard framing longitude $\tilde{\lambda}$ is the writhe $\mathrm{wr}(D)$ of the diagram D, which can be computed by $\mathrm{wr}(D) = c_+ - c_-$, where c_+ and c_- are the number of positive and negative crossings of the diagram. We then have $\tilde{\lambda} = \lambda \mu^{wr(D)}$ and thus

$$w_\lambda = h_\lambda = w_\mu^{-wr(D)} \prod_{X(D)} \frac{z''_{uo} z'_{ui}}{z'_{lo} z''_{li}}. \tag{17}$$

2.5 Labeled knot diagrams

In this section we introduce a labeling of the crossings in a knot diagram, closely related to the Dowker-Thistlethwaite notation of knots.

Recall that D is a planar diagram of an oriented knot K and that we have chosen two special crossings 1 and 2 that are successive in the diagram, such that such crossing 1

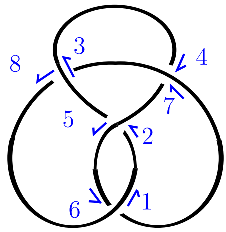


Fig. 9 A labeling of the crossings of a figure eight knot diagram. The 4 distinct crossings of the diagram have labels (1, 6), (2, 5), (3, 8) and (4, 7)

corresponds to an underpass and crossing 2 to an overpass. This choice determines a labeling of crossings of D as follows.

Following the knot, we label the other crossings $3, 4, \ldots$. Note that as the knot passes through each crossing twice, each crossing c of D gets two labels $j < j'$. Exactly one of those two labels correspond to the overpass and the other one to the underpass. Arcs of the diagram join two successive over- or underpasses labeled l and $l+1$ (or $2c(D)$ and 1). We write $[l, l+1]$ for the arc joining crossings l and $l+1$.

This labeling is illustrated in Fig. 9 in the case of the figure eight knot.

2.6 Analysis of triangle and shingle relations

In this section, we show that the triangle and shingle equations allow us to eliminate variables in the gluing variety $\mathcal{G}_D$. We have the following:

Proposition 2.2 *In $\mathcal{G}_D$, each of the variables $w_c, z_{c,li}, z_{c,lo}, z_{c,ui}, z_{c,uo}$ are monomials in the variables w_c, w_μ and $w_0 = z_{1,lo}$.*

Proof Fix a labeled knot diagram D as in Sect. 2.5. Before eliminating variables, we start by assigning to each arc $[l, l+1]$ of the diagram a new parameter $z_{l,l+1}$. These parameters are expressed in terms of the previous parameters by the following rules:

$$z_{1,2} = z_{1,lo} = w_0 \text{ and } z_{l,l+1} = z_{1,lo} \prod_{j \in [\![2,l]\!] \cap O(D)} w_j \prod_{j \in [\![2,l]\!] \cap U(D)} w_j^{-1}.$$

We recall that in the above $O(D)$ (resp. $U(D)$) is the set of overpasses (resp. underpasses) in the diagram D. Also, given integers $a, b \in \mathbb{Z}$ with $a \leq b$, we denote

$$[\![a, b]\!] = \{a, a+1, \ldots, b\}.$$

Note that the arc parameters $z_{l,l+1}$ are all clearly monomials in w_0 and the w_c's.

We claim that each of the shape parameters $z_{c,li}, z_{c,lo}, z_{c,ui}, z_{c,uo}$ are monomials in the $z_{l,l+1}$'s and w_μ. This will imply the proposition. Indeed, let $[k, k+1]$ be an arc of K. Then we claim that:

$$z_{k,k+1} = \begin{cases} z_{k,lo} & \text{if } k \text{ is an underpass} \\ \frac{1}{z_{k+1,li}} & \text{if } k+1 \text{ is an underpass} \\ \frac{w_\mu}{z_{k,uo}} & \text{if } k \text{ is an overpass} \\ \frac{z_{k+1,ui}}{w_\mu} & \text{if } k+1 \text{ is an overpass} \end{cases}$$

Note $z_{1,2} = z_{1,lo}$ by definition. If k is an underpass, the formula

$$z_{k,k+1} = z_{1,lo} \prod_{j \in [\![2,k]\!] \cap O(D)} w_j \prod_{j \in [\![2,k]\!] \cap U(D)} w_j^{-1}$$

matches with the upper shingle equation expressing $z_{k,lo}$ in terms of $z_{1,lo}$. Indeed, if k is the underpass coming immediately after underpass 1, Eq. (12) says:

$$z_{k,lo} = z_{1,lo} w_k^{-1} \prod_{j \in [\![2,k-1]\!]} w_j.$$

As crossings $2, 3, \ldots k-1$ correspond to overpasses and k to an underpass, we also have

$$z_{k,k+1} = z_{1,lo} w_k^{-1} \prod_{j \in [\![2,k-1]\!]} w_j.$$

By induction, we find that $z_{k,k+1} = z_{k,lo}$ for any underpass k.

The second case is then a consequence of the lower triangle equation $z_{k+1,li} = \frac{1}{w_{k+1} z_{k+1,lo}}$, and the fact that $z_{k,k+1} = z_{k+1,k+2} w_{k+1}$ as $k+1$ is an underpass.

Note that $z_{2,ui} = \frac{z_{1,lo}}{w_\mu}$ by Eq. (16), so the fourth case is valid for the arc $[1,2]$. Similarly to case 1, we can prove case 4 for other arcs ending in an overpass from the lower shingle equations by induction.

Finally, the third case follows as $z_{k,uo} = \frac{1}{w_k z_{k,ui}}$, and $z_{k,k+1} = w_k z_{k-1,k}$. □

In the rest of the paper, we will often use the arc parameters $z_{k,k+1}$ defined above to express equations in $\mathcal{G}_D$.

For instance, thanks to Proposition 2.2, we can rewrite the big region equations $r_k = 1$ as equations $r_k(w) = 1$, where $r_k(w)$ is expressed in terms of the variables w only.

Remark 2.3 Although the arc parameters $z_{l,l+1}$ are just monomials in the w variables, they are helpful for writing down the equations defining $\mathcal{G}_D$ in a more compact way. When the choice of a crossing c is implicit, we introduce a simplified notation for the parameters associated to arcs neighboring c. We will write $z_a, z_b, z_{a'}, z_{b'}$ for the parameters associated to the inward half of the overpass, inward half of underpass, outward half of underpass and outward half of underpass.

With this convention, at any crossing we have:

$$z_{ui} = \frac{z_a}{w_\mu}, \ z_{li} = \frac{1}{z_b}, \ z_{uo} = \frac{w_\mu}{z_{a'}}, \text{ and } z_{lo} = z_{b'}.$$

For instance, we get a new expression of the holonomy of the longitude:

Proposition 2.4 *With the convention of Remark 2.3, the holonomy of the zero-winding number longitude is expressed by:*

$$w_\lambda = w_\mu^{-wr(D)} \prod_{X(D)} w \left(\frac{1 - \frac{w_\mu}{z_{a'}}}{1 - \frac{w_\mu}{z_a}} \right) \left(\frac{1 - z_{b'}}{1 - z_b} \right). \tag{18}$$

Proof By Eq. (17) we have:

$$\begin{aligned} w_\lambda &= w_\mu^{-wr(D)} \prod_{X(D)} \frac{z''_{uo} z'_{ui}}{z'_{lo} z''_{li}} \\ &= w_\mu^{-wr(D)} \prod_{X(D)} \left(\frac{1 - \frac{z_{a'}}{w_\mu}}{1 - \frac{z_a}{w_\mu}} \right) \left(\frac{1 - z_{b'}}{1 - z_b} \right) \\ &= w_\mu^{-wr(D)} \prod_{X(D)} \frac{z_{a'}}{z_a} \left(\frac{1 - \frac{w_\mu}{z_{a'}}}{1 - \frac{w_\mu}{z_a}} \right) \left(\frac{1 - z_{b'}}{1 - z_b} \right) \\ &= w_\mu^{-wr(D)} \prod_{X(D)} w \left(\frac{1 - \frac{w_\mu}{z_{a'}}}{1 - \frac{w_\mu}{z_a}} \right) \left(\frac{1 - z_{b'}}{1 - z_b} \right). \end{aligned}$$

□

2.7 Analysis of big region equations

Recall that the big region equations are parametrized by the regions of the planar diagram D, i.e., by the connected components of $S^2 \backslash D$. In this section, we give an alternative set of equations which are parametrized by the crossings of D, and we call those the loop equations.

Our motivation comes from the fact that we will later match the loop equations with equations that come from a state sum formula for the colored Jones polynomial.

Consider a crossing c in the labeled diagram D. Recall from Sect. 2.5 that c has two labels $j < j'$. The arc $[j, j']$ starts and ends at the same crossing, hence one may close it up to obtain a loop γ_c. For a region R_i of the diagram, let us pick a point p_i in the interior of R_i. We write $w(\gamma_c, p_i)$ for the winding number of γ relative to the point p_i. The big region equation corresponding to the region R_i is $r_i = 1$, where r_i is the product of corners factors, see Eq. (14) and Fig. 6. The loop equation $L_c = 1$ is then defined by

$$L_c = \prod_{R_i \text{ region}} r_i^{w(\gamma_c, p_i)}. \tag{19}$$

We also introduce

$$L_0 = \prod_{R_i \text{ region of } D} r_i^{w(K,p_i)}. \tag{20}$$

Proposition 2.5 *The set of equations $L_0 = 1$, $L_c = 1$ for all $c \in X(D)$ is equivalent to the set of equations $r_i = 1$ for all region R_i of D.*

Proof The equations $L_0 = 1$, $L_c = 1$ are clearly implied by the big region equations $r_i = 1$ as the L_c's and L_0 are monomials in the r_i's. We will show that the r_i's are also monomials in L_0 and the L_c's, and thus equations $r_i = 1$ are a consequence of loop equations.

