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An Ansatz for the asymptotics of hypergeometric multisums

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Abstract

Sequences that are defined by multisums of hypergeometric terms with compact support occur frequently in enumeration problems of combinatorics, algebraic geometry and perturbative quantum field theory. The standard recipe to study the asymptotic expansion of such sequences is to find a recurrence satisfied by them, convert it into a differential equation satisfied by their generating series, and analyze the singularities in the complex plane. We propose a shortcut by constructing directly from the structure of the hypergeometric term a finite set, for which we conjecture (and in some cases prove) that it contains all the singularities of the generating series. Our construction of this finite set is given by the solution set of a balanced system of polynomial equations of a rather special form, reminiscent of the Bethe ansatz. The finite set can also be identified with the set of critical values of a potential function, as well as with the evaluation of elements of an additive K-theory group by a regulator function. We give a proof of our conjecture in some special cases, and we illustrate our results with numerous examples.

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1. Introduction

1.1. The problem

The problem considered here is the following: given a balanced hypergeometric term $\mathfrak{t}_{n,k_1,\dots,k_r}$ with compact support for each $n \in \mathbb{N}$, we wish to find an asymptotic expansion of the sequence

$$a_{\mathfrak{t},n} = \sum_{(k_1,\dots,k_r)\in\mathbb{Z}^r} \mathfrak{t}_{n,k_1,\dots,k_r}. \tag{1}$$

Such sequences occur frequently in enumeration problems of combinatorics, algebraic geometry and perturbative quantum field theory; see [10,20,31]. The standard recipe for this is to find a recurrence satisfied by (a_n), convert it into a differential equation satisfied by the generating series

$$G_{\mathfrak{t}}(z)=\sum_{n=0}^{\infty}a_{\mathfrak{t},n}z^n$$

and analyze the singularities of its analytic continuation in the complex plane. We propose a shortcut by constructing directly from the structure of the hypergeometric term $\mathfrak{t}_{n,k_1,\dots,k_r}$ a finite set $S_{\mathfrak{t}}$, for which we conjecture (and in some cases prove) that it contains all the singularities of $G(z)$. Our construction of $S_{\mathfrak{t}}$ is given by the solution set of a balanced system of polynomial equations of a rather special form, reminiscent of the Bethe ansatz. $S_{\mathfrak{t}}$ can also be identified with the set of critical values of a potential function, as well as with the evaluation of elements of an additive K-theory group by a regulator function. We give a proof of our conjecture in some special cases, and we illustrate our results with numerous examples.

1.2. *Existence of asymptotic expansions*

A general existence theorem for asymptotic expansions of sequences discussed above was recently given in [14]. To phrase it, we need to recall what is a G-function in the sense of Siegel [30].

Definition 1.1. We say that series $G(z)=\sum_{n=0}^{\infty}a_n z^n$ is a *G-function* if:

(a) the coefficients a_n are algebraic numbers, and
(b) there exists a constant $C>0$ so that for every $n\in\mathbb{N}$ the absolute value of every conjugate of a_n is less than or equal to C^n, and
(c) the common denominator of $a_0,\dots,a_n$ is less than or equal to C^n.
(d) $G(z)$ is holonomic, i.e., it satisfies a linear differential equation with coefficients polynomials in z.

The main result of [14] is the following theorem.

Theorem 1. *(See [14, Theorem 3].) For every balanced term $\mathfrak{t}$, the generating series $G_{\mathfrak{t}}(z)$ is a G-function.*

Using the fact that the local monodromy of a G-function around a singular point is *quasi-unipotent* (see [2,3,5,9,18]), an elementary application of Cauchy's theorem implies the following corollary; see [14,17] and [6, Section 7].

Corollary 1.2. *If $G(z)=\sum_{n=0}^{\infty}a_n z^n$ is a G-function, then (a_n) has a* transseries expansion*, that is an expansion of the form*

$$a_n \sim \sum_{\lambda\in\Sigma}\lambda^{-n}n^{\alpha_\lambda}(\log n)^{\beta_\lambda}\sum_{s=0}^{\infty}\frac{c_{\lambda,s}}{n^s} \tag{2}$$

where Σ is the set of singularities of $G(z)$, $\alpha_\lambda\in\mathbb{Q}$, $\beta_\lambda\in\mathbb{N}$, and $c_{\lambda,s}\in\mathbb{C}$. In addition, Σ is a finite set of algebraic numbers, and generates a number field $E=\mathbb{Q}(\Sigma)$.

1.3. Computation of asymptotic expansions

Theorem 1 and its Corollary 1.2 are not constructive. The usual way for computing the asymptotic expansion for sequences $(a_{\mathfrak{t},n})$ of the form (1) is to find a linear recurrence, and convert it into a differential equation for the generating series $G_{\mathfrak{t}}(z)$. The singularities of $G_{\mathfrak{t}}(z)$ are easily located from the roots of coefficient of the leading derivative of the ODE. This approach is taken by Wimp, Zeilberger, following Birkhoff, Trjitzinsky; see [4,34] and also [23]. A recurrence for a multisum sequence $(a_{\mathfrak{t},n})$ follows from Wilf–Zeilberger's constructive theorem, and its computer implementation; see [26–28,33,35]. Although constructive, these algorithms are impractical for multisums with, say, more than three summation variables.

On the other hand, it seems wasteful to compute an ODE for $G_{\mathfrak{t}}(z)$, and then discard all but a small part of it in order to determine the singularities $\Sigma_{\mathfrak{t}}$ of $G_{\mathfrak{t}}(z)$.

The main result of the paper is a construction of a finite set $S_{\mathfrak{t}}$ of algebraic numbers directly from the summand $\mathfrak{t}_{n,k_1,\dots,k_r}$, which we conjecture that it includes the set $\Sigma_{\mathfrak{t}}$. We give a proof of our conjecture in some special cases, as well as supporting examples.

Our definition of the set $\Sigma_{\mathfrak{t}}$ is reminiscent of the Bethe ansatz, and is related to critical values of potential functions and additive K-theory.

Before we formulate our conjecture let us give an instructive example.

Example 1.3. Consider the *Apery sequence* (a_n) defined by

$$a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2. \tag{3}$$

It turns out that (a_n) satisfies a linear recursion relation with coefficients in $\mathbb{Q}[n]$ (see [34, p. 174])

$$(n+2)^3 a_{n+2} - (2n+3)\big(17n^2+51n+39\big)a_{n+1} + (n+1)^3 a_n = 0 \tag{4}$$

for all $n \in \mathbb{N}$, with initial conditions $a_0 = 1, a_1 = 5$. It follows that $G(z)$ is *holonomic*; i.e., it satisfies a linear ODE with coefficients in $\mathbb{Q}[z]$:

$$\begin{aligned} &z^2\big(z^2-34z+1\big)G'''(z) + 3z\big(2z^2-51z+1\big)G''(z) + \big(7z^2-112z+1\big)G'(z) \\ &\quad + (z-5)G(z) = 0 \end{aligned} \tag{5}$$

with initial conditions $G(0) = 1$, $G'(0) = 5$, $G''(0) = 146$.

This implies that the possible singularities of $G(z)$ are the roots of the equation:

$$z^2\big(z^2-34z+1\big) = 0. \tag{6}$$

Thus, $G(z)$ has analytic continuation as a multivalued function in $\mathbb{C} \setminus \{0, 17+12\sqrt{2}, 17-12\sqrt{2}\}$. Since the Taylor series coefficients of $G(z)$ at $z=0$ are positive integers, and $G(z)$ is analytic at $z=0$, it follows that $G(z)$ has a singularity inside the punctured unit disk. Thus, $G(z)$ is singular at $17-12\sqrt{2}$. By Galois invariance, it is also singular at $17+12\sqrt{2}$. The proof of Corollary 1.2 implies that (a_n) has an asymptotic expansion of the form:

$$a_n \sim \big(17+12\sqrt{2}\big)^n n^{-3/2} \sum_{s=0}^{\infty} \frac{c_{1s}}{n^s} + \big(17-12\sqrt{2}\big)^n n^{-3/2} \sum_{s=0}^{\infty} \frac{c_{2s}}{n^s} \tag{7}$$

for some constants $c_{1s}, c_{2s} \in \mathbb{C}$ with $c_{10}c_{20} \neq 0$. A final calculation shows that

$$c_{10} = \frac{1}{\pi^{3/2}} \frac{3+2\sqrt{2}}{4\sqrt[4]{2}}, \qquad c_{20} = \frac{1}{\pi^{3/2}} \frac{3-2\sqrt{2}}{4\sqrt[4]{2}}.$$

Our paper gives an ansatz that quickly produces the numbers $17 \pm 2\sqrt{2}$ given the expression (3) of (a_n), bypassing Eqs. (4) and (5). This is explained in Section 5.2. In a forthcoming publication we will explain how to compute the *Stokes constants* c_{s0} for $s = 0, 1$ in terms of the expression (3).

1.4. Hypergeometric terms

We have already mentioned multisums of balanced hypergeometric terms. Let us define what those are.

Definition 1.4. An r-dimensional *balanced hypergeometric term* $\mathfrak{t}_{n,k}$ (in short, *balanced term*, also denoted by $\mathfrak{t}$) in variables (n, k), where $n \in \mathbb{N}$ and $k = (k_1, \ldots, k_r) \in \mathbb{Z}^r$, is an expression of the form:

$$\mathfrak{t}_{n,k} = C_0^n \prod_{i=1}^{r} C_i^{r_i} \prod_{j=1}^{J} A_j(n,k)!^{\epsilon_j} \tag{8}$$

where C_i are algebraic numbers for $i = 0, \ldots, r$, $\epsilon_j = \pm 1$ for $j = 1, \ldots, J$, and A_j are integral linear forms in (n, k) that satisfy the *balance condition*:

$$\sum_{j=1}^{J} \epsilon_j A_j = 0. \tag{9}$$

Remark 1.5. An alternative way of encoding a balanced term is to record the vector $(C_0, \ldots, C_r)$, and the $J \times (r + 2)$ matrix of the coefficients of the linear forms $A_j(n, k)$, and the signs ϵ_j for $j = 1, \ldots, r$.