Let us consider the diagram D as an oriented 4-valent graph embedded in S^2. For any $\delta \in H_1(D, \mathbb{Z})$, we can also introduce a loop equation

$$L_\delta = \prod_{R_i \text{ region}} r_i^{w(\delta,p_i)}.$$

Note that $\delta \to L_\delta$ is a morphism of group $H_1(D, \mathbb{Z}) \to \mathbb{C}^*$ and that the equation r_i can be presented in this form too:

Indeed, chose $\delta = \partial R_i$ with positive orientation. Then $w(\delta, p_j) = 0$ if $j \neq i$, and $w(\delta, p_i) = 1$, hence $L_\delta = r_i$.

Thus we only need to prove that $H_1(D, \mathbb{Z})$ is generated by K and the classes γ_c. The diagram D has $c(D)$ vertices and $2c(D)$ edges, and thus $H_1(D, \mathbb{Z}) = \mathbb{Z}^{c(D)+1}$. So we need to show that K and the loops γ_c are a $\mathbb{Z}$-basis of $H_1(D, \mathbb{Z})$. To do this we first show that they are linearly independent in the space of 1-chains $C_1(D, \mathbb{Z})$.

Recall that we fixed a labeling of overpasses and underpasses in $[1, 2c(D)]$ following the knot K. Note that the arcs $[1, 2], [2, 3], \dots [2c(D), 1]$ give a basis of $C_1(D, \mathbb{Z})$. We order this basis with the convention $[1, 2] < [2, 3] < \cdots < [2c(D), 1]$.

Then $K = [1, 2] + [2, 3] + \cdots + [2c(D), 1]$ in $C_1(D, \mathbb{Z})$, and if a crossing c has labels $j < j'$, then $\gamma_c = [j, j+1] + \cdots + [j'-1, j']$.

We see that K is not in the space generated by the γ_c as it is the only one with non-zero coordinate along $[2c(D), 1]$.

Moreover, the loops γ_c are linearly independent as the indices of their first non-zero coordinates are all different.

So K and the γ_c are linearly independent in $H_1(D, \mathbb{Z})$, and thus a $\mathbb{Q}$-basis of $H_1(D, \mathbb{Q})$. We can actually show that they form a $\mathbb{Z}$-basis of $H_1(D, \mathbb{Z})$. Indeed if $\delta \in H_1(D, \mathbb{Z})$, we can subtract a $\mathbb{Z}$-linear combination of K and the γ_c's to δ to obtain an element with 0 coordinate on $[2c(D), 1]$ and each $[j, j+1]$ for each crossing with labels $j < j'$. This element has then to be zero as (K, γ_c) is a $\mathbb{Q}$-basis of $H_1(D, \mathbb{Q})$.

Thus K and the γ_c's generate $H_1(D, \mathbb{Z})$, and the r_i's are monomials in the L_0, L_c. $\square$

2.8 Formulas for the loop equations

In this section, we simplify the equations L_0, L_c which we defined as monomials in the big region equations. Our goal is to express those equations in terms of the arc parameters $z_{k,k+1}$ introduced in Sect. 2.6, which we recall are monomials in the w variables.

Proposition 2.6 *Let c be a crossing of D with labels $j < j'$. For $k \in [j, j']$, let $\varepsilon(k) = 1$ if k corresponds to a positive crossing and $\varepsilon(k) = -1$ otherwise. Let also $u_+(k) = \frac{1+\varepsilon(k)}{2}$ and $u_-(k) = \frac{1-\varepsilon(k)}{2}$. Then we have:*

$$L_c = K_c \prod_{k \in [\![j+1, j'-1]\!] \cap O(D)} \left(\frac{z_b^{u_-(k)}}{z_{b'}^{u_+(k)}} \right) \left(\frac{1 - \frac{w_\mu}{z_{a'}}}{1 - \frac{w_\mu}{z_a}} \right)$$
$$\times \prod_{k \in [\![j+1, j'-1]\!] \cap U(D)} w_\mu^{\varepsilon(k)} \left(\frac{z_a^{u_-(k)}}{z_{a'}^{u_+(k)}} \right) \left(\frac{1 - z_b}{1 - z_{b'}} \right), \tag{21}$$

where in the above we set

$$K_c = \begin{cases} \left(\frac{1}{z_{b'_c}}\right) \frac{\left(1 - \frac{w_\mu}{z_{a'_c}}\right)(1 - z_{b_c})}{(1 - w_c)} & \text{if } j \text{ is an overpass and} \varepsilon(c) = +1, \\ \left(-\frac{z_{a'_c}}{w_\mu}\right) \frac{\left(1 - \frac{w_\mu}{z_{a'_c}}\right)(1 - z_{b_c})}{(1 - w_c)} & \text{if } j \text{ is an overpass and} \varepsilon(c) = -1, \\ \left(\frac{w_\mu}{z_{a'_c}}\right) \frac{(1 - w_c)}{\left(1 - \frac{w_\mu}{z_{a_c}}\right)\left(1 - z_{b'_c}\right)} & \text{if } j \text{ is an underpass and} \varepsilon(c) = +1, \\ (-z_{b'_c}) \frac{(1 - w_c)}{\left(1 - \frac{w_\mu}{z_{a_c}}\right)(1 - z_{b'_c})} & \text{if } j \text{ is an underpass and} \varepsilon(c) = -1. \end{cases}$$

Proof We recall that γ is the loop obtained from the arc $[j, j']$ of D by gluing its two ends together. Let also γ' be the complementary loop of γ, which is obtained from the arc $[j', j]$ by gluing the two ends. Note that γ' goes through the underpass labeled 1.

As $L_c = \prod_{R_i \text{ region}} r_i^{w(\gamma, p_i)}$ is a product of big region equations, and each big region factor is a product of corner factors, we can rewrite L_c as a product of corner factors. Each corner v of D appears in one region R_i only , and the winding number $w(\gamma, v)$ of γ around v is the same as $w(\gamma, p_i)$. Thus we may rewrite L_c as

$$L_c = \prod_{v \text{ corner of } D} f(v)^{w(\gamma, v)},$$

where the corner factors $f(v)$ are those of Fig. 6.

Figure 10 shows the local pattern of winding numbers of corners near a crossing of D, depending which neighboring arcs belong to γ and γ'. First let us note for a crossing between two strands of γ', all local winding numbers are equal, thus the crossing contributes by the product of all 4 corners factors to some power. However, at any positive crossing, the product of corner factors is

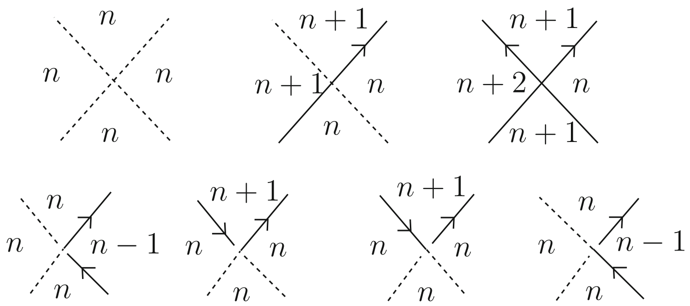


Fig. 10 The local pattern of winding numbers near a crossing. Strands of γ are represented by solid lines, strands of γ' by dashed lines. The bottom row corresponds to the 4 different possibilities for over/underpass j: positive overpass, negative overpass, positive underpass, or negative underpass

$$(w'z''_{uo}z''_{lo})(w''z'_{uo}z'_{li})(w'z''_{ui}z''_{li})(w''z'_{ui}z'_{lo}) = \frac{1}{w^2 z_{ui}z_{uo}z_{li}z_{lo}} = 1$$

by the rule $zz'z'' = -1$ and the triangle equations. Similarly, at any positive crossing, the product of corner factors is

$$(w''z'_{uo}z'_{lo})(w'z''_{uo}z''_{li})(w''z'_{ui}z'_{li})(w'z''_{ui}z''_{lo}) = \frac{1}{w^2 z_{ui}z_{uo}z_{li}z_{lo}} = 1.$$

So crossings between two strands of γ' do not contribute to L_c.

Next we consider a crossing between one strand of γ and one strand of γ'. By the local winding numbers shown in Fig. 10 and that fact that the product of the 4 corner factors at a crossing is 1, such a crossing contributes by the product of the two corner factors to the left of γ. Similarly, for a crossing between two strands of γ, we get the product of the two corner factors to the left of the first strand times tFig. he two corner factors to the left of the other strand.

Hence, each overpass or underpass $l \in [\![j+1, j'-1]\!]$ of γ contributes to one factor K_l which is the product of the two left corner factors. By the rule described in Fig. 6, for a positive overpass we get

$$\begin{aligned} K_l &= (w'z''_{uo}z''_{lo})(w''z'_{ui}z'_{lo}) = \frac{z'_{ui}z''_{uo}}{wz_{lo}} = z_{li}z'_{ui}z''_{uo} \\ &= \frac{1}{z_b}\left(\frac{1-\frac{z_{a'}}{w_\mu}}{1-\frac{z_a}{w_\mu}}\right) = \frac{z_{a'}}{z_a z_b}\left(\frac{1-\frac{w_\mu}{z_{a'}}}{1-\frac{w_\mu}{z_a}}\right) = \frac{1}{z_{b'}}\left(\frac{1-\frac{w_\mu}{z_{a'}}}{1-\frac{w_\mu}{z_a}}\right), \end{aligned}$$

where the last equality comes from the fact that, at any crossing, $\frac{z_{a'}}{z_a} = \frac{z_b}{z_{b'}} = w$. Similarly, at a negative overpass we get:

$$\begin{aligned} K_l &= (w'z''_{uo}z''_{li})(w''z'_{ui}z'_{li}) = \frac{z''_{uo}z'_{ui}}{wz_{li}} = z_{lo}z'_{uo}z''_{ui} \\ &= z_{b'}\left(\frac{1-\frac{z_{a'}}{w_\mu}}{1-\frac{z_a}{w_\mu}}\right) = \frac{z_{b'}z_{a'}}{z_a}\left(\frac{1-\frac{w_\mu}{z_{a'}}}{1-\frac{w_\mu}{z_a}}\right) = z_b\left(\frac{1-\frac{w_\mu}{z_{a'}}}{1-\frac{w_\mu}{z_a}}\right). \end{aligned}$$