1.5. Balanced terms, generating series and singularities

Given a balanced term $\mathfrak{t}$, we will assign a sequence $(a_{\mathfrak{t},n})$ and a corresponding generating series $G_{\mathfrak{t}}(z)$ to a balanced term $\mathfrak{t}$, and will study the set $\Sigma_{\mathfrak{t}}$ of singularities of the analytic continuation of $G_{\mathfrak{t}}(z)$ in the complex plane.

Let us introduce some useful notation. Given a linear form $A(n, k)$ in variables (n, k) where $k = (k_1, \ldots, k_r)$, and $i = 0, \ldots, r$, let us define

$$v_i(A) = a_i, \qquad v_0(A) = a_0, \quad \text{where } A(n,k) = a_0 n + \sum_{i=1}^{r} a_i k_i. \tag{10}$$

For $w=(w_1,\dots,w_r)$ we define:

$$A(w)=a_0+\sum_{i=1}^{r}a_i w_i. \tag{11}$$

Definition 1.6. Given a balanced term $\mathfrak{t}$ as in (8), define its *Newton polytope* $P_{\mathfrak{t}}$ by

$$P_{\mathfrak{t}}=\left\{w\in\mathbb{R}^r \;\middle|\; A_j(w)\geqslant 0 \text{ for } j=1,\dots,J\right\}\subset\mathbb{R}^r. \tag{12}$$

We will assume that $P_{\mathfrak{t}}$ is a compact rational convex polytope in $\mathbb{R}^r$ with non-empty interior. It follows that for every $n\in\mathbb{N}$ we have:

$$\operatorname{support}(\mathfrak{t}_{n,k})=nP_{\mathfrak{t}}\cap\mathbb{Z}^r. \tag{13}$$

Definition 1.7. Given a balanced term $\mathfrak{t}$ consider the sequence:

$$a_{\mathfrak{t},n}=\sum_{k\in nP_{\mathfrak{t}}\cap\mathbb{Z}^r}\mathfrak{t}_{n,k} \tag{14}$$

(the sum is finite for every $n\in\mathbb{N}$) and the corresponding generating function:

$$G_{\mathfrak{t}}(z)=\sum_{n=0}^{\infty}a_{\mathfrak{t},n}z^n\in\bar{\mathbb{Q}}[\![z]\!]. \tag{15}$$

Here $\bar{\mathbb{Q}}$ denote the field of algebraic numbers. Let $\Sigma_{\mathfrak{t}}$ denote the finite set of singularities of $G_{\mathfrak{t}}$, and $E_{\mathfrak{t}}=\mathbb{Q}(\Sigma_{\mathfrak{t}})$ denote the corresponding number field, following Corollary 1.2.

Remark 1.8. Notice that $\mathfrak{t}$ determines $G_{\mathfrak{t}}$ but *not* vice-versa. Indeed, there are nontrivial identities among multisums of balanced terms. Knowing a complete set of such identities would be very useful in constructing invariants of knotted objects, as well as in understanding relations among periods; see [21].

Remark 1.9. The balance condition of Eq. (9) is imposed so that for every balanced term $\mathfrak{t}$ the corresponding sequence $(a_{\mathfrak{t},n})$ grows at most exponentially. This follows from Stirling's formula (see Corollary 2.1) and it implies that the power series $G_{\mathfrak{t}}(z)$ is the germ of an analytic function at $z=0$. Given a proper hypergeometric term $\mathfrak{t}_{n,k}$ in the sense of [33], we can find $\alpha\in\mathbb{Q}$ and $\epsilon=\pm 1$ so that $\mathfrak{t}_{n,k}(\alpha n)!^{\epsilon}$ is a balanced term.

1.6. The definition of $S_{\mathfrak{t}}$ and $K_{\mathfrak{t}}$

Let us observe that if $\mathfrak{t}$ is a balanced term and Δ is a face of its Newton polytope $P_{\mathfrak{t}}$, then $\mathfrak{t}|_{\Delta}$ is also a balanced term.

Definition 1.10. Given a balanced term $\mathfrak{t}$ as in Eq. (8) consider the following system of *variational equations*:

$$C_i \prod_{j=1}^{J} A_j(w)^{\epsilon_j v_i(A_j)} = 1 \quad \text{for } i = 1, \dots, r \tag{16}$$

in the variables $w = (w_1, \dots, w_r)$. Let $X_{\mathfrak{t}}$ denote the set of complex solutions of (16), with the convention that when $r = 0$ we set $X_{\mathfrak{t}} = \{0\}$, and define

$$\mathrm{CV}_{\mathfrak{t}} = \left\{ C_0^{-1} \prod_{j: A_j(w) \neq 0} A_j(w)^{-\epsilon_j v_0(A_j)} \;\middle|\; w \in X_{\mathfrak{t}} \right\}, \tag{17}$$

$$S_{\mathfrak{t}} = \{0\} \cup \cup_{\Delta \text{ face of } P_{\mathfrak{t}}} \mathrm{CV}_{\mathfrak{t}|_\Delta}, \tag{18}$$

$$K_{\mathfrak{t}} = \mathbb{Q}(S_{\mathfrak{t}}). \tag{19}$$

Remark 1.11. There are two different incarnations of the set $\mathrm{CV}_{\mathfrak{t}}$: it coincides with

(a) the set of critical values of a potential function; see Theorem 5.
(b) the evaluation of elements of an additive K-theory group under the entropy regulator function; see Theorem 6.

It is unknown to the author whether $X_{\mathfrak{t}}$ is always a finite set. Nevertheless, $S_{\mathfrak{t}}$ is always a finite subset of $\bar{\mathbb{Q}}$; see Theorem 5. Eqs. (16) are reminiscent of the *Bethe ansatz*.

1.7. The conjecture

Section 1.5 constructs a map:

$$\text{Balanced terms } \mathfrak{t} \longrightarrow (E_{\mathfrak{t}}, \Sigma_{\mathfrak{t}})$$

via generating series $G_{\mathfrak{t}}(z)$ and their singularities, where $E_{\mathfrak{t}}$ is a number field and $\Sigma_{\mathfrak{t}}$ is a finite subset of $E_{\mathfrak{t}}$. Section 1.6 constructs a map:

$$\text{Balanced terms } \mathfrak{t} \longrightarrow (K_{\mathfrak{t}}, S_{\mathfrak{t}})$$

via solutions of polynomial equations. We are now ready to formulate our main conjecture.

Conjecture 1. *For every balanced term* $\mathfrak{t}$ *we have*: $\Sigma_{\mathfrak{t}} \subset S_{\mathfrak{t}}$ *and consequently,* $E_{\mathfrak{t}} \subset K_{\mathfrak{t}}$.

1.8. Partial results

Conjecture 1 is known to hold in the following cases:

(a) For 0-dimensional balanced terms, see Theorem 2.
(b) For positive special terms, see Theorem 3.
(c) For 1-dimensional balanced terms, see [13].

Since the finite sets $S_{\mathfrak{t}}$ and $\Sigma_{\mathfrak{t}}$ that appear in Conjecture 1 are in principle computable (as explained in Section 1.3), one may try to check random examples. We give some evidence in Section 5. We refer the reader to [16] for an interesting class of 1-dimensional examples related to $6j$-symbols, and of interest to atomic physics and low-dimensional topology.

The following proposition follows from the classical fact concerning singularities of hypergeometric series; see for example [25, Section 5] and [24].

Theorem 2. *Suppose that* $\mathfrak{t}_{n,k}$ *is* 0*-dimensional balanced term as in* (8)*, with* $k = \emptyset$*. Then* $G_{\mathfrak{t}}(z)$ *is a hypergeometric series and Conjecture* 1 *holds.*

When $\mathfrak{t}$ is positive-dimensional, the generating series $G_{\mathfrak{t}}(z)$ is no longer hypergeometric in general. To state our next result, recall that a finite subset $S \subset \bar{\mathbb{Q}}$ of algebraic numbers is *irreducible* over $\mathbb{Q}$ if the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}, \mathbb{Q})$ acts transitively on S.

Definition 1.12. A *special hypergeometric term* $\mathfrak{t}_{n,k}$ (in short, *special term*) is an expression of the form:

$$\mathfrak{t}_{n,k} = C_0^n \prod_{i=1}^{r} C_i^{r_i} \prod_{j=1}^{J} \binom{B_j(n,k)}{C_j(n,k)} \tag{20}$$

where $C \in \mathbb{Q}$, L and B_j and C_j are integral linear forms in (n,k). We will assume that for every $n \in \mathbb{N}$, the support of $\mathfrak{t}_{n,k} = 0$ as a function of $k \in \mathbb{Z}^r$ is finite. We will call such a term *positive* if $C_i > 0$ for $i = 0, \ldots, r$.

Lemma 1.13.

(a) *A balanced term is the ratio of two special terms. In other words, it can always be written in the form*:

$$\mathfrak{t}_{n,k} = C^{L(n,k)} \prod_{j=1}^{s} \binom{B_j(n,k)}{C_j(n,k)}^{\epsilon_j} \tag{21}$$

for some integral linear forms B_j, C_j *and signs* ϵ_j.