At a positive underpass we get:

$$K_l = (w'' z'_{ui} z'_{lo})(w' z''_{ui} z''_{li}) = \frac{z'_{lo} z''_{li}}{w z_{ui}} = z_{uo} z'_{lo} z''_{li} = \frac{w_\mu}{z_{a'}} \left(\frac{1 - z_b}{1 - z_{b'}} \right),$$

and, finally, at a negative underpass we get:

$$K_l = (w' z''_{uo} z''_{li})(w'' z'_{uo} z'_{lo}) = \frac{z'_{lo} z''_{li}}{w z_{uo}} = z_{ui} z'_{lo} z''_{li} = \frac{z_a}{w_\mu} \left(\frac{1 - z_b}{1 - z_{b'}} \right).$$

All those overpass/underpass factors correspond to the ones in Eq. (21). Finally we turn to the contribution K_c of crossing c. By the local pattern of winding numbers in Fig. 10, and the corner factors rule of Fig. 6, we have, if j is a positive overpass:

$$\begin{aligned} K_c &= \frac{1}{w'' z'_{uo} z'_{li}} = \frac{(1 - \frac{w_\mu}{z_{a'}})(1 - \frac{1}{z_b})}{(1 - \frac{1}{w})} \\ &= \left(\frac{w}{z_b} \right) \frac{(1 - \frac{w_\mu}{z_{a'}})(1 - z_b)}{(1 - w)} = \left(\frac{1}{z_{b'}} \right) \frac{(1 - \frac{w_\mu}{z_{a'}})(1 - z_b)}{(1 - w)}. \end{aligned}$$

If j is a negative overpass, we have:

$$K_c = w' z''_{uo} z''_{li} = \left(\frac{1}{1 - w} \right) (1 - \frac{z_{a'}}{w_\mu})(1 - z_b) = \left(- \frac{z_{a'}}{w_\mu} \right) \frac{(1 - \frac{w_\mu}{z_{a'}})(1 - z_b)}{(1 - w)}.$$

If j is a positive underpass, then:

$$\begin{aligned} K_c &= w'' z'_{ui} z'_{lo} = \frac{(1 - \frac{1}{w})}{(1 - \frac{z_a}{w_\mu})(1 - z_{b'})} \\ &= \left(\frac{w_\mu}{w z_a} \right) \frac{1 - w}{(1 - \frac{w_\mu}{z_a})(1 - z_{b'})} = \left(\frac{w_\mu}{z_{a'}} \right) \frac{1 - w}{(1 - \frac{w_\mu}{z_a})(1 - z_{b'})}. \end{aligned}$$

Finally if j is a negative underpass, then:

$$K_c = \frac{1}{w' z''_{ui} z''_{lo}} = \frac{(1 - w)}{(1 - \frac{w_\mu}{z_a})(1 - \frac{1}{z_{b'}})} = (-z_{b'}) \frac{(1 - w)}{(1 - \frac{w_\mu}{z_a})(1 - z_{b'})}.$$

We clearly see that in each case the factor K_c matches with that of Proposition 2.6. □

We want to rearrange the loop equations slightly, grouping together the factors $w_\mu^{\varepsilon(k)}$ on the one side and the factors $\frac{z_b^{u_-(k)}}{z_{b'}^{u_+(k)}}$ and $\frac{z_a^{u_-(k)}}{z_{a'}^{u_+(k)}}$ on the other side. For the former we claim:

Lemma 2.7 *Let c be a crossing of D with labels $j < j'$, γ the loop $[j, j']/_{j=j'}$, and γ' the loop $[j', j]/_{j=j'}$. For $l \in (j, j')$ an over- or underpass, let $\varepsilon(l)$ be the sign of the corresponding crossing. Then we have*

$$\sum_{l \in U(D) \cap (j,j')} \varepsilon(l) = \sum_{l \in (j,j')} \frac{\varepsilon(l)}{2} = \mathrm{wr}(\gamma) + \mathrm{lk}(\gamma, \gamma').$$

Remark 2.8 By the above lemma, the factors $w_\mu^{\varepsilon(k)}$ in the product on the right of Eq. (21) group up to one factor $w_\mu^{\mathrm{wr}(\gamma)+\mathrm{lk}(\gamma,\gamma')}$.

Proof The crossings of D that are in (j, j') are of two types: self-crossings of γ and crossing between γ and γ'. Self-crossings of γ belong to both an overpass and an underpass $l \in (j, j')$, hence in both sums in the lemma, those crossings contribute to $c_+(\gamma) - c_-(\gamma) = \mathrm{wr}(\gamma)$.

Moreover the linking number of γ and γ' can be computed in two ways as $\sum_{l \in \gamma \cap \gamma'} \frac{\varepsilon(l)}{2}$ or as $\sum_{l \in \gamma \cap \gamma' \cap U(D)} \varepsilon(l)$. Thus hence in both sums in the lemma mixed crossings contribute to $\mathrm{lk}(\gamma, \gamma')$. □

Lemma 2.9 *Let c be a crossing of D with labels $j < j'$. Then:*

$$\begin{aligned}\prod_{k \in (j,j') \cap O(D)} \frac{z_b^{u_-(k)}}{z_b'^{u_+(k)}} \prod_{k \in (j,j') \cap U(D)} \frac{z_a^{u_-(k)}}{z_{a'}^{u_+(k)}} \\ = C \prod_{k \in (j,j') \cap O(D)} (z_b z_{b'})^{-\frac{\varepsilon(k)}{2}} \prod_{k \in (j,j') \cap U(D)} (z_a z_{a'})^{-\frac{\varepsilon(k)}{2}},\end{aligned}$$

where $C = \left(\frac{z_{b_c}}{z_{a'_c}}\right)^{\frac{1}{2}} = \left(\frac{z_{b'_c}}{z_{a_c}}\right)^{\frac{1}{2}}$ if j is an overpass and $C = \left(\frac{z_{a_c}}{z_{b'_c}}\right)^{\frac{1}{2}} = \left(\frac{z_{a'_c}}{z_{b_c}}\right)^{\frac{1}{2}}$.

Proof We have by definition of $u_+(k)$ and $u_-(k)$:

$$\frac{z_b^{u_-(k)}}{z_{b'}^{u_+(k)}} = \left(\frac{z_b}{z_{b'}}\right)^{\frac{1}{2}} (z_b z_b')^{-\frac{\varepsilon(k)}{2}}, \text{ and } \frac{z_a^{u_-(k)}}{z_{a'}^{u_+(k)}} = \left(\frac{z_a}{z_{a'}}\right)^{\frac{1}{2}} (z_a z_a')^{-\frac{\varepsilon(k)}{2}}.$$

Moreover, as at any crossing $\frac{z_b}{z_{b'}} = \frac{z_{a'}}{z_a} = w$, we have:

$$\begin{aligned}\prod_{k \in (j,j') \cap O(D)} \left(\frac{z_b}{z_{b'}}\right)^{\frac{1}{2}} \prod_{k \in (j,j') \cap U(D)} \left(\frac{z_a}{z_{a'}}\right)^{\frac{1}{2}} &= \prod_{k \in (j,j') \cap O(D)} \left(\frac{z_{a'}}{z_a}\right)^{\frac{1}{2}} \prod_{k \in (j,j') \cap U(D)} \left(\frac{z_{b'}}{z_b}\right)^{\frac{1}{2}} \\ &= \prod_{k \in (j,j')} \left(\frac{z_{k,k+1}}{z_{k-1,k}}\right)^{\frac{1}{2}} = \left(\frac{z_{j'-1,j'}}{z_{j,j+1}}\right)^{\frac{1}{2}}.\end{aligned}$$

Finally, if j is an overpass then $\left(\frac{z_{j'-1,j'}}{z_{j,j+1}}\right)^{\frac{1}{2}} = \left(\frac{z_{b_c}}{z_{a'_c}}\right)^{\frac{1}{2}}$ as $z_{a'_c} = z_{j,j+1}$ and $z_{b_c} = z_{j'-1,j'}$. Similarly, $\left(\frac{z_{j'-1,j'}}{z_{j,j+1}}\right)^{\frac{1}{2}} = \left(\frac{z_{a_c}}{z_{b'_c}}\right)^{\frac{1}{2}}$ if j is an underpass. □

From Proposition 2.6 together with Lemmas 2.7 and 2.9 , we obtain another formula for the loop equation:

Proposition 2.10 *Let c be a crossing of D with labels $j < j'$ and let L_c be the associated loop equation. If $k \in (j, j')$, let $\varepsilon(k)$ be the sign of the corresponding crossing. Then:*

$$L_c = K'_c \prod_{k \in (j,j') \cap O(D)} \left(\frac{w_\mu}{z_b z_{b'}}\right)^{\frac{\varepsilon(k)}{2}} \left(\frac{1 - \frac{w_\mu}{z_{a'}}}{1 - \frac{w_\mu}{z_a}}\right) \times \prod_{k \in (j,j') \cap U(D)} \left(\frac{w_\mu}{z_a z_{a'}}\right)^{\frac{\varepsilon(k)}{2}} \left(\frac{1 - z_b}{1 - z_{b'}}\right) \tag{22}$$

where K'_c is obtained from K_c of Proposition 2.6 by replacing respectively a factor $\left(\frac{1}{z_{b'_c}}\right)$, $z_{a'_c}$, $\left(\frac{1}{z_{a'_c}}\right)$, or $z_{b'_c}$ by $\frac{1}{(z_{a_c} z_{b'_c})^{\frac{1}{2}}}$, $(z_{a'_c} z_{b_c})^{\frac{1}{2}}$, $\frac{1}{(z_{a'_c} z_{b_c})^{\frac{1}{2}}}$, or $(z_{a_c} z_{b'_c})^{\frac{1}{2}}$ if j is a positive overpass, a negative overpass, a positive underpass or a negative underpass.

Finally, we turn to the expression of the last loop equation $L_0 = \prod_{R_i \text{ region}} r_i^{w(K,R_i)}$ that we introduced in Section 2.7.

Proposition 2.11 *We have the formula:*

$$L_0 = \prod_{c \in X(D)} \left(\frac{w_\mu}{z_a z_b}\right)^{\varepsilon(c)} \frac{(1 - \frac{w_\mu}{z_{a'}})(1 - z_b)}{(1 - \frac{w_\mu}{z_a})(1 - z_{b'})}$$

Proof We proceed similarly as in the proof of 2.6. As we are taking the whole knot K instead of one of the loops γ_c, the local pattern of winding numbers at any crossing looks like the third drawing in Fig. 10.

By the corner factor rule of Fig. 6, we get a factor

$$\frac{z'_{ui} z'_{lo}}{z'_{uo} z'_{li}} = \frac{(1 - \frac{w_\mu}{z_{a'}})(1 - \frac{1}{z_b})}{(1 - \frac{z_a}{w_\mu})(1 - z_{b'})} = \left(\frac{w_\mu}{z_a z_b}\right) \frac{(1 - \frac{w_\mu}{z_{a'}})(1 - z_b)}{(1 - \frac{w_\mu}{z_a})(1 - z_{b'})}$$

at a positive crossing and a factor:

$$\frac{z''_{uo}z''_{li}}{z''_{ui}z''_{lo}} = \frac{(1-\frac{z_{a'}}{w_\mu})(1-z_b)}{(1-\frac{w_\mu}{z_a})(1-\frac{1}{z_{b'}})}$$
$$= \left(\frac{z_{a'}z_{b'}}{w_\mu}\right)\frac{(1-\frac{w_\mu}{z_{a'}})(1-z_b)}{(1-\frac{w_\mu}{z_a})(1-z_{b'})} = \left(\frac{z_a z_b}{w_\mu}\right)\frac{(1-\frac{w_\mu}{z_{a'}})(1-z_b)}{(1-\frac{w_\mu}{z_a})(1-z_{b'})},$$

at a negative crossing, using that $z_a z_b = z_{a'} z_{b'}$ at any crossing. □

2.9 A square root of the holonomy of the longitude

In this section, we show that the holonomy of the longitude w_λ admits a square root in $\mathbb{C}[\mathcal{G}_D]$. We prove the following.

Proposition 2.12 *Let s be defined by*

$$s = \prod_{X(D)} \frac{(1-\frac{w_\mu}{z_a})}{(1-\frac{w_\mu}{z_{a'}})} w^{-1/2} (z_a z_b)^{\frac{\varepsilon(c)}{2}}. \tag{23}$$

Then $s \in \mathbb{C}(w_\mu, w_0, w_c)$ and $s^2 = \frac{1}{w_\lambda L_0}$.

Proof By Eq. (18),

$$w_\lambda = \prod_{X(D)} w_\mu^{-\varepsilon(c)} w \left(\frac{1-\frac{w_\mu}{z_{a'}}}{1-\frac{w_\mu}{z_a}}\right)\left(\frac{1-z_{b'}}{1-z_b}\right),$$

and by Eq. (22):

$$L_0 = \prod_{X(D)} \left(\frac{w_\mu}{z_a z_b}\right)^{\varepsilon(c)} \left(\frac{1-\frac{w_\mu}{z_{a'}}}{1-\frac{w_\mu}{z_a}}\right)\left(\frac{1-z_b}{1-z_{b'}}\right).$$

Those two equations clearly imply that $s^2 = \frac{1}{w_\lambda L_0}$. The non-trivial part is to show that s is actually in $\mathbb{C}(w_\mu, w_0, w_c)$, which is equivalent to showing the degree of the monomial $\prod_{X(D)} w z_a z_b$ is even in each of the variable w_μ, w_0 and w_c.

First we note that all arc parameters z_a, z_b have degree 0 along w_μ and degree 1 along w_0. So what we need to show is that the product $\prod_{X(D)} z_a z_b$ has odd degree along each variable w_c associated to a crossing. We remark that this product is also the product of all arc parameters as each arc is an inward arc of exactly one crossing.

Let c be a crossing with labels $j < j'$. Then for any arc $[k, k+1]$ the arc parameter $z_{k,k+1}$ is of the form $z_{k,k+1} = w_0 w_c^{\varepsilon} \prod_{c' \neq c} w_{c'}^{\varepsilon_{c'}}$, where $\varepsilon \in \{-1, 0, 1\}$, and $\varepsilon \neq 0$ if and only if $[k, k+1] \subset [j, j']$. So all we have to show is that $j' - j$ is always odd for any crossing c. The reason is that the loop $\gamma = [j, j']/_{j \sim j'}$ has $j' - j - 1$

intersection points with the rest of K, and those intersection points bound a collection of segments, which are the intersection of K with a disk bounded by γ. So $j' - j - 1$ is always even. $\square$

3 q-holonomic functions, creative telescoping and certificates

In this section we recall some properties of q-holonomic functions, creative telescoping and certificates, which we will combine with a state sum formula for the colored Jones polynomial to prove our main Theorem 1.1. Recall that a q-holonomic function $f : \mathbb{Z} \to \mathbb{Q}(q)$ is one that satisfies a non-zero recursion relation of the form (1), i.e., a function with annihilator (3) satisfying $\mathrm{Ann}(f) \neq 0$. q-holonomic functions of several variables are defined using a notion of Hilbert series dimension, and are closed under sums, products as well as summation of some of their variables. Building blocks of q-holonomic functions are the proper q-hypergeometric functions of [37]. For a detailed discussion of q-holonomic functions, we refer the reader to the survey article [17].

The following proposition is the fundamental theorem of q-holonomic functions. When F is proper q-hypergeometric, a proof was given in Wilf-Zeilberger [37]. A detailed proof of the next proposition, as well as a self-contained introduction to q-holonomic functions, we refer the reader to [17].

Proposition 3.1 (a) *Proper q-hypergeometric functions are q-holonomic.*

(b) *Let $F : \mathbb{Z}^{r+1} \to \mathbb{Q}(q)$ be q-holonomic in the variables $(n, k) \in \mathbb{Z} \times \mathbb{Z}^r$ such that $F(n, \cdot)$ has finite support for any n and let $f : \mathbb{Z} \to \mathbb{Q}(q)$ be defined by*

$$f(n) = \sum_{k \in \mathbb{Z}^r} F(n, k).$$

Then f is q-holonomic.

The above proposition combined with an R-matrix state-sum formula for the colored Jones polynomial implies that the colored Jones polynomial of a knot (or link, colored by representations of a fixed simple Lie algebra) is q-holonomic [15].

With the notation of the above proposition, a natural question is how to compute $\mathrm{Ann}(f)$ given $\mathrm{Ann}(F)$. This is a difficult problem practically unsolved. However, an easier question can be solved: namely given $\mathrm{Ann}(F)$, how to compute a nonzero element in $\mathrm{Ann}(f)$. The answer to this question is given by certificates, which are synonymous to the method of creative telescoping, coined by Zeilberger [39]. The latter aims at computing recursions for holonomic functions obtained by summing/integrating all but one variables. For a detailed discussion and applications, see [30,37] and also [3].

Proposition 3.2 (a) *Let F and f be as in Proposition 3.1, and consider the map φ from (6). Let*

$$P \in \mathrm{Ann}(F) \cap \mathbb{Q}[q, Q]\langle E, E_i \rangle. \tag{24}$$

Then $\varphi(P) \in \mathrm{Ann}(f)$.

(b) *There exists* P *as above with* $\varphi(P) \neq 0$.

Nonzero elements P as in (24) are called "certificates", and those that satisfy $\varphi(P) \neq 0$ are called "good certificates". Certificates are usually computed in the intersection $\mathrm{Ann}(F) \cap \mathbb{Q}(q, Q)\langle E, E_i \rangle$, where membership reduces to a linear algebra question over the field $\mathbb{Q}(q, Q)$ and then lifted to the ring $\mathbb{Q}[q, Q]\langle E, E_i \rangle$ by clearing denominators.

Part (b) is shown in Zeilberger [38] and in detail in Koutschan's thesis [23, Thm.2.7]. In the latter reference, this is called the "elimination property" of holonomic ideals. Part (a) is easy and motivates the name "creative telescoping". Indeed, one may write

$$P(E, Q, E_i) = \tilde{P}(E, Q) + \sum_{i=1}^{d} (E_i - 1) R_i(E, Q, E_i).$$

A recurrence relation of this form is also called a certificate. After expanding the sum $\sum_{k \in \mathbb{Z}^d} P(E, Q, E_i) F(n, k) = 0$, the terms

$$\sum_{k \in \mathbb{Z}^d} (E_i - 1) R_i(E, Q, E_i) F(n, k),$$

are telescoping sums and thus equal to 0. Finally, note that when F is proper q-hypergeometric, an operator P as above may be found by using its monomials as unknowns and solving a system of linear equations of PF/F. Hence, once P is found (and that is the difficult part), it is easy to check that it satisfies the relation $PF = 0$, which reduces to an identity in a field of finitely many variables–hence the name "certificate".

Part (b) follows by multiplying an element of $\mathrm{Ann}(F)$ on the left if necessary by a monomial in Q_i. We thank Koutschan for pointing this out to us.

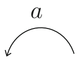
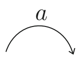
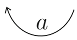
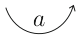
4 The colored Jones polynomial of a knot

4.1 State sum formula for the colored Jones polynomial of a knot diagram

In this section, we use a diagram D of an oriented knot K to give a (state sum) formula for the n-th colored Jones polynomial $J_K(n) \in \mathbb{Z}[q^{\pm 1}]$ of K. Such a formula is obtained by placing an R-matrix at each crossing, coloring the arcs of the diagram with integers, and contracting tensors as described for instance in Turaev's book [34]. The formula described in this section follows the conventions introduced in [16]; we also refer to [16] for all proofs.