(b) *The set of special terms is an abelian monoid with respect to multiplication, whose corresponding abelian group is the set of balanced terms.*

The proof of (a) follows from writing a balanced term in the form:

$$\mathfrak{t}_{n,k} = C_0^n \prod_{i=1}^{r} C_i^{r_i} \frac{\prod_{j:\epsilon_j=1}^{J} A_j(n,k)!}{A(n,k)!} \frac{A(n,k)!}{\prod_{j:\epsilon_j=-1}^{J} A_j(n,k)!}$$

where

$$A(n,k) = \sum_{j:\epsilon_j=1}^{J} A_j(n,k) = \sum_{j:\epsilon_j=-1}^{J} A_j(n,k).$$

To illustrate part (a) of the above lemma for 0-dimensional balanced terms, we have:

$$\frac{(30n)!n!}{(16n)!(10n)!(5n)!} = \binom{30n}{16n}\binom{14n}{10n}\binom{5n}{4n}^{-1}.$$

The above identity also shows that if a balanced term takes integer values, it need not be a special term. This phenomenon was studied by Rodriguez and Villegas; see [29].

Theorem 3. *Fix a positive special term $\mathfrak{t}_{n,k}$ such that $\Sigma_{\mathfrak{t}} \setminus \{0\}$ is irreducible over $\mathbb{Q}$. Then, $\Sigma_{\mathfrak{t}} \subset \mathrm{CV}_{\mathfrak{t}} \subset S_{\mathfrak{t}}$ and Conjecture* 1 *holds.*

In a forthcoming publication we will give a proof of Conjecture 1 for 1-dimensional balanced terms; see [13]. Let us end this section with an inverse type (or geometric realization) problem.

Problem 1.14. Given $\lambda \in \bar{\mathbb{Q}}$, does there exist a special term $\mathfrak{t}$ so that $\lambda \in S_{\mathfrak{t}}$?

1.9. Laurent polynomials: A source of special terms

This section, which is of independent interest, associates an special term $\mathfrak{t}_F$ to a Laurent polynomial F with the property that the generating series $G_{\mathfrak{t}_F}(z)$ is identified with the trace of the resolvent $R_F(z)$ of F. Combined with Theorem 1, this implies that $R_F(z)$ is a G-function.

If $F \in M_N(\bar{\mathbb{Q}}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}])$ is a square matrix of size N with entries Laurent polynomials in d variables, let $\mathrm{Tr}(F)$ denote the *constant term* of its usual trace. The *moment generating series* of F is the power series

$$G_F(z) = \sum_{n=0}^{\infty} \mathrm{Tr}\big(F^n\big) z^n. \tag{22}$$

Theorem 4.

(a) *For every $F \in \bar{\mathbb{Q}}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$ there exists a special term $\mathfrak{t}_F$ so that*

$$G_F(z) = G_{\mathfrak{t}_F}(z). \tag{23}$$

Consequently, $G_F(z)$ is a G-function.

(b) *The Newton polytope $P_{\mathfrak{t}_F}$ depends is a combinatorial simplex which depends only on the monomials that appear in F and not on their coefficients.*

(c) *For every $F \in M_N(\bar{\mathbb{Q}}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}])$, $G_F(z)$ is a G-function.*

We thank C. Sabbah for providing an independent proof of part (c) when $N = 1$ using the regularity of the *Gauss–Manin connection*. Compare also with [8].

1.10. Plan of the proof

In Section 2, we introduce a potential function associated to a balanced term $\mathfrak{t}$ and we show that the set of its critical values coincides with the set $S_{\mathfrak{t}}$ that features in Conjecture 1. This also implies that $S_{\mathfrak{t}}$ is finite.

In Section 3 we assign elements of an extended additive K-theory group to a balanced term $\mathfrak{t}$, and we show that the set of their values (under the entropy regulator map) coincides with the set $S_{\mathfrak{t}}$ that features in Conjecture 1.

In Section 4 we give a proof of Theorems 3 (using results from hypergeometric functions) and 2 (using an application of Laplace's method), which are partial case of our Conjecture 1.

In Section 5 we give several examples that illustrate Conjecture 1.

In Section 6 we study a special case of Conjecture 1, with input a Laurent polynomial in many commuting variables.

2. Balanced terms and potential functions

2.1. *The Stirling formula and potential functions*

As a motivation of a potential function associated to a balanced term, recall *Stirling formula*, which computes the asymptotic expansion of $n!$ (see [25]):

$$\log n! \sim n\log n - n + \frac{1}{2}\log n + \frac{1}{2}\log(2\pi) + O\left(\frac{1}{n}\right). \tag{24}$$

For $x > 0$, we can define $x! = \Gamma(x+1)$. The Stirling formula implies that for $a > 0$ fixed and $n \in \mathbb{N}$ large, we have:

$$\log(an)! = (n\log n - n)a + a\log(a) + O\left(\frac{\log n}{n}\right). \tag{25}$$

The next corollary motivates our definition of the potential function.

Corollary 2.1. *For every balanced term* $\mathfrak{t}$ *as in* (8) *and every* w *in the interior of* $P_{\mathfrak{t}}$ *we have*:

$$\mathfrak{t}_{n,nw} = e^{nV_{\mathfrak{t}}(w) + O\left(\frac{\log n}{n}\right)} \tag{26}$$

where the potential function $V_{\mathfrak{t}}$ *is defined below.*

Definition 2.2. Given a balanced term $\mathfrak{t}$ as in (8), define its corresponding *potential function* $V_{\mathfrak{t}}$ by

$$V_{\mathfrak{t}}(w) = C(w) + \sum_{j=1}^{J} \epsilon_j A_j(w)\log\bigl(A_j(w)\bigr) \tag{27}$$

where $w = (w_1, \ldots, w_r)$,

$$C(n,k) = \log C_0 \cdot n + \log C_1 \cdot k_1 + \cdots + \log C_r \cdot k_r, \tag{28}$$

and

$$C(w) = \log C_0 + \log C_1 w_1 + \cdots + \log C_r w_r. \tag{29}$$

$V_{\mathfrak{t}}$ is a multivalued analytic function on the complement of the *linear hyperplane arrangement* $\mathbb{C}^r \setminus \mathcal{A}_{\mathfrak{t}}$, where

$$\mathcal{A}_{\mathfrak{t}} = \left\{ w \in \mathbb{C}^r \;\Big|\; \prod_{j=1}^{J} A_j(w) = 0 \right\}. \tag{30}$$

In fact, let

$$\varpi : \hat{\mathbb{C}}^r \longrightarrow \mathbb{C}^r \setminus \mathcal{A}_{\mathfrak{t}} \tag{31}$$

denote the universal abelian cover of $\mathbb{C}^r \setminus \mathcal{A}_{\mathfrak{t}}$.

The next theorem relates the critical points and critical values of $V_{\mathfrak{t}}$ with the sets $X_{\mathfrak{t}}$ and $\mathrm{CV}_{\mathfrak{t}}$ from Definition 1.10.

Theorem 5.

(a) *For every balanced term* $\mathfrak{t}$, $\varpi^{-1}(X_{\mathfrak{t}})$ *coincides with the set of critical points of* $V_{\mathfrak{t}}$.
(b) *If* w *is a critical point of* $V_{\mathfrak{t}}$, *then*

$$e^{-V_{\mathfrak{t}}(w)} = C^{-v_0(L)} \prod_{j=1}^{J} A_j(w)^{-\epsilon_j v_0(A_j)} = C^{-v_0(L)} \prod_{j:A_j(w)\neq 0} A_j(w)^{-\epsilon_j v_0(A_j)}. \tag{32}$$

Thus $\mathrm{CV}_{\mathfrak{t}}$ *coincides with the exponential of the set of the negatives of the critical values of* $V_{\mathfrak{t}}$.
(c) *For every face* Δ *of the Newton polytope of* $\mathfrak{t}$ *we have*:

$$V_{(\mathfrak{t}|_{\Delta})} = (V_{\mathfrak{t}})|_{\Delta}.$$

(d) $S_{\mathfrak{t}}$ *is a finite subset of* $\bar{\mathbb{Q}}$ *and* $K_{\mathfrak{t}}$ *is a number field.*

2.2. *Proof of Theorem 5*

Let us fix a balanced term $\mathfrak{t}$ as in (8) and a face Δ of its Newton polytope $P_{\mathfrak{t}}$. Without loss of generality, assume that $\Delta = P_{\mathfrak{t}}$. Since

$$\frac{d}{dx}\big(x\log(x)\big) = \log(x) + 1 \tag{33}$$

it follows that for every $i = 1, \dots, r$ and every $j = 1, \dots, J$ we have:

$$\frac{\partial}{\partial w_i} A_j(w)\log\big(A_j(w)\big) = v_i(A_j)\log\big(A_j(w)\big) + v_i(A_j). \tag{34}$$

Adding up with respect to j, using the balancing condition of Eq. (9), and the notation of Section 1.5 applied to the linear form of Eq. (28), it follows that

$$\begin{aligned}\frac{\partial}{\partial w_i} V_{\mathfrak{t}}(w) &= \log C_i + \sum_{j=1}^{J} \epsilon_j v_i(A_j) \log\bigl(A_j(w)\bigr) + \sum_{j=1}^{J} \epsilon_j v_i(A_j) \\ &= \log C_i + \sum_{j=1}^{J} \epsilon_j v_i(A_j) \log\bigl(A_j(w)\bigr).\end{aligned}$$

This proves that the critical points $w = (w_1, \dots, w_r)$ of $V_{\mathfrak{t}}$ are the solutions to the following system of *logarithmic variational equations*:

$$\log C_i + \sum_{j=1}^{J} \epsilon_j v_i(A_j) \log\bigl(A_j(w)\bigr) \in \mathbb{Z}(1) \quad \text{for } i = 1, \dots, r \tag{35}$$

where, for a subgroup K of $(\mathbb{C}, +)$ and an integer $n \in \mathbb{Z}$, we define

$$K(n) = (2\pi i)^n K. \tag{36}$$

Exponentiating, it follows that w satisfies variational equations (16), and concludes the proof of part (a).