For $n \geq 0$, we define the n-th quantum factorial by

$$(q)_n = \prod_{i=1}^{n} (1 - q^i).$$

Part				
Weight	$q^{\frac{2a-n}{4}}$	$q^{-\frac{(2a-n)}{4}}$	$q^{-\frac{(2a-n)}{4}}$	$q^{\frac{2a-n}{4}}$

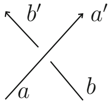
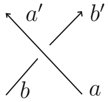
Part		
Weight	$q^{\frac{(n+na+nb'-a'b'-ab)}{2}} \frac{(q)_{n-a}}{(q)_{n-a'}} \frac{(q)_b}{(q)_{b'}(q)_k}$	$(-1)^k q^{\frac{(-n-na'-nb+a'b+ab'-a'+a)}{2}} \frac{(q)_{n-a}}{(q)_{n-a'}} \frac{(q)_b}{(q)_{b'}(q)_k}$

Fig. 11 The local parts X of D, their arc-colors r and their weights $w(X, r)$

Note that quantum factorials satisfy the recurrence relation $(q)_{n+1} = (1 - q^{n+1})(q)_n$ for any $n \geq 0$. As it will be helpful for us to have recurrence relations that are valid for any $n \in \mathbb{Z}$, we will use the following convention of quantum factorials and their inverses:

$$(q)_n = \begin{cases} \prod_{j=1}^{n}(1-q^i) & \text{if } n \geq 0, \\ 0 & \text{if } n < 0, \end{cases} \qquad \frac{1}{(q)_n} = \begin{cases} \frac{1}{\prod_{i=1}^{n}(1-q^i)} & \text{if } n \geq 0, \\ 0 & \text{if } n < 0. \end{cases}$$

With the above definition and with the notation of (2) we have:

$$(1 - qQ)(E - (1 - qQ)) \in \operatorname{Ann}((q)_n), \qquad ((1 - qQ)E - 1) \in \operatorname{Ann}(1/(q)_n).$$

Fix a labeled diagram D of an oriented knot K as in Sect. 2.5. After possibly performing a local rotation, one can arrange D so that at each crossing the two strands of K are going upwards. The diagram D is then composed of two types of pieces: the crossings (which can be possible or negative) and local extrema. Let $\operatorname{arc}(D)$ be the set of arcs of the diagram D, we say that a coloring

$$r : \operatorname{arc}(D) \longrightarrow \mathbb{Z}$$

is n-admissible if the color of any arc is in $[0, n]$ and for any crossing, if a, a', b, b' are the color of the neighboring arcs in shown in Fig. 11, then $a' - a = b - b' = k \geq 0$. Let $S_{D,n}$ be the set of all n-admissible colorings of the arcs of D. Note that $S_{D,n}$ coincides with the set of lattice points in the n-th dilatation of a rational convex polytope P_D defined by the n-admissibility conditions.

For a proof of the next proposition, we refer to [16, Sec.2].

Proposition 4.1 *The normalized n-th colored Jones polynomial of K is obtained by the formula:*

$$J_K(n) = q^{n/2} \sum_{r \in S_{D,n}} \widetilde{w}_D(n, r), \tag{25}$$

where $\widetilde{w}_D(n,r) = \prod_{X \text{ piece}} w(X,r)$ *is a product of weights associated to crossings and extrema of* D *as shown in Fig.* 11.

The insertion of the factor $q^{n/2}$ in front of the above sum is done for convenience only, so that $J_K(n)$ is a Laurent polynomial in q rather than one in $q^{1/2}$. This normalization plays no role in the AJ Conjecture . Note that we have $J_K(0) = 1$ for every knot K and $J_{\text{Unknot}}(n) = \frac{1-q^{n+1}}{1-q}$ for any $n \geq 0$ and $J_K(1, q^{-1})/J_{\text{Unknot}}(1, q^{-1})$ is the Jones polynomial of K.

Note that the color of all arcs are completely determined by the shifts $(k_1, \dots, k_{c(D)}) \in \mathbb{Z}^{c(D)}$ associated to crossings and the color k_0 of the arc $[1, 2]$. In other words, $r = r(k)$ is a linear function of $k = (k_0, \dots, k_{c(D)}) \in \mathbb{Z}^{c(D)+1}$. Suppressing the dependence on q, we introduce the notation

$$w_D(n,k) = q^{n/2}\widetilde{w}_D(n, r(k))(q).$$

When examining recurrence relations for the colored Jones it will be more convenient to express $J_K(n)$ as a sum over all $k \in \mathbb{Z}^{c(D)+1}$ rather than a sum over colorings r in the set $S_{D,n}$ of lattice points in the rational convex polytope P_D. For this we have the lemma:

Lemma 4.2 *For any knot* K*, we have:*

$$J_K(n) = \sum_{k \in \mathbb{Z}^{c(D)+1}} w_D(n,k). \tag{26}$$

Proof We recall that we have set the convention $\frac{1}{(q)_n} = 0$ if $n < 0$. From the definition of weights associated to crossings, we see that at any crossing the weight vanishes unless $k \geq 0$, $b' \geq 0$ and $a' \leq n$.

Pick a coloring so that the associated weight is non-zero. Consider the color $c_{i,i+1}$ of the arc $[i, i+1]$. If i is an underpass, then we get that $c_{i,i+1} \geq 0$. If on the other hand i is an overpass, then $c_{i,i+1} = c_{i-1,i} + k_i$, so $c_{i,i+1} \geq c_{i-1,i}$. If $i - 1$ is an underpass, one concludes that $c_{i,i+1} \geq c_{i-1,i} \geq 0$, else, one can continue until we meet an underpass k, and write

$$c_{i,i+1} \geq c_{i-1,i} \geq \cdots \geq c_{k,k+1} \geq 0.$$

Thus if the weight is non-zero, the color of all arcs must be non-negative.

Similarly, we can show that the color of all arcs muss be at most n. We already know that $c_{i,i+1} \leq n$ if i is an overpass. Else, if k is the overpass immediately before i, we have

$$c_{i,i+1} \leq c_{i-1,i} \cdots \leq c_{k,k+1} \leq n.$$

Thus any non-zero weight corresponds to an element of $S_{D,n}$. $\square$

4.2 The annihilator ideal of the summand of the state sum

It is easy to see that the summand $w_D(n, k)$ of the state sum (25) is a q-proper hypergeometric function in the sense of [37]. In this section we compute generators of its annihilator ideal. To do so, we compute the effect of the shift operators E, E_0 and E_c on $w_D(n, k)$. Each operator is acting on exactly one of the $c(D) + 2$ variables (n, k) leaving all others fixed.

- E shifts n to $n + 1$.
- E_0 shifts k_0 to $k_0 + 1$. As the color of any other arc of D is of the form $k_0 + \sum_{c \in X(D)} \varepsilon_c k_c$ with $\varepsilon_c \in \{-1, 0, 1\}$, the operator E_0 actually shifts the color of all arcs up by 1.
- E_c for each crossing c shifts k_c to $k_c + 1$.

The propositions of this section will match, after setting $q = 1$, with the gluing equations of the $5T$-spine of the knot projection.

Because we will later reduce our equations by plugging $q = 1$, it will only matter to us that they are exact up to fixed powers of q. We will write q^* for a power of q which does not depend on (n, k).

Let us start by considering the effect of E_0 on w.

Proposition 4.3 *The summand $w_D(n, k)$ of the colored Jones polynomial satisfies:*

$$\frac{E_0 w_D(n, k)}{w_D(n, k)} = q^* \prod_{c \in X(D)} \left(\frac{q^n}{q^a q^b}\right)^{\varepsilon(c)} \frac{(1 - q^{n-a'})(1 - q^{b+1})}{(1 - q^{n-a})(1 - q^{b'+1})} \tag{27}$$

Remark 4.4 The denominators in the above equations actually vanish if $k \notin S_{D,n}$. To obtain recurrence relations that are valid for any (n, k), we can simply move each denominator to the other side of the equation. The convention $\frac{1}{(q)_i} = 0$ if $i < 0$ will ensure that the equations still hold.

Proof Let us note first that the weights of local extrema are linear powers of q. When computing the ratio $\frac{E_0 w_D(n,k)}{w_D(n,k)}$ those weights will only contribute to a q^* factor. Thus we can discard those weights while trying to prove Proposition 4.3. We can also discard any linear power q from the weights of crossing, as well as the contribution of the factor $q^{n/2}$ in w_D for the same reason.

We also note that one can separate the weights $w(c)$ of crossings into a product of two factors $w_>(c)$ and $w_<(c)$, where

$$w_>(c) = \frac{(q)_{n-a}}{(q)_{n-a'}} \frac{(q)_b}{(q)_{b'}(q)_k}$$

and

$$w_<(c) = \begin{cases} q^{(n+na+nb'-a'b'-ab)/2} \text{ if } \varepsilon(c) = +1, \\ (-1)^k q^{(-n-na'-nb+a'b+ab')/2} \text{ if } \varepsilon(c) = -1. \end{cases}$$

where a, a', b, b' are the colors of arcs neighboring the crossing c, following the convention described in Fig. 11.

Recall that E_0 shifts the color of all arcs up by 1. Up to q^*, the ratio $\frac{E_0 w_D(n,k)}{w_D(n,k)}$ is a product of factors $\mu(c) = \frac{E_0 w_>(c)}{w_>(c)}$ and $\nu(c) = \frac{E_0 w_<(c)}{w_<(c)}$ for every crossing. We compute that:

$$\mu(c) = \frac{(q)_{n-a-1}(q)_{n-a'}}{(q)_{n-a}(q)_{n-a'-1}} \frac{(q)_{b+1}(q)_{b'}}{(q)_b(q)_{b'+1}} = \frac{(1-q^{n-a'})(1-q^{b+1})}{(1-q^{n-a})(1-q^{b'+1})},$$

and

$$\nu(c) = q^* \frac{q^{(n(a+1)+n(b'+1)-(a'+1)(b'+1)-(a+1)(b+1))/2}}{q^{(na+nb'-a'b'-ab)/2}} = q^* \frac{q^n}{q^{(a+a'+b+b')/2}} = q^* \frac{q^n}{q^{a+b}},$$

if c is positive and

$$\nu(c) = q^* \frac{q^{(-n(a'+1)-n(b+1)+(a'+1)(b+1)+(a+1)(b'+1))/2}}{q^{(-na'-nb+a'b+ab')/2}} = q^* \frac{q^{(a+a'+b+b')/2}}{q^n} = q^* \frac{q^{a+b}}{q^n},$$

if c is negative. This gives Eq. (27). □

Let us now turn to the effect of operator E.

Proposition 4.5 *The summand $w_D(n,k)$ of the colored Jones state sum satisfies:*

$$\frac{E w_D(n,k)}{w_D(n,k)} = q^* \prod_{X(D)} q^{\varepsilon(c)\left(\frac{a+b}{2}\right)-\frac{k}{2}} \left(\frac{1-q^{n+1-a}}{1-q^{n+1-a'}}\right). \tag{28}$$

Proof Again, we can safely ignore the contribution of weights of local extrema and any linear power of q in the weights of crossings as they just contribute to a q^* factor. First, note that the effect of E is to shift n up by 1 and leave the colors of all arcs invariant. Then, as in the previous Proposition, any crossing c contributes to the ratio by the product of two factors $\mu(c)$ and $\nu(c)$, where

$$\mu(c) = \frac{E w_>(c)}{w_>(c)} = \frac{(q)_{n+1-a}(q)_{n-a'}}{(q)_{n+1-a'}(q)_{n-a}} = \frac{(1-q^{n+1-a})}{(1-q^{n+1-a'})},$$

and

$$\nu(c) = \frac{E w_<(c)}{w_<(c)} = q^* \frac{q^{\frac{(n+1)a+(n+1)b'-a'b'-ab}{2}}}{q^{\frac{na+nb'-a'b'-ab}{2}}} = q^* q^{(a+b')/2} = q^* q^{(a+b)/2-k/2},$$

as $b' = b - k$, if c is a positive crossing. For c a negative crossing, we have:

$$\nu(c) = q^* \frac{(-1)^k q^{\frac{(-(n+1)a' - (n+1)b + a'b + ab')}{2}}}{(-1)^k q^{\frac{(-na' - nb + a'b + ab')}{2}}} = q^* q^{(-a'-b)/2} = q^* q^{-(a+b)/2 - k/2}.$$

as $a' = a + k$. Combining the factors $\mu(c)$ and $\nu(c)$ we get Eq. (28). □

Proposition 4.6 *Fix a labeled diagram D as in Sect. 2.5. Let c be a crossing of D with labels $j < j'$. Then the summand $w_D(n, k)$ of the colored Jones polynomial satisfies:*

$$\frac{E_c w_D(n,k)}{w_D(n,k)} = q^* F_c \prod_{l \in O(D) \cap (j,j')} \left(\frac{q^n}{q^b q^{b'}}\right)^{\frac{\varepsilon(l)}{2}} \frac{1 - q^{n-a'}}{1 - q^{n-a}} \prod_{l \in U(D) \cap (j,j')} \left(\frac{q^n}{q^a q^{a'}}\right)^{\frac{\varepsilon(l)}{2}} \frac{1 - q^{b+1}}{1 - q^{b'+1}}, \tag{29}$$

if j is an overpass and

$$\frac{E_c w_D(n,k)}{w_D(n,k)} = q^* F_c \prod_{l \in O(D) \cap (j,j')} \left(\frac{q^n}{q^b q^{b'}}\right)^{-\frac{\varepsilon(l)}{2}} \frac{1 - q^{n+1-a}}{1 - q^{n+1-a'}} \prod_{l \in U(D) \cap (j,j')} \left(\frac{q^n}{q^a q^{a'}}\right)^{-\frac{\varepsilon(l)}{2}} \frac{1 - q^{b'}}{1 - q^{b}}, \tag{30}$$

if j is an underpass. In the above, we set

$$F_c = \begin{cases} q^{-\frac{a_c + b'_c}{2}} \left(\frac{(1-q^{b_c+1})(1-q^{n-a'_c})}{1-q^{k_c+1}}\right) & \text{if } j \text{ is an overpass and} \varepsilon(c) = +1, \\ -q^{\frac{a'_c + b_c}{2} - n} \left(\frac{(1-q^{b_c+1})(1-q^{n-a'_c})}{1-q^{k_c+1}}\right) & \text{if } j \text{ is an overpass and} \varepsilon(c) = -1, \\ q^{\frac{a'_c + b_c}{2} - n} \left(\frac{(1-q^{b'_c})(1-q^{n-a_c+1})}{1-q^{k_c+1}}\right) & \text{if } j \text{ is an underpass and} \varepsilon(c) = +1, \\ -q^{-\frac{a_c + b'_c}{2}} \left(\frac{(1-q^{b'_c})(1-q^{n-a_c+1})}{1-q^{k_c+1}}\right) & \text{if } j \text{ is an underpass and} \varepsilon(c) = -1\,. \end{cases}$$

Proof Let c be a crossing with labels $j < j'$. The effect of E_c is to shift k_c up by 1. Note that the colors of arcs $[k, k+1] \subset [1, j] \cup [j', 1]$ do not depend on k_c, while the colors of arcs $[k, k+1] \subset [j, j']$ are of the form $c_0 + \varepsilon k_c$, where c_0 does not depend on k_c and $\varepsilon = 1$ if j is an overpass, $\varepsilon = -1$ else. Thus the effect of E_c is to shift the colors of arcs in $[j, j']$ up by 1 (if j is an overpass) or down by 1 (if j is an underpass).

As before we neglect the weights of local extrema and any linear power q in the weights of crossings. Let us write a, a', b, b' for the colors of the arcs neighboring a crossing $c' \in (j, j')$ with labels $l < l'$, let $k = a' - a = b - b'$.

First we note that the weights $w_>(c') = \frac{(q)_{n-a}}{(q)_{n-a'}} \frac{(q)_b}{(q)_{b'}(q)_k}$ can separated into a factor $w_>(l) = \frac{(q)_{n-a}}{(q)_{n-a'}}$ associated to the overpass l and a factor $w_>(l') = \frac{(q)_b}{(q)_{b'}(q)_k}$ associated to the underpass l'. The weights $w_<(c')$ are not separable in the same way; however the ratios $\nu(c') = \frac{E_c w_<(c')}{w_<(c')}$ are linear powers of q and thus we can compute those factors up to q^* as a product of two factors $\nu(l), \nu(l')$, where in $\nu(l)$ we apply the shift only to the colors a, a' and in $\nu(l')$ we apply the shift only to the colors b, b'.

Now we compute the factors $\mu(l) = \frac{E_c w_>(l)}{w_>(l)}$ and $\nu(l)$ associated to over- or underpasses.

Note that if $l \notin [\![j, j']\!]$, then no arc of the over- or underpass l has its color changed under the shift E_c. Thus $\mu(l), \nu(l) = 1$ in this case.

Consider $l \in (j, j')$ that corresponds to a positive crossing. Assume first that l is an overpass. If j is an overpass, the operator E_c shifts the colors a, a' up by 1, and we have

$$\mu(l) = \frac{(q)_{n-(a+1)}(q)_{n-a'}}{(q)_{n-(a'+1)}(q)_{n-a}} = \frac{1-q^{n-a'}}{1-q^{n-a}},$$
$$\text{and } \nu(l) = q^* \frac{q^{\frac{n(a+1)-(a'+1)b'-(a+1)b}{2}}}{q^{\frac{na-a'b-ab}{2}}} = q^* q^{\frac{n-b-b'}{2}}.$$

If j was an underpass instead, colors a, a' are shifted down by 1 under E_c, so that

$$\mu(l) = \frac{(q)_{n-(a-1)}(q)_{n-a'}}{(q)_{n-(a'-1)}(q)_{n-a}} = \frac{1-q^{n+1-a}}{1-q^{n+1-a'}},$$
$$\text{and } \nu(l) = q^* \frac{q^{\frac{n(a-1)-(a'-1)b'-(a-1)b}{2}}}{q^{\frac{na-a'b-ab}{2}}} = q^* q^{-\frac{n-b-b'}{2}}.$$

Now if $l \in (j, j')$ is an underpass and j is an overpass, the colors b, b' are shifted up by 1 under E_c and we get:

$$\mu(l) = \frac{(q)_{b+1}(q)_{b'}}{(q)_{b'+1}(q)_b} = \frac{1-q^{b+1}}{1-q^{b'+1}}, \text{ and } \nu(l) = q^* \frac{q^{\frac{n(b'+1)-a'(b'+1)-a(b+1)}{2}}}{q^{\frac{(nb'-a'b'-ab}{2}}} = q^* q^{\frac{(n-a'-a)}{2}}.$$

Finally if j is an underpass instead, colors b, b' are shifted down by 1 and:

$$\mu(l) = \frac{(q)_{b-1}(q)_{b'}}{(q)_{b'-1}(q)_b} = \frac{1-q^{b'}}{1-q^b}, \text{ and } \nu(l) = q^* \frac{q^{\frac{n(b'-1)-a'(b'-1)-a(b-1)}{2}}}{q^{\frac{nb'-a'b'-ab}{2}}} = q^* q^{\frac{-(n-a'-a)}{2}}.$$

We see that those factors match with the ones in Eqs. (29) and (30) considering $\varepsilon(l) = +1$. If l corresponds to a negative crossing, only the $\nu(l)$ factor is changed. The computation of the $\nu(l)$ factors is similar and left to the reader.

There is now just one factor to be considered: the factor $F_c = \frac{E_c w(c)}{w(c)}$ coming from crossing c. Assume that j is a positive overpass, then E_c shifts the colors a'_c and b_c up by one and leaves colors a_c, b'_c invariant. Also here E_c shifts k_c up by one. We get

$$\mu(c) = \frac{E_c w_>(c)}{w_>(c)} = \frac{(q)_{n-a'_c}}{(q)_{n-(a'_c+1)}} \frac{(q)_{b_c+1}(q)_k}{(q)_{b_c}(q)_{k+1}} = \frac{(1-q^{n-a'_c})(1-q^{b_c+1})}{(1-q^{k_c+1})}$$

and

$$\nu(c) = \frac{E_c w_<(c)}{w_<(c)} = q^* \frac{q^{\frac{-(a'_c+1)b_c - a_c(b_c+1)}{2}}}{q^{\frac{-a'_c b_c - a_c b_c}{2}}} = q^* q^{-\frac{a_c+b'_c}{2}}.$$

Thus $F_c = \mu(c)\nu(c)$ matches with the formula of Proposition 4.5. The other possibilities for j (negative overpass, positive underpass, negative underpass) yield similar computations and are left to the reader. □

Recall that the annihilator ideal $\mathrm{Ann}(w_D)$ is a left ideal of the ring $\mathbb{Q}[q, Q, Q_{\mathbf{c}}]\langle E, E_{\mathbf{c}}\rangle$ where $Q_{\mathbf{c}} = (Q_0, \dots, Q_{c(D)})$ and $E_{\mathbf{c}} = (E_0, \dots, E_{c(D)})$. Let $\mathrm{Ann}_{\mathrm{rat}}(w_D)$ denote the corresponding ideal of the ring $\mathbb{Q}(q, Q, Q_{\mathbf{c}})\langle E, E_{\mathbf{c}}\rangle$. Let R, R_c (for $c = 1, \dots, c(D)$) and R_0 denote the expressions on the right hand side of Eqs. (28), (27) and (29), (30) respectively.