For part (b), we will show that if w is a critical point of $V_{\mathfrak{t}}$, the corresponding critical value is given by

$$V_{\mathfrak{t}}(w) = \log C_0 + \sum_{j=1}^{J} \epsilon_j v_0(A_j) \log\bigl(A_j(w)\bigr) \in \mathbb{C}/\mathbb{Z}(1). \tag{37}$$

Exponentiating, we deduce the first equality of Eq. (32). The second equality follows from the fact that $A_j(w) \neq 0$ for all critical points w of $V_{\mathfrak{t}}$.

To show (37), observe that for any linear form $A(n,k)$ we have:

$$A(w) = v_0(A)w_0 + \sum_{i=1}^{r} v_i(A)w_i.$$

Suppose that w satisfies logarithmic variational equations (35). Using the definition of the potential function, and collecting terms with respect to $w_1, \dots, w_r$ it follows that

$$\begin{aligned}V_{\mathfrak{t}}(w) &= \log C_0 + \sum_{j=1}^{J} \epsilon_j v_0(A_j) \log\bigl(A_j(w)\bigr) + \sum_{i=1}^{r} w_i \left(\log C_i + \sum_{j=1}^{J} \epsilon_j v_i(A_j) \log\bigl(A_j(w)\bigr) \right) \\ &= \log C_0 + \sum_{j=1}^{J} \epsilon_j v_0(A_j) \log\bigl(A_j(w)\bigr).\end{aligned}$$

This concludes part (b). Part (c) follows from set-theoretic considerations, and part (d) follows from the following facts:

(i) an analytic function is constant on each component of its set of critical points,

(ii) the set of critical points are the complex points of an affine variety defined over $\mathbb{Q}$ by (16),
(iii) every affine variety has finitely many connected components.

This concludes the proof of Theorem 5. $\square$

3. Balanced terms, the entropy function and additive K-theory

In this section we will assign elements of an extended additive K-theory group to a balanced term, and using them, we will identify our finite set $\mathrm{CV}_{\mathfrak{t}}$ from Definition 1.10 with the values of the constructed elements under a regulator map; see Theorem 6.

3.1. A brief review of the entropy function and additive K-theory

In this section we will give a brief summary of an extended version of additive K-theory and the entropy function following [15], and motivated by [12]. This section is independent of the rest of the paper, and may be skipped at first reading.

Definition 3.1. Consider the *entropy function* Φ, defined by

$$\Phi(x) = -x\log(x) - (1-x)\log(1-x) \tag{38}$$

for $x \in (0,1)$.

$\Phi(x)$ is a multivalued analytic function on $\mathbb{C} \setminus \{0,1\}$, given by the double integral of a rational function as follows from

$$\Phi''(x) = -\frac{1}{x} - \frac{1}{1-x}. \tag{39}$$

For a detailed description of the analytic continuation of Φ, we refer the reader to [15]. Let $\hat{\mathbb{C}}$ denote the *universal abelian cover* of $\mathbb{C}^{**}$. In [15, Section 1.3] we show that Φ has an analytic continuation:

$$\Phi : \hat{\mathbb{C}} \longrightarrow \mathbb{C}. \tag{40}$$

In [15, Definition 1.7] we show that Φ satisfies three 4-term relations, one of which is the analytic continuation of (48). The other two are dictated by the variation of Φ along the cuts $(1,\infty)$ and $(-\infty,0)$.

Using the three 4-term relations of Φ, we introduce an extended version $\widehat{\beta_2(\mathbb{C})}$ in [15, Definition 1.7].

Definition 3.2. The *extended group* $\widehat{\beta_2(\mathbb{C})}$ is the $\mathbb{C}$-vector space generated by the symbols $\langle x\rangle$ with $x=(z;p,q)\in\hat{\mathbb{C}}$, subject to the *extended 4-term relation*:

$$\langle x_0;p_0,q_0\rangle - \langle x_1;p_1,q_1\rangle + (1-x_0)\left\langle \frac{x_1}{1-x_0};p_2,q_2\right\rangle - (1-x_1)\left\langle \frac{x_0}{1-x_1};p_3,q_3\right\rangle = 0 \tag{41}$$

for $((x_0; p_0, q_0), \ldots, (x_3; p_3, q_3)) \in \widehat{4\mathrm{T}}$, and the relations:

$$\langle x; p, q\rangle - \langle x; p, q'\rangle = \langle x; p, q-2\rangle - \langle x; p, q'-2\rangle, \tag{42}$$

$$\langle x; p, q\rangle - \langle x; p', q\rangle = \langle x; p-2, q\rangle - \langle x; p'-2, q\rangle \tag{43}$$

for $x \in \mathbb{C}^{**}$, $p, q, p', q' \in 2\mathbb{Z}$.

Since the three 4-term relations in the definition of $\widehat{\beta_2(\mathbb{C})}$ are satisfied by the entropy function, it follows that Φ gives rise to a *regulator map*:

$$R : \widehat{\beta_2(\mathbb{C})} \longrightarrow \mathbb{C}. \tag{44}$$

For a motivation of the extended group $\widehat{\beta_2(\mathbb{C})}$ and its relation to additive (i.e., infinitesimal) K-theory and infinitesimal polylogarithms, see [15, Section 1.1] and references therein.

3.2. Balanced terms and additive K-theory

In this section, it will be more convenient to use the presentation (21) of balanced terms. In this case, we have:

$$t_{n,k} = C_0^n \prod_{i=1}^{r} C_i^{k_i} \prod_{j=1}^{J} B_j(n,k)!^{\epsilon_j} C_j(n,k)!^{-\epsilon_j} (B_j - C_j)(n,k)!^{-\epsilon_j}. \tag{45}$$

The Stirling formula motivates the constructions in this section. Indeed, we have the following:

Lemma 3.3. *For $a > b > 0$, we have*:

$$a\Phi\left(\frac{b}{a}\right) = a\log(a) - b\log(b) - (a-b)\log(a-b) \tag{46}$$

and

$$\binom{an}{bn} \sim e^{na\Phi(\frac{b}{a})} \sqrt{\frac{a}{2b(a-b)\pi n}} \left(1 + O\left(\frac{1}{n}\right)\right). \tag{47}$$

Proof. Eq. (46) is elementary. Eq. (47) follows from the Stirling formula (24). □

The next lemma gives a combinatorial proof of the 4-term relation of the entropy function.

Lemma 3.4. *For $a, b, a+b \in (0,1)$, Φ satisfies the 4-term relation*:

$$\Phi(b) - \Phi(a) + (1-b)\Phi\left(\frac{a}{1-b}\right) - (1-a)\Phi\left(\frac{b}{1-a}\right) = 0. \tag{48}$$

Proof. The 4-term relation follows from the associativity of the multinomial coefficients

$$\binom{\alpha+\beta+\gamma}{\alpha}\binom{\beta+\gamma}{\beta}=\binom{\alpha+\beta+\gamma}{\beta}\binom{\alpha+\gamma}{\alpha}=\frac{(\alpha+\beta+\gamma)!}{\alpha!\,\beta!\,\gamma!} \tag{49}$$

applied to $(\alpha,\beta,\gamma)=(an,bn,cn)$ and Lemma 3.3, and the specialization to $a+b+c=1$. In fact, the 4-term relation (48) and a local integrability assumption uniquely determines Φ up to multiplication by a complex number. See for example, [7] and [1, Section 5.4, p. 66]. □

Corollary 3.5. *If* $\mathfrak{t}$ *is a balanced term as in* (21)*, then its potential function is given by*

$$V_{\mathfrak{t}}(w)=C(w)+\sum_{j=1}^{J}\epsilon_j B_j(w)\Phi\left(\frac{C_j(w)}{B_j(w)}\right). \tag{50}$$

Consider the complement $\mathbb{C}^r\setminus\mathcal{A}'_{\mathfrak{t}}$ of the linear hyperplane arrangement given by

$$\mathcal{A}'_{\mathfrak{t}}=\left\{w\in\mathbb{C}^r \;\Big|\; \prod_{j=1}^{J}B_j(w)C_j(w)\big(B_j(w)-1\big)\big(B_j(w)-C_j(w)\big)=0\right\}. \tag{51}$$

Let

$$\varpi:\hat{\mathbb{C}}^r\longrightarrow\mathbb{C}^r\setminus\mathcal{A}'_{\mathfrak{t}} \tag{52}$$

denote the universal abelian cover of $\mathbb{C}^r\setminus\mathcal{A}'_{\mathfrak{t}}$. For $j=1,\dots,r$, the functions $B_j(w)$, $C_j(w)/B_j(w)$ have analytic continuation

$$B_j:\hat{\mathbb{C}}^r\longrightarrow\mathbb{C},\qquad \frac{C_j}{B_j}:\hat{\mathbb{C}}^r\longrightarrow\hat{\mathbb{C}}.$$

It follows that the potential function given by (50) has an analytic continuation

$$V_{\mathfrak{t}}:\hat{\mathbb{C}}^r\longrightarrow\mathbb{C}. \tag{53}$$

Let $\mathcal{C}_{\mathfrak{t}}$ denote the set of critical points of $V_{\mathfrak{t}}$.

Definition 3.6. For a balanced term $\mathfrak{t}$ as in (21), we define the map

$$\beta_{\mathfrak{t}}:\mathcal{C}_{\mathfrak{t}}\longrightarrow\widehat{\mathcal{B}_2(\mathbb{C})} \tag{54}$$

by

$$\beta_{\mathfrak{t}}(w)=\sum_{j=1}^{J}\epsilon_j B_j(w)\left\langle\frac{C_j(w)}{B_j(w)}\right\rangle.$$

Theorem 6. *For every balanced term* $\mathfrak{t}$, *we have a commutative diagram*:

$$\begin{tikzcd} \mathcal{C}_{\mathfrak{t}} \arrow[r, "\beta_{\mathfrak{t}}"] \arrow[d, "\varpi"] & \widehat{\beta_2(\mathbb{C})} \arrow[d, "e^R"] \\ X_{\mathfrak{t}} \arrow[r, "C_0 e^{-V_{\mathfrak{t}}}"] & \mathrm{CV}_{\mathfrak{t}} \end{tikzcd}$$

3.3. *Proof of Theorem 6*

We begin by observing that the analytic continuation of the logarithm function gives an analytic function

$$\mathrm{Log} : \hat{\mathbb{C}} \longrightarrow \mathbb{C}.$$

The proof of part (a) of Theorem 5 implies that $w \in \mathcal{C}_{\mathfrak{t}}$ if and only if w satisfies the logarithmic variational equations

$$\begin{aligned} \log C_i + \sum_{j=1}^{J} \epsilon_j \bigl(v_i(B_j) \,\mathrm{Log}\bigl(B_j(w)\bigr) - v_i(C_j) \,\mathrm{Log}\bigl(C_j(w)\bigr) \\ - v_i(B_j - C_j) \,\mathrm{Log}\bigl((B_j - C_j)(w)\bigr)\bigr) = 0 \end{aligned} \tag{55}$$

for $i = 1, \dots, r$. Exponentiating, this implies that $\mathcal{C}_{\mathfrak{t}} = \varpi^{-1}(X_{\mathfrak{t}})$.

The proof of part (b) of Theorem 5 implies that if $w \in \mathcal{C}_{\mathfrak{t}}$, then

$$\begin{aligned} V_{\mathfrak{t}}(w) = \log C_0 + \sum_{j=1}^{J} \epsilon_j \bigl(v_0(B_j) \,\mathrm{Log}\bigl(B_j(w)\bigr) - v_0(C_j) \,\mathrm{Log}\bigl(C_j(w)\bigr) \\ - v_0(B_j - C_j) \,\mathrm{Log}\bigl((B_j - C_j)(w)\bigr)\bigr). \end{aligned}$$

On the other hand, if $w \in \mathcal{C}_{\mathfrak{t}}$, we have:

$$\begin{aligned} R\bigl(\beta_{\mathfrak{t}}(w)\bigr) &= -\sum_{j=1}^{J} \epsilon_j B_j(w) \Phi\left(\frac{C_j(w)}{B_j(w)}\right) \\ &= -\sum_{j=1}^{J} \epsilon_j \bigl(B_j(w) \,\mathrm{Log}\bigl(B_j(w)\bigr) - C_j(w) \,\mathrm{Log}\bigl(C_j(w)\bigr) \\ &\quad - \bigl(B_j(w) - C_j(w)\bigr) \,\mathrm{Log}\bigl((B_j - C_j)(w)\bigr)\bigr) \end{aligned}$$

where the last equality follows from the analytic continuation of (46). Expanding the linear forms B_j, C_j, and $B_j - C_j$ with respect to the variables w_i for $i = 0, \dots, r$, and using the logarithmic variational equations (55) (as in the proof of part (b) of Theorem 5), it follows that

$$R\big(\beta_{\mathfrak{t}}(w)\big) = -\sum_{j=1}^{J} \epsilon_j \big(v_0(B_j)\operatorname{Log}\big(B_j(w)\big) - v_0(C_j)\operatorname{Log}\big(C_j(w)\big) - v_0(B_j - C_j)\operatorname{Log}\big((B_j - C_j)(w)\big)\big).$$

Thus,

$$V_{\mathfrak{t}}(w) = \log C_0 - R\big(\beta_{\mathfrak{t}}(w)\big).$$

This concludes the proof of Theorem 6.

4. Proof of Theorems 2 and 3

4.1. Proof of Theorem 2

A 0-dimensional balanced term is of the form

$$\mathfrak{t}_n = C_0^n \prod_{j=1}^{J} (b_j n)!^{\epsilon_j}$$

where $b_j \in \mathbb{N}$, $\epsilon_j = \pm 1$ for $j = 1, \dots, J$ satisfying $\sum_{j=1}^{J} \epsilon_j b_j = 0$. Since $\mathbb{Z}^0 = \{0\}$, it follows that

$$a_{\mathfrak{t},n} = \mathfrak{t}_n$$

is so-called closed form. The Newton polytope of $\mathfrak{t}$ is given by $P_{\mathfrak{t}} = \{0\} \subset \mathbb{R}^0$. In addition, $A_j(w) = b_j$ and $v_0(A_j) = b_j$ for $j = 1, \dots, J$. By definition, we have $X_{\mathfrak{t}} = \{0\}$, and

$$\mathrm{CV}_{\mathfrak{t}} = \left\{ C_0^{-1} \prod_{j=1}^{J} b_j^{-\epsilon_j b_j} \right\}. \tag{56}$$

On the other hand, $G_{\mathfrak{t}}(z)$ is a *hypergeometric series* with singularities $\{0, \mathrm{CV}_{\mathfrak{t}}\}$; see for example [25, Section 5] and [24]. The result follows.

4.2. Proof of Theorem 3

The proof of Theorem 3 is a variant of Laplace's method and uses the positivity of the restriction of the potential function to $P_{\mathfrak{t}}$. See also [19, Section 5.1.4]. Suppose that $\mathfrak{t}_{n,k} \geqslant 0$ for all n, k. Recall the corresponding polytope $P_{\mathfrak{t}} \subset \mathbb{R}^r$ and consider the restriction of the potential function to $P_{\mathfrak{t}}$:

$$V_{\mathfrak{t}} : P_{\mathfrak{t}} \longrightarrow \mathbb{R}.$$

It is easy to show that $\Phi(x) > 0$ for $x \in (0, 1)$. This is illustrated by the plot of the entropy function for $x \in [0, 1]$ (see Fig. 1).

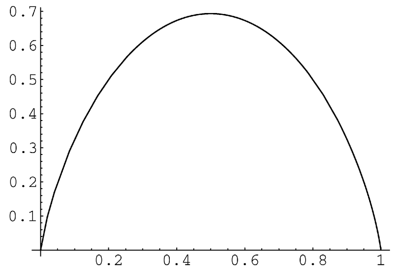

Fig. 1.

It follows that the restriction of the potential function on $P_{\mathfrak{t}}$ is nonnegative and continuous. By compactness it follows that the function achieves a maximum in $\hat{w}$ in the interior of $P_{\mathfrak{t}}$. It follows that for every $k \in nP_{\mathfrak{t}} \cap \mathbb{Z}^r$ we have:

$$0 \leqslant \mathfrak{t}_{n,k} \leqslant \mathfrak{t}_{n,n\hat{w}}.$$

Summing up over the lattice points $k \in nP_{\mathfrak{t}} \cap \mathbb{Z}^r$, and using the fact that the number of lattice points in a rational convex polyhedron (dilated by n) is a polynomial function of n, it follows that there exist a polynomial $p(n) \in \mathbb{Q}[n]$ so that for all $n \in \mathbb{N}$ we have:

$$\mathfrak{t}_{n,n\hat{w}} \leqslant a_{\mathfrak{t},n} \leqslant p(n)\mathfrak{t}_{n,n\hat{w}}.$$

Using Corollary 2.1, it follows that there exist polynomials $p_1(n), p_2(n) \in \mathbb{Q}[n]$ so that for all $n \in \mathbb{N}$ we have:

$$p_1(n)e^{nV_{\mathfrak{t}}(\hat{w})} \leqslant a_{\mathfrak{t},n} \leqslant p_2(n)e^{nV_{\mathfrak{t}}(\hat{w})}.$$

This implies that $G_{\mathfrak{t}}(z)$ has a singularity at $z = e^{-V_{\mathfrak{t}}(\hat{w})} > 0$. Since the maximum lies in the interior of $P_{\mathfrak{t}}$, it follows that $\hat{w}$ is a critical point of $V_{\mathfrak{t}}$. Thus, $\hat{w} \in S_{\mathfrak{t}}$ and consequently, $e^{-V_{\mathfrak{t}}(\hat{w})} \in \mathrm{CV}_{\mathfrak{t}}$.

If in addition $\Sigma_{\mathfrak{t}} \setminus \{0\}$ is irreducible over $\mathbb{Q}$, it follows that the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts transitively on $\Sigma_{\mathfrak{t}}$. This implies that $\Sigma_{\mathfrak{t}} \subset \mathrm{CV}_{\mathfrak{t}}$. □

Remark 4.1. When $\mathfrak{t}_{n,k} \geqslant 0$ for all n, k, then $G_{\mathfrak{t}}(z)$ has a singularity at $\rho > 0$ where $1/\rho$ is the radius of convergence of $G_{\mathfrak{t}}(z)$. This is known as Pringsheim's theorem, see [32, Section 7.21].

5. Some examples

5.1. A closed form example

As a warm-up example, consider the 1-dimensional special term

$$\mathfrak{t}_{n,k} = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

whose corresponding sequence is closed form and the generating series is a rational function:

$$a_{\mathfrak{t},n} = 2^n, \qquad G_{\mathfrak{t}}(z) = \frac{1}{1-2z}.$$

In that case $\Sigma_{\mathfrak{t}} = \{1/2\}$ and $E_{\mathfrak{t}} = \mathbb{Q}$.