Proposition 4.7 *The ideal* $\mathrm{Ann}_{\mathrm{rat}}(w_D)$ *is generated by the set*

$$\{E_c - R_c(q, Q, Q_c),\ c = 0, \dots, c(D),\ E - R_c(q, Q, Q_c)\}. \tag{31}$$

Below, we will need to specialize our operators to $q = 1$. To make this possible, we introduce the subring $Q_{\mathrm{loc}}(q, Q, Q_c)$ of the field $\mathbb{Q}(q, Q, Q_c)$ that consists of all rational functions that are regular (i.e., well-defined) at $q = 1$.

Let $\mathrm{Ann}_{\mathrm{rat,loc}}(w_D) = \mathrm{Ann}_{\mathrm{rat}}(w_D) \cap \mathbb{Q}_{\mathrm{loc}}(q, Q, Q_c)\langle E, E_c\rangle$ denote the left ideal of the ring $\mathbb{Q}_{\mathrm{loc}}(q, Q, Q_c)\langle E, E_c\rangle$.

Proposition 4.8 *The ideal* $\mathrm{Ann}_{\mathrm{rat,loc}}(w_D)$ *is generated by the set* (31).

Proof First, let us note that $\mathbb{Q}_{\mathrm{loc}}(q, Q, Q_{\mathbf{c}})\langle E, E_{\mathbf{c}}\rangle$ is a subring of $\mathbb{Q}(q, Q, Q_{\mathbf{c}})$ $\langle E, E_{\mathbf{c}}\rangle$.

Indeed, if $P(q, Q, Q_{\mathbf{c}})$ is in $\mathbb{Q}_{\mathrm{loc}}(q, Q, Q_{\mathbf{c}})$ then $EP(q, Q, Q_{\mathbf{c}}) = P(q, q^{-1}Q, Q_{\mathbf{c}})E$ is also in $\mathbb{Q}_{\mathrm{loc}}(q, Q, Q_{\mathbf{c}})\langle E, E_{\mathbf{c}}\rangle$, as the denominator of $P(q, q^{-1}$ $Q, Q_{\mathbf{c}})$ evaluated at $q = 1$ is the same as that of $R(q, Q, Q_{\mathbf{c}})$. The same can said for multiplication by one of the $E_{\mathbf{c}}$'s.

Secondly, it is easy to see that the elements $R(q, Q, Q_{\mathbf{c}})$ and $R_c(q, Q, Q_{\mathbf{c}})$ (for $c = 1, \dots, c(D)$) are in $\mathbb{Q}_{\mathrm{loc}}(q, Q, Q_{\mathbf{c}})$. Let I be the left $\mathbb{Q}_{\mathrm{loc}}(q, Q, Q_{\mathbf{c}})\langle E, E_{\mathbf{c}}\rangle$ ideal generated by those elements.

Let us order monomials in E and the $E_{\mathbf{c}}$'s using a lexicographic order. We claim that I contains a monic element in each non zero $(E, E_{\mathbf{c}})$-degree. Indeed, if $E - R(q, Q, Q_{\mathbf{c}})$ is one of the above described generators and $(\alpha, \beta_{\mathbf{c}}) \in \mathbb{N}^{c(D)+2}$, multiplying by $E^{\alpha}E_{\mathbf{c}}^{\beta_{\mathbf{c}}}$ on the left we get that I contains an element of the form $E^{\alpha+1}E_{\mathbf{c}}^{\beta_{\mathbf{c}}} - \tilde{R}(q,Q,Q_{\mathbf{c}})E^{\alpha}E_{\mathbf{c}}^{\beta_{\mathbf{c}}}$ where $\tilde{R}(q,Q,Q_{\mathbf{c}}) \in \mathbb{Q}_{\mathrm{loc}}(q,Q,Q_{\mathbf{c}})$. Using also the generators $E_{\mathbf{c}} - R_c(q,Q,Q_{\mathbf{c}})$ the claim follows.

Now let $P(q,Q,Q_{\mathbf{c}},E,E_{\mathbf{c}})$ be an arbitrary element $\mathrm{Ann}_{\mathrm{rat,loc}}(w_D)$. We may write

$$P(q,Q,Q_{\mathbf{c}},E,E_{\mathbf{c}}) = \sum_{(\alpha,\beta_{\mathbf{c}})\in\mathbb{N}^{c(D)+2}} R_{\alpha,\beta_{\mathbf{c}}}(q,Q,Q_{\mathbf{c}})E^{\alpha}E_{\mathbf{c}}^{\beta_{\mathbf{c}}}.$$

As I contains a monic element in each non-zero $(E,E_{\mathbf{c}})$ degree, one may subtrack elements of I to $P(q,Q,Q_{\mathbf{c}},E,E_{\mathbf{c}})$ to drop its degree until we get that $P-S \in \mathbb{Q}_{\mathrm{loc}}(q,Q,Q_{\mathbf{c}})$ for some element $S \in I$. But $P-S$ must also be in $\mathrm{Ann}_{\mathrm{loc}}(w_D)$, and as $w_D \neq 0$ it must be zero. Thus we can conclude that $I = \mathrm{Ann}_{\mathrm{rat,loc}}(w_D)$. □

5 Matching the annihilator ideal and the gluing equations

5.1 From the annihilator of the state summand to the gluing equations variety

In the previous sections we studied the gluing equations variety $\mathcal{G}_D$ of a knot diagram D and the state summand $w_D(n,k)$ of the colored Jones polynomial of K. In this section we compare the annihilator ideal of the summand with the defining ideal of the gluing equations variety, once we set $q=1$, and conclude that they exactly match. Let us abbreviate the evaluation of a rational function $f(q)$ at $q=1$ by $\mathrm{ev}_q f(q) = f(1)$.

Consider the map ψ defined by:

$$\psi : \mathbb{Q}[Q,Q_{\mathbf{c}}][E] \to \mathbb{C}[\mathcal{G}_D], \qquad (E,Q,Q_c) \mapsto (w_{\lambda}^{-1/2}, w_{\mu}, w_c) \tag{32}$$

where $\mathbb{C}[\mathcal{G}_D]$ denotes the coordinate ring of the affine variety $\mathcal{G}_D$ and $w_{\lambda}^{-1/2}$ is the element of $\mathbb{C}[\mathcal{G}_D]$ described in Proposition 2.12.

The main result which connects the quantum invariant with the classical one can be summarized in the following.

Theorem 5.1 (a) *We have:*

$$(\psi \circ \mathrm{ev}_q \circ \varphi)(\mathrm{Ann}_{\mathrm{rat,loc}}(w_D)) = 0. \tag{33}$$

(b) If $P(q,Q,E) \in \varphi(\mathrm{Ann}(F) \cap \mathbb{Q}[q,Q]\langle E,E_k\rangle)$ as in (7), then $P(q,Q,E)$ annihilates the colored Jones polynomial and $P(1,w_{\mu},w_{\lambda}^{-1/2}) = 0$.

Proof Recall the generators of the annihilator ideal $\mathrm{Ann}_{\mathrm{rat,loc}}(w_D)$ given by Eq. (31), as well as the functions $L_k - 1$ for $k = 0,\ldots,c(D)$ of the coordinate ring of $\mathcal{G}_D$ defined in Sect. 2.7. We will match the two.

For an arc of the diagram with color a, let Q_a be the multiplication by q^a. We claim that $\varphi(Q_a) = z_a$, the corresponding arc parameter. Indeed, for the arc $[1,2]$ we have $\varphi(Q_0) = w_0$ is the arc parameter of the arc $[1,2]$, and going from arc $[k-1,k]$ to

$[k, k+1]$ we shift the multiplication operator by $Q_c^{\pm 1}$ and the arc parameter by $w_c^{\pm 1}$, depending on whether k is an over- or underpass.

By Eqs. (28) and (27):

$$R(1, Q, Q_{\mathbf{c}}) = \prod_{X(D)} \left(\frac{1 - \frac{Q}{Q_a}}{1 - \frac{Q}{Q_{a'}}} \right) (Q_a Q_b)^{\frac{\varepsilon(c)}{2}} \, Q^{-\frac{1}{2}}$$

$$R_0(1, Q, Q_{\mathbf{c}}) = \prod_{c \in X(D)} \left(\frac{Q}{Q_a Q_b} \right)^{\varepsilon(c)} \frac{(1 - \frac{Q}{Q_{a'}})(1 - Q_b)}{(1 - \frac{Q}{Q_a})(1 - Q_{b'})}.$$

If c is a crossing with labels $j < j'$, and j is an overpass, we have by Eq. (29):

$$R_c(1, Q, Q_{\mathbf{c}}) = \mathrm{ev}_q(F_c) \prod_{k \in (j, j') \cap O(D)} \left(\frac{Q}{Q_b Q_{b'}} \right)^{\frac{\varepsilon(k)}{2}} \left(\frac{1 - \frac{Q}{Q_{a'}}}{1 - \frac{Q}{Q_a}} \right)$$
$$\times \prod_{k \in (j, j') \cap U(D)} \left(\frac{Q}{Q_a Q_{a'}} \right)^{\frac{\varepsilon(k)}{2}} \left(\frac{1 - Q_b}{1 - Q_{b'}} \right)$$

where

$$\mathrm{ev}_q(F_c) = \frac{1}{Q_{b'_c}} \frac{(1 - \frac{Q}{Q_{a'_c}})(1 - Q_{b_c})}{(1 - Q_c)}$$

if c is a positive crossing for example. Similarly by Eq. (30), if j is an underpass, then

$$R_c(1, Q, Q_{\mathbf{c}})) = \mathrm{ev}_q(F_c) \prod_{l \in O(D) \cap (j, j')} \left(\frac{Q}{Q_b Q_{b'}} \right)^{-\frac{\varepsilon(k)}{2}} \left(\frac{1 - \frac{Q}{Q_a}}{1 - \frac{Q}{Q_{a'}}} \right)$$
$$\times \prod_{l \in U(D) \cap (j, j')} \left(\frac{Q}{Q_a Q_{a'}} \right)^{-\frac{\varepsilon(k)}{2}} \left(\frac{1 - Q_{b'}}{1 - Q_b} \right).$$