To compare with our ansatz, the Newton polytope is given by

$$P_{\mathfrak{t}} = [0, 1] \subset \mathbb{R}.$$

Variational equations (16) are

$$\frac{1}{w(1-w)^{-1}} = 1$$

in the variable $w = w_1$, with solution set

$$X_{\mathfrak{t}} = \{1/2\}.$$

Thus,

$$\mathrm{CV}_{\mathfrak{t}} = \left\{ \frac{1}{(1-w)^{-1}} \;\middle|\; w = 1/2 \right\} = \{1/2\}.$$

For the other two faces $\Delta_0 = \{0\}$ and $\Delta_1 = \{1\}$ of the Newton polytope $P_{\mathfrak{t}}$, the restriction is a 0-dimensional balanced term. Eq. (56) gives that

$$\mathfrak{t}|_{\Delta_0} = \mathfrak{t}_{n,k}|_{k=0} = 1, \qquad \mathfrak{t}|_{\Delta_1} = \mathfrak{t}_{n,k}|_{k=n} = 1.$$

Thus,

$$\mathrm{CV}_{\mathfrak{t}|_{\Delta_0}} = \mathrm{CV}_{\mathfrak{t}|_{\Delta_1}} = \{1\}.$$

This implies that

$$S_{\mathfrak{t}} = \{0, 1, 1/2\}, \qquad K_{\mathfrak{t}} = \mathbb{Q},$$

confirming Conjecture 1. For completeness, the potential function is given by

$$V_{\mathfrak{t}}(w) = \Phi(w).$$

5.2. The Apery sequence

As an illustration of Conjecture 1 and Theorem 3, let us consider the special term

$$\mathfrak{t}_{n,k} = \binom{n}{k}^2 \binom{n+k}{k}^2 = \left(\frac{(n+k)!}{k!^2(n-k)!} \right)^2 \tag{57}$$

and the corresponding sequence (3). Eq. (5) implies that the singularities of $G(z)$ are a subset of the roots of the equation

$$z^2\left(z^2-34z+1\right)=0.$$

In addition, a nonzero singularity exists. Thus, we obtain that

$$\left\{0, 17+12\sqrt{2}, 17-12\sqrt{2}\right\} \subset \Sigma_{\mathfrak{t}} \subset \left\{0, 17+12\sqrt{2}, 17-12\sqrt{2}\right\}, \qquad E_{\mathfrak{t}}=\mathbb{Q}\left(\sqrt{2}\right). \tag{58}$$

Thus $E_{\mathfrak{t}}$ is a quadratic number field of type [2, 0] and discriminant 8.

On the other hand, the Newton polytope is given by

$$P_{\mathfrak{t}}=[0,1]\subset\mathbb{R}. \tag{59}$$

Variational equations (16) are

$$\left(\frac{1-w}{w}\frac{1+w}{w}\right)^2=1 \tag{60}$$

in the variable $w=w_1$, with solution set

$$X_{\mathfrak{t}}=\left\{1/\sqrt{2}, -1/\sqrt{2}\right\}.$$

Thus,

$$\mathrm{CV}_{\mathfrak{t}}=\left\{\left(\frac{1-w}{1+w}\right)^2 \,\middle|\, w=\pm 1/\sqrt{2}\right\}=\left\{17+12\sqrt{2}, 17-12\sqrt{2}\right\}.$$

For the other two faces $\Delta_0=\{0\}$ and $\Delta_1=\{1\}$ of the Newton polytope $P_{\mathfrak{t}}$ the restriction is a 0-dimensional balanced term:

$$\mathfrak{t}|_{\Delta_0}=\mathfrak{t}_{n,k}|_{k=0}=1, \qquad \mathfrak{t}|_{\Delta_1}=\mathfrak{t}_{n,k}|_{k=n}=\left(\frac{(2n)!}{n!^2}\right)^2.$$

Eq. (56) implies that

$$\mathrm{CV}_{\mathfrak{t}|_{\Delta_0}}=\{1\}, \qquad \mathrm{CV}_{\mathfrak{t}|_{\Delta_1}}=\{16\}.$$

Therefore,

$$S_{\mathfrak{t}}=\left\{0, 1, 16, 17+12\sqrt{2}, 17-12\sqrt{2}\right\}, \qquad K_{\mathfrak{t}}=\mathbb{Q}\left(\sqrt{2}\right),$$

confirming Conjecture 1. For completeness, the potential function is given by

$$V_{\mathfrak{t}}(w)=2\Phi(w)+2(1+w)\Phi\left(\frac{w}{1+w}\right).$$

5.3. An example with critical points at the boundary

In this example we find critical points at the boundary of the Newton polytope even though the balanced term is positive. This shows that Theorem 3 is sharp. Consider the balanced term

$$\mathfrak{t}_{n,k} = \frac{1}{\binom{n}{k}} = \frac{(n-k)!k!}{n!} \tag{61}$$

and the corresponding sequence $(a_{\mathfrak{t},n})$. The `zb.m` implementation of the WZ algorithm (see [26, 27]) gives that $(a_{\mathfrak{t},n})$ satisfies the inhomogeneous recursion relation:

$$-2(n+1)a_{n+1} + (n+2)a_n = -2 - 2n \tag{62}$$

for all $n \in \mathbb{N}$ with initial conditions $a_0 = 1$. It follows that (a_n) satisfies the homogeneous recursion relation:

$$-2(n+3)a_{n+3} + (12+5n)a_{n+2} - 4(n+2)a_{n+1} + (n+2)a_n = 0 \tag{63}$$

for all $n \in \mathbb{N}$ with initial conditions $a_0 = 1, a_1 = 2, a_2 = 5/2$. The corresponding generating series $G_{\mathfrak{t}}(z)$ satisfies the inhomogeneous ODE:

$$(z-1)^2(z-2)f'(z) + 2(z-1)^2 f(z) + 2 = 0 \tag{64}$$

with initial conditions $f(0) = 1$. We can convert (64) into a homogeneous ODE by differentiating once. It follows that

$$\Sigma_{\mathfrak{t}} \subset \{1, 2\}, \qquad E_{\mathfrak{t}} = \mathbb{Q}. \tag{65}$$

On the other hand, $P_{\mathfrak{t}} = [0, 1] \subset \mathbb{R}$. Variational equations (16) are

$$\frac{w}{1-w} = 1$$

in the variable $w = w_1$ with solution set $X_{\mathfrak{t}} = \{1/2\}$ and

$$\mathrm{CV}_{\mathfrak{t}} = \left\{ \frac{1}{1-w} \;\middle|\; w = \frac{1}{2} \right\} = \{2\}.$$

For the other two faces $\Delta_0 = \{0\}$ and $\Delta_1 = \{1\}$ of the Newton polytope $P_{\mathfrak{t}}$ we have:

$$\mathfrak{t}|_{\Delta_0} = \mathfrak{t}_{n,k}|_{k=0} = 1, \qquad \mathfrak{t}|_{\Delta_1} = \mathfrak{t}_{n,k}|_{k=n} = 1.$$

Thus,

$$\mathrm{CV}_{\mathfrak{t}|_{\Delta_0}} = \mathrm{CV}_{\mathfrak{t}|_{\Delta_1}} = \{1\}.$$

This implies that

$$S_{\mathfrak{t}} = \{0, 1, 2\}, \qquad K_{\mathfrak{t}} = \mathbb{Q},$$

confirming Conjecture 1. For completeness, the potential function is given by

$$V_{\mathfrak{t}}(w) = -\Phi(w).$$

6. Laurent polynomials and special terms

6.1. Proof of Theorem 4

In this section we will prove Theorem 4. Consider $F \in \bar{\mathbb{Q}}[x_1^{\pm 1}, \dots, x_d^{\pm 1}]$. Let us decompose F into a sum of monomials with coefficients

$$F = \sum_{\alpha \in \mathcal{A}} c_\alpha x^\alpha \tag{66}$$

where $\mathcal{A}$ is the finite set of monomials of F, and where for every $\alpha = (\alpha_1, \dots, \alpha_d)$ we denote $x^\alpha = \prod_{j=1}^d x_j^{\alpha_j}$. Let

$$r = |\mathcal{A}| \tag{67}$$

denote the number of monomials of F. Recall that

$$\Delta_n = \left\{ (y_1, \dots, y_{n+1}) \in \mathbb{R}^{n+1} \;\middle|\; \sum_{j=1}^{n+1} y_j = 1,\ y_i \geqslant 0,\ i = 1, \dots, n+1 \right\}$$

denotes the standard n-dimensional *simplex* in $\mathbb{R}^{n+1}$. Part (a) of Theorem 4 follows easily from the multinomial coefficient theorem. Indeed, for every $n \in \mathbb{N}$ we have:

$$\begin{aligned} F^n &= \sum_{\sum_{\alpha \in \mathcal{A}} k_\alpha = n} \frac{n!}{\prod_{\alpha \in \mathcal{A}} k_\alpha!} \prod_{\alpha \in \mathcal{A}} c_\alpha^{k_\alpha} x^{k_\alpha \alpha} \\ &= \sum_{\sum_{\alpha \in \mathcal{A}} k_\alpha = n} \frac{n!}{\prod_{\alpha \in \mathcal{A}} k_\alpha!} \prod_{\alpha \in \mathcal{A}} c_\alpha^{k_\alpha} \cdot x^{\sum_{\alpha \in \mathcal{A}} k_\alpha \alpha}. \end{aligned}$$

It follows that for every $n \in \mathbb{N}$ we have

$$\begin{aligned} \operatorname{Tr}\bigl(F^n\bigr) &= \sum_{\sum_{\alpha \in \mathcal{A}} k_\alpha = n, \sum_{\alpha \in \mathcal{A}} k_\alpha \alpha = 0} \frac{n!}{\prod_{\alpha \in \mathcal{A}} k_\alpha!} \prod_{\alpha \in \mathcal{A}} c_\alpha^{k_\alpha} \\ &= \sum_{k \in n P_{\mathfrak{t}_F} \cap \mathbb{Z}^r} \mathfrak{t}_{F,n,k} \end{aligned}$$

where

$$\mathfrak{t}_{F,n,k} = \frac{n!}{\prod_{\alpha \in \mathcal{A}} k_\alpha!} \prod_{\alpha \in \mathcal{A}} c_\alpha^{k_\alpha} \tag{68}$$

and the Newton polygon $P_{\mathfrak{t}_F}$ is given by

$$P_{\mathfrak{t}_F} = \Delta_{r-1} \cap W \tag{69}$$

where

$$W = \left\{ (x_\alpha) \in \mathbb{R}^r \;\Big|\; \sum_{\alpha \in \mathcal{A}} x_\alpha \alpha = 0 \right\} \tag{70}$$

is a linear subspace of $\mathbb{R}^r$.