Comparing $(\mathrm{ev}_q \circ \varphi)(E - R(q, Q, Q_{\mathbf{c}}))$ with Eq. (23), we get that

$$(\mathrm{ev}_q \circ \varphi)(E - R(q, Q, Q_{\mathbf{c}})) = s - s = 0.$$

Comparing $(\mathrm{ev}_q \circ \varphi)(E_0 - R_0(q, Q, Q_{\mathbf{c}}))$ with Eq. (22), we get that

$$(\mathrm{ev}_q \circ \varphi)(E_0 - R_0(q, Q, Q_{\mathbf{c}})) = 1 - L_0.$$

Finally, if c is a crossing with labels $j < j'$, comparing $(\mathrm{ev}_q \circ \varphi)(E_c - R_c(q, Q, Q_{\mathbf{c}}))$ with Eq. (21), we get that

$$(\mathrm{ev}_q \circ \varphi)(E_c - R_c(q, Q, Q_{\mathbf{c}})) = 1 - L_c$$

if j is an overpass, while if j is an underpass we get that

$$(\mathrm{ev}_q \circ \varphi)(E_c - R_c(q, Q, Q_{\mathbf{c}})) = 1 - L_c^{-1} = L_c^{-1}(L_c - 1)\,.$$

Thus the image of the generators of the ideal $\mathrm{Ann}_{\mathrm{rat,loc}}(w_D)$ by $\mathrm{ev}_q \circ \varphi$ are generators of the ideal I_D. This proves part (a) of Theorem 5.1. Part (b) follows from part (a) and Eq. (7). □

5.2 Proof of Theorem 1.1

Proof Fix a labeled, oriented planar projection D of K. Then, the certificate recursion $\hat{A}^c_D(q, Q, E)$ annihilates the colored Jones polynomial, as this is true for all q-holonomic sums (5). This concludes part (a).

For part (b), Theorem 5.1 implies that $\hat{A}^c_D(1, w_\mu, w_\lambda^{-1/2}) = 0 \in \mathbb{C}[\mathcal{G}_D]$. In other words, the function $\hat{A}^c_D(1, w_\mu, w_\lambda^{-1/2})$ in the coordinate ring of $\mathcal{G}_D$ is identically zero. Since this is true for every labeled, oriented diagram D of a knot K, this concludes part (b) of Theorem 1.1. □

Acknowledgements Open access funding provided by Projekt DEAL. S.G. was supported in part by DMS-18-11114. S.G. wishes to thank Dylan Thurston for enlightening conversations in several occasions regarding the octahedral decomposition of knot complements and the state sum formulas for the colored Jones polynomial and Christoph Koutchan for enlightening conversations on q-holonomic functions. The paper was conceived and completed while both authors were visiting the Max-Planck Institute for Mathematics in Bonn. The authors wish to thank the institute for its hospitality.

References

1. Andersen, J.E., Malusa, A.: The AJ-conjecture for the Teichmüller TQFT, Preprint (2017)
2. Boyd, D.W., Rodriguez-Villegas, F.: Mahler's measure and the dilogarithm (II), Preprint (2005)
3. Bostan, A., Lairez, P., Salvy, B: Creative telescoping for rational functions using the Griffiths–Dwork method. In: ISSAC 2013–Proceedings of the 38th International Symposium on Symbolic and Algebraic Computation, pp. 93–100. ACM, New York (2013)
4. Bar-Natan, D., Garoufalidis, S.: On the Melvin–Morton–Rozansky conjecture. Invent. Math. **125**(1), 103–133 (1996)
5. Cooper, D., Culler, M., Gillet, H., Long, D., Shalen, P.: Plane curves associated to character varieties of 3-manifolds. Invent. Math. **118**(1), 47–84 (1994)
6. Culler, M., Dunfield, N.M., Weeks, J.R.,: SnapPy. http://www.math.uic.edu/t3m/SnapPy
7. Champanerkar, A.A.: A-polynomial and Bloch invariants of hyperbolic 3-manifolds, ProQuest LLC, Ann Arbor, MI, Thesis (Ph.D.)–Columbia University (2003)
8. Dimofte, T.: Quantum Riemann surfaces in Chern–Simons theory. Adv. Theor. Math. Phys. **17**(3), 479–599 (2013)

9. Garoufalidis, S.: On the characteristic and deformation varieties of a knot. In: Proceedings of the Casson Fest, Geom. Topol. Monogr., vol. 7, pp. 291–309. Geom. Topol. Publ., Coventry (2004)
10. Garoufalidis, S.: The degree of a q-holonomic sequence is a quadratic quasi-polynomial. Electron. J. Combin. **182**, 4, 23 (2011)
11. Garoufalidis, S.: The Jones slopes of a knot. Quantum Topol. **2**(1), 43–69 (2011)
12. Garoufalidis, S., Goerner, M., Zickert, C.K.: Gluing equations for $\mathrm{PGL}(n, \mathbb{C})$-representations of 3-manifolds. Algebraic Geom. Topol. **15**(1), 565–622 (2015)
13. Garoufalidis, S., Koutschan, C.: The noncommutative A-polynomial of $(-2, 3, n)$ pretzel knots. Exp. Math. **21**(3), 241–251 (2012)
14. Garoufalidis, S., Koutschan, C.: Irreducibility of q-difference operators and the knot 7_4. Algebraic Geom. Topol. **13**(6), 3261–3286 (2013)
15. Garoufalidis, S., Lê, T.T.Q.: The colored Jones function is q-holonomic. Geom. Topol **9**, 1253–1293 (2005)
16. Garoufalidis, S., Lê, T.T.Q.: Nahm sums, stability and the colored Jones polynomial. Res. Math. Sci. **2**, Art. 1, 55 (2015)
17. Garoufalidis, S., Lê, T.T.Q.: A survey of q-holonomic functions. Enseign. Math. **62**(3–4), 501–525 (2016)
18. Garoufalidis, S., Mattman, T.W.: The A-polynomial of the $(-2, 3, 3+2n)$ pretzel knots. N. Y. J. Math. **17**, 269–279 (2011)
19. Garoufalidis, S., Sun, X.: The non-commutative A-polynomial of twist knots. J. Knot Theory Ramif. **19**(12), 1571–1595 (2010)
20. Jones, V.: Hecke algebra representations of braid groups and link polynomials. Ann. Math. (2) **126**(2), 335–388 (1987)
21. Kashaev, R.M.: A link invariant from quantum dilogarithm. Mod. Phys. Lett. A **10**(19), 1409–1418 (1995)
22. Kim, H., Kim, S., Yoon, S.: Octahedral developing of knot complement I: pseudo-hyperbolic structure. Geom. Dedicata **197**, 123–172 (2018)
23. Koutschan, C.: Advanced applications of the holonomic systems approach. RISC, Johannes Kepler University, Linz, Austria. Ph.D. Thesis (2009)
24. Kim, S., Park, I.: Octahedral developing of knot complement III: simplicial polyhedralization and solution variety **(In preparation)**
25. Lê, T.T.Q.: The colored Jones polynomial and the A-polynomial of knots. Adv. Math. **207**(2), 782–804 (2006)
26. Lairez, P.: Computing periods of rational integrals. Math. Comput. **85**(300), 1719–1752 (2016)
27. Lê, T.T.Q., Zhang, X.: Character varieties, A-polynomials and the AJ conjecture. Algebraic Geom. Topol. **17**(1), 157–188 (2017)
28. Murakami, H., Murakami, J.: The colored Jones polynomials and the simplicial volume of a knot. Acta Math. **186**(1), 85–104 (2001)
29. Neumann, W.D., Zagier, D.: Volumes of hyperbolic three-manifolds. Topology **24**(3), 307–332 (1985)
30. Petkovšek, M., Wilf, H.S., Zeilberger, D.: $A = B$, A K Peters Ltd, Wellesley, MA: With a foreword by Donald E. Knuth, With a separately available computer disk (1996)
31. Rinat, M.: The hyperbolic volume of knots from the quantum dilogarithm. Lett. Math. Phys. **39**(3), 269–275 (1997)
32. Thurston, W.: The Geometry and Topology of 3-manifolds, Universitext. Springer, Berlin, Lecture notes (1977)
33. Thurston, D.: Hyperbolic volume and the Jones polynomial. Grenoble Notes (1999)
34. Turaev, V.: Quantum Invariants of Knots and 3-Manifolds, de Gruyter Studies in Mathematics, vol. 18. Walter de Gruyter & Co., Berlin (1994)
35. Weeks, J.: Computation of Hyperbolic Structures in Knot Theory, Handbook of Knot Theory, pp. 461–480. Elsevier B. V., Amsterdam (2005)
36. Witten, E.: Quantum field theory and the Jones polynomial. Commun. Math. Phys. **121**(3), 351–399 (1989)
37. Wilf, H.S., Zeilberger, D.: An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral identities. Invent. Math. **108**(3), 575–633 (1992)
38. Zeilberger, D.: A holonomic systems approach to special functions identities. J. Comput. Appl. Math. **32**(3), 321–368 (1990)
39. Zeilberger, D.: The method of creative telescoping. J. Symb. Comput. **11**(3), 195–204 (1991)

40. Zickert, C.K.: Hyperbolic 3-manifolds and the Cheeger–Chern–Simons class, ProQuest LLC, Ann Arbor, MI, 2008, Thesis (Ph.D.)–Columbia University