To prove part (b) of Theorem 4, let us assume that the origin is in the interior of the Newton polytope of F. Such F are also called *convenient* in singularity theory [22]. If F is not convenient, we can replace it with its the restriction F_f to a face f of its Newton polytope that contains the origin and observe that $\mathrm{Tr}((F_f)^n) = \mathrm{Tr}(F^n)$.

When F is convenient, it follows that W has dimension $r - d$ and $W \cap C^o \neq \emptyset$, where C^o is the interior of the cone C which is spanned by the coordinate vectors in $\mathbb{R}^r$. (b) follows from Lemma 6.3 below.

For part (c) of Theorem 4, fix $F \in M_N(\bar{\mathbb{Q}}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}])$, $G_F(z)$. Recall that for an invertible matrix A we have $A^{-1} = \det(A)^{-1}\mathrm{Cof}(A)$, where $\mathrm{Cof}(A)$ is the co-factor matrix. It follows that

$$\sum_{n=0}^{\infty} F^n z^n = (I - zF)^{-1} = \frac{1}{\det(I - zF)} \mathrm{Ad}(I - zF) \tag{71}$$

where $\mathrm{Ad}(I - zF) \in M_N(\bar{\mathbb{Q}}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}, z])$. Now

$$\det(I - zF) = 1 - \sum_{\alpha \in \mathcal{S}} c_\alpha z^{d_\alpha} x^\alpha$$

where $\mathcal{S}$ is a finite set, $c_\alpha \in \bar{\mathbb{Q}}$ and $d_\alpha \in \mathbb{N} \setminus \{0\}$. Thus,

$$\frac{1}{\det(I - zF)} = \sum_{n=0}^{\infty} \sum_{\sum_{\alpha \in \mathcal{S}} k_\alpha = n} \frac{n!}{\prod_{\alpha \in \mathcal{S}} k_\alpha!} z^{\sum_{\alpha \in \mathcal{S}} d_\alpha k_\alpha} \prod_{\alpha \in \mathcal{S}} x^{\sum_{\alpha \in \mathcal{S}} k_\alpha \alpha}.$$

Substituting the above into Eq. (71) and taking the constant term, it follows that there exists a finite set S and a finite collection $\mathfrak{t}_F^{(j)}$ of balanced terms for $j \in S$ such that for every $n \in \mathbb{N}$ we have:

$$\mathrm{Tr}\big(F^n\big) = \sum_{j \in S} a_{\mathfrak{t}_{F,n}^{(j)}}.$$

It follows that the moment generating series $G_F(z)$ (defined in Eq. (22)) is given by

$$G_F(z) = \sum_{j \in S} G_{\mathfrak{t}_F^{(j)}}(z).$$

Since $G_{\mathfrak{t}_F^{(j)}}(z)$ is a G-function (by Theorem 1), and the set of G-functions is closed under addition (see [2]), this concludes the proof Theorem 4.

Remark 6.1. The balanced terms $\mathfrak{t}_F^{(j)}$ in the above proof use affine linear forms, rather than linear ones. In other words, they are given by

$$\mathfrak{t}_{n,k} = C_0^n \prod_{i=1}^{r} C_i^{r_i} \prod_{j=1}^{J} A_j(n,k)!^{\epsilon_j} \tag{72}$$

where C_i are algebraic numbers for $i = 0, \dots, r$, $\epsilon_j = \pm 1$ for $j = 1, \dots, J$, and A_j are affine linear forms, given by $A_j(n,k) = A_j^{\mathrm{lin}}(n,k) + b_j$ where $A_j^{\mathrm{lin}}(n,k)$ are linear forms that satisfy the balance condition

$$\sum_{j=1}^{J} \epsilon_j A_j^{\mathrm{lin}} = 0. \tag{73}$$

Theorem 1 remains true for such balanced terms. For an balanced term $\mathfrak{t}$ of the form (72) consider the balanced term $\mathfrak{t}^{\mathrm{lin}}$ defined by

$$\mathfrak{t}_{n,k}^{\mathrm{lin}} = C_0^n \prod_{i=1}^{r} C_i^{r_i} \prod_{j=1}^{J} A_j^{\mathrm{lin}}(n,k)!^{\epsilon_j}. \tag{74}$$

We define $S^{\mathfrak{t}} = S^{\mathfrak{t}^{\mathrm{lin}}}$.

Remark 6.2. In the notation of Theorem 4, $P_{\mathfrak{t}_F}$ is a *simple* polytope and the associated toric variety is projective and has *quotient singularities*; see [11]. In other words, the stabilizers of the torus action on the toric variety are finite abelian groups.

The following lemma was communicated to us by J. Pommersheim.

Lemma 6.3. *If W is a linear subspace of $\mathbb{R}^n$ of dimension s, and $W \cap C^o \neq \emptyset$, where C is the cone spanned by the n coordinate vectors in $\mathbb{R}^n$, then $\Delta_{n-1} \cap W$ is a combinatorial simplex in V.*

Proof. We can prove the claim by downward induction on s. When $s = n-1$, W is a hyperplane. Write $D_{n-1} = C \cap H$ where H is the hyperplane given by $\sum_{i=1}^n x_i = 0$. Observe that C is a simplicial cone and the intersection $C \cap W$ is a simplicial cone in W. Since the intersection of a simplicial cone inside C with H is a simplicial cone, it follows that $D_{n-1} \cap W = (C \cap W) \cap H$ is a simplicial cone. This proves the claim when $s = n - 1$. A downward induction on s concludes the proof. $\square$

6.2. The Newton polytope of a Laurent polynomial and its associated balanced term

In a later publication we will study the close relationship between the Newton polytope of a Laurent polynomial F and the Newton polytope of the corresponding special term $\mathfrak{t}_F$. In what follows, fix a Laurent polynomial F as in (66) and its associated balanced term $\mathfrak{t}_F$ as in (68). With the help of the commutative diagram of Proposition 6.5 below, we will compare the extended critical values of F with the set $S_{\mathfrak{t}_F}$. Keep in mind that [8] use the Newton polytope of F, whereas our ansatz uses the Newton polytope of $\mathfrak{t}_F$.

Consider the polynomial map

$$\phi : \bar{\mathbb{Q}}\big[w_\alpha^{\pm 1} \bigm| \alpha \in \mathcal{A}\big] \longrightarrow \bar{\mathbb{Q}}\big[x_1^{\pm 1}, \dots, x_d^{\pm 1}\big] \tag{75}$$

given by $\phi(w_\alpha) = x^\alpha$. Its kernel $\ker(\phi)$ is a *monomial ideal*. Adjoin the monomial w_{α_0} to $\mathcal{A}$ (if it not already there), where $\alpha_0 = 0 \in \mathbb{Z}^d$, and consider the homogeneous ideal $\ker^h(\phi)$, where the degree of w_α is α.

Lemma 6.4. *Variational equations* (16) *for the balanced term* $\mathfrak{t}_F$ *are equivalent to the following system of equations*:

$$\begin{cases} \prod_{\alpha \in \mathcal{A}} x_\alpha^{-p_\alpha} c_\alpha^{p_\alpha} = 1 & \textit{for } \big(w_\alpha^{p_\alpha}\big) \in \ker^h(\phi), \\ \sum_{\alpha \in \mathcal{A}} x_\alpha \alpha = 0, \\ \sum_{\alpha \in \mathcal{A}} x_\alpha = 1 \end{cases} \tag{76}$$

in the variables $(x_\alpha)_{\alpha \in \mathcal{A}}$.

Proof. This follows easily from the description of the Newton polytope $P_{\mathfrak{t}_F}$ and the shape of the balanced term $\mathfrak{t}$. □

Consider the rational function

$$\Psi_F : (\mathbb{C}^*)^d \setminus F^{-1}(0) \longrightarrow (\mathbb{C}^*)^r, \quad u \mapsto \Psi_F(u) = \left(\frac{c_\alpha u^\alpha}{F(u)}\right)_{\alpha \in \mathcal{A}}. \tag{77}$$

Observe that the image of Ψ_F lies in the complex affine simplex

$$\left\{(x_\alpha) \in (\mathbb{C}^*)^r \Bigm| \sum_{\alpha \in \mathcal{A}} x_\alpha = 1\right\}.$$

Proposition 6.5.

(a) *If* $u = (u_1, \dots, u_d)$ *is a critical point of the restriction of* F *on* $(\mathbb{C}^*)^d$ *with nonvanishing critical value, then* $\Psi_F(u)$ *satisfies variational equations* (16) *for the maximal face* $P_{\mathfrak{t}_F}$ *of the Newton polytope* $P_{\mathfrak{t}_F}$ *of* $\mathfrak{t}_P$.

(b) *Restricting to those critical points, we have a commutative diagram*:

$$\begin{tikzcd} \textit{Critical points of } F \textit{ on } (\mathbb{C}^*)^d \arrow[r, "\Psi_F"] \arrow[d, "F"] & X_{\mathfrak{t}_F, P_{\mathfrak{t}_F}} \arrow[d, "\text{Map (17)}"] \\ \textit{Critical values of } F \arrow[r] & S_{\mathfrak{t}_F, P_{\mathfrak{t}_F}} \end{tikzcd} \tag{78}$$

(c) *The top horizontal map of the above diagram is* 1–1 *and onto.*

Proof. For (a) we will use the alternative system of variational equations given in (76). Suppose that $u = (u_1, \dots, u_d)$ is a critical point of F with nonzero critical value, and let $(w_\alpha)_{\alpha\in\mathcal{A}}$ denote the tuple $(c_\alpha u^\alpha / F(u))_{\alpha\in\mathcal{A}}$. We need to show that $(w_\alpha)_{\alpha\in\mathcal{A}}$ is a solution to Eq. (76). Consider a point $(p_\alpha)_{\alpha\in\mathcal{A}} \in P_{\mathrm{t}_F}$. Then, we can verify the first line of Eq. (76) as follows

$$\begin{aligned}\prod_{\alpha\in\mathcal{A}} w_\alpha^{-p_\alpha} c_\alpha^{p_\alpha} &= \prod_{\alpha\in\mathcal{A}} \left(\frac{c_\alpha u^\alpha}{F(u)}\right)^{-p_\alpha} c_\alpha^{p_\alpha} \\ &= u^{-\sum_{\alpha\in\mathcal{A}} p_\alpha \alpha} F(u)^{-\sum_{\alpha\in\mathcal{A}} p_\alpha} \\ &= 1\end{aligned}$$

since $(w_\alpha^{p_\alpha})_{\alpha\in\mathcal{A}}$ in the kernel of ϕ. To verify the second line of Eq. (76), recall that

$$z\partial_z\big(z^k\big) = kz^k.$$

Since u is a critical point of F, it follows that for every $i = 1, \dots, d$ we have

$$u_i \partial_{u_i} F(u) = \sum_{\alpha\in\mathcal{A}} v_i(\alpha) c_\alpha u^\alpha = 0$$

where $v_i(\alpha)$ is the ith coordinate of α. It follows that $\sum_{\alpha\in\mathcal{A}} w_\alpha \alpha = 0$, which verifies the second line of (76). To verify the third line, we compute:

$$\sum_{\alpha\in\mathcal{A}} \frac{c_\alpha u^\alpha}{F(u)} = \frac{1}{F(u)} \sum_{\alpha\in\mathcal{A}} c_\alpha u^\alpha = \frac{F(u)}{F(u)} = 1.$$

(b) and (c) follow from similar computations. □

Let us end this section with an example that illustrates Lemma 6.4 and Proposition 6.5.

Example 6.6. Consider the Laurent polynomial

$$F(x_1, x_2) = ax_1 + bx_1^{-1} + cx_2 + dx_2^{-1} + fx_1x_2 + g.$$

Its Newton polytope is given in Fig. 2.

With the notation from the previous section, we have $d = 2$, $r = 6$. Moreover, for every $n \in \mathbb{N}$, we have

$$F^n = \sum_{k_1+\cdots+k_6=n} \frac{n!}{k_1!\dots k_6!} a^{k_1} b^{k_2} c^{k_3} d^{k_4} f^{k_5} g^{k_6} x_1^{k_1-k_2+k_5} x_2^{k_3-k_4+k_5}.$$

It follows that

$$\mathrm{Tr}\big(F^n\big) = \sum_{(k_1,\dots,k_6)\in nP_{\mathrm{t}_F}\cap\mathbb{Z}^6} \frac{n!}{k_1!\dots k_6!} a^{k_1} b^{k_2} c^{k_3} d^{k_4} f^{k_5} g^{k_6}$$

where the Newton polytope $P_{\mathrm{t}_F} \subset \mathbb{R}^6$ is given by

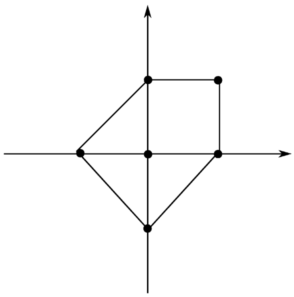

Fig. 2.

$$P_{\mathrm{t}_F} = \{(x_1, \ldots, x_6) \in \mathbb{R}^6 \mid x_1 - x_2 + x_5 = 0,\ x_3 - x_4 + x_5 = 0,\ x_1 + \cdots + x_6 = 1,\ x_i \geqslant 0\}. \tag{79}$$

We can use the variables (x_1, x_3, x_5) to parametrize P_{t_F} as follows:

$$P_{\mathrm{t}_F} = \{(x_1, x_3, x_5) \in \mathbb{R}^3 \mid 1 \geqslant 2x_1 + 2x_3 + 3x_5,\ x_i \geqslant 0\}. \tag{80}$$

This confirms that P_{t_F} is a combinatorial 2-dimensional simplex.

If $(k_1, \ldots, k_6) \in P_{\mathrm{t}_F}$, then $(k_2, k_4, k_6) = (k_1 + k_5, k_3 + k_5, n - 2k_1 - 2k_3 - 3k_5)$, and

$$\begin{aligned}\operatorname{Tr}(F^n) = & \sum_{2k_1+2k_3+2k_5=n} \frac{n!}{k_1!(k_1+k_5)!k_3!(k_3+k_5)!k_5!(n-2k_1-2k_3-3k_5)!} \\ & \times \left(\frac{ab}{g^2}\right)^{k_1} \left(\frac{cd}{g^2}\right)^{k_3} \left(\frac{bdf}{g^3}\right)^{k_5} g^n.\end{aligned}$$

Variational equations (16) are:

$$\begin{aligned}\frac{1}{w_1(w_1+w_5)(1-2w_1-2w_3-3w_5)^{-2}} \frac{ab}{g^2} &= 1, \\ \frac{1}{w_3(w_3+w_5)(1-2w_1-2w_3-3w_5)^{-2}} \frac{cd}{g^2} &= 1, \\ \frac{1}{(w_1+w_5)(w_3+w_5)w_5(1-2w_1-2w_3-2w_5)^{-3}} \frac{bdf}{g^3} &= 1,\end{aligned} \tag{81}$$

in the variables (w_1, w_3, w_5). Reintroducing the variables w_2, w_4 and w_6 defined by

$$w_2 = w_1 + w_5, \qquad w_4 = w_3 + w_5, \qquad w_6 = 1 - 2w_1 - 2w_3 - 2w_5,$$

variational equations become

$$\begin{aligned} w_2 &= w_1 + w_5, \\ w_4 &= w_3 + w_5, \\ w_6 &= 1 - 2w_1 - 2w_3 - 2w_5, \end{aligned} \tag{82}$$

$$\frac{1}{w_1 w_2 w_6^{-2}} \frac{ab}{g^2} = 1,$$
$$\frac{1}{w_3 w_4 w_6^{-2}} \frac{cd}{g^2} = 1,$$
$$\frac{1}{w_2 w_4 w_5 w_6^{-3}} \frac{bdf}{g^3} = 1. \tag{83}$$

On the other hand, $w_{\alpha_0} = w_6$ and $\ker(\phi)$ is generated by the relations $w_6 = 1$, $w_1 w_2 = 1$, $w_3 w_4 = 1$ and $w_2 w_4 w_5 = 1$ which lead to the homogeneous relations $w_1 w_2 w_6^{-2} = 1$, $w_3 w_4 w_6^{-2} = 1$ and $w_2 w_4 w_5 w_6^{-3}$ for $\ker^h(\phi)$ that appear in the above variational equations. This illustrates Lemma 6.4.

Suppose now that $u = (u_1, u_2)$ is a critical point of F. Then, u satisfies the equations:

$$\frac{\partial F}{\partial u_1} = a - \frac{b}{u_1^2} + f u_2 = 0, \tag{84}$$
$$\frac{\partial F}{\partial u_2} = c - \frac{d}{u_2^2} + f u_1 = 0. \tag{85}$$

The map Ψ_F is defined by $\Psi_F(u_1, u_2) = (w_1, \dots, w_6)$ where

$$w_1 = \frac{a u_1}{F(u)}, \qquad w_2 = \frac{b}{u_1 F(u)}, \qquad w_3 = \frac{c u_2}{F(u)},$$
$$w_4 = \frac{d}{u_2 F(u)}, \qquad w_5 = \frac{f u_1 u_2}{F(u)}, \qquad w_5 = \frac{g}{F(u)}.$$

If $u = (u_1, u_2)$ satisfies Eqs. (84), (85), it is easy to see that $w = \Psi_F(u)$ satisfies variational equations (82) and (83). For example, we have

$$w_3 + w_5 - w_4 = \frac{c u_2}{F(u)} + \frac{f u_1 u_2}{F(u)} - \frac{d}{u_2 F(u)} = \frac{u_2}{F(u)} \left(c + f u_1 - \frac{d}{u_2^2} \right) = 0$$

and

$$\frac{1}{w_3 w_4 w_6^{-2}} \frac{cd}{g^2} = \frac{1}{\frac{c u_2}{F(u)} \frac{d}{u_2 F(u)} (\frac{g}{F(u)})^{-2}} \frac{cd}{g^2} = 1.$$

This illustrates Proposition 6.5.

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