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Gevrey series in quantum topology

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Abstract. Our aim is to prove that two formal power series of importance to quantum topology are Gevrey. These series are the Kashaev invariant of a knot (reformulated by Huynh and the second author) and the Gromov norm of the LMO of an integral homology 3-sphere. It follows that the power series associated to a simple Lie algebra and a homology sphere is Gevrey. Contrary to the case of analysis, our formal power series are not solutions to differential equations with polynomial coefficients. The first author has conjectured (and in some cases proved, in joint work with Costin) that our formal power series have resurgent Borel transform, with geometrically interesting set of singularities.

1. Introduction

1.1. Gevrey series. A formal power series

$$f(x) = \sum_{n=0}^{\infty} a_n \frac{1}{x^n} \in \mathbb{C}[[1/x]] \tag{0}$$

is called *Gevrey-s* if there exists a positive constant C, such that

$$|a_n| \leqq C^n n!^s$$

for all $n > 0$. Here, x is supposed to be large. In other words, we will order power series so that $1/x^n \gg 1/x^m$ iff $0 \leqq n < m$. Gevrey-0 series are well known: they are precisely the convergent power series for x in a neighborhood of infinity. We will abbreviate Gevrey-1 by Gevrey. For example, the series

$$\sum_{n=0}^{\infty} n! \frac{1}{x^{n+1}} \tag{1}$$

is Gevrey. Typically, Gevrey power series are divergent (for x in a neighborhood of infinity), and developing a meaningful calculus of Gevrey power series is a well-studied subject;

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see [Ha], [Ra], [Ec] and also [Ba]. Gevrey power series appear naturally as formal power series solutions to differential equations–linear or not. For example, the unique formal power series solution to *Euler's equation*

$$f'(x) + f(x) = \frac{1}{x} \tag{2}$$

is the series of Equation (1). One can construct actual solutions of the ODE (2) by suitably resumming the factorially divergent series (1), resulting in analytic functions with an essential singularity at infinity; see [Ha], [Ra], [Ec]. The *resummation process* of a Gevrey formal power series $f(x) \in \mathbb{Q}[[1/x]]$ as in Equation (0) consists of the following steps:

- Consider its Borel transform $G(p)$, defined by

$$G(p) = \sum_{n=1}^{\infty} a_n \frac{p^{n-1}}{(n-1)!} \in \mathbb{C}[[p]].$$

Since $f(x)$ is Gevrey, it follows that $G(p)$ is analytic in a neighborhood of $p = 0$.

- Endless analytically continue $G(p)$ to a so-called resurgent function.

- Medianize if needed.

- Define the Laplace transform of $G(p)$ by

$$(\mathscr{L}G)(x) = \int_0^{\infty} e^{-xp} G(p)\, dp.$$

In the example the power series of (1), its Borel transform $G(p)$ is given by

$$G(p) = \frac{1}{1-p}$$

which is a resurgent (in fact, meromorphic) function with a single singularity at $p = 1$.

In general, the output of a resummation is an analytic function (defined at least in a right half-plane), constructed in a canonical way from the divergent formal power series $f(x)$. In analysis, the resummation process commutes with differentiation, and as a result one constructs actual solutions of differential equations which are asymptotic to the formal power series that one starts with.

A side corollary of resurgence (of importance to quantum topology) is the existence of an asymptotic expansion of the coefficients of the power series $f(x)$. For a thorough discussion and examples, see [CG1].

The above description highlights the necessity of the Gevrey property, as a starting point of the resummation.

1.2. Formal power series in quantum topology. As mentioned before, a usual source of Gevrey series is a differential equation or a fixed-point problem. Quantum topology offers a different source of Gevrey series that do not seem to come from differential equations with polynomial coefficients, due to the different structure of singularities of their Borel transforms. For example (and getting a little ahead of us), the Kashaev invariant of two simplest knots (the trefoil (3_1), and the figure eight (4_1)) are the power series:

$$F_{3_1}(x) = \sum_{n=0}^{\infty} (e^{1/x})_n, \tag{3}$$

$$F_{4_1}(x) = \sum_{n=0}^{\infty} (e^{1/x})_n (e^{-1/x})_n, \tag{4}$$

where

$$(q)_n = (1-q)\dots(1-q^n).$$

Notice that $(e^{1/x})_n \in 1/x^n \mathbb{Q}[[1/x]]$, thus the power series $F_{3_1}(x)$ and $F_{4_1}(x)$ are well-defined elements of the formal power series ring $\mathbb{Q}[[1/x]]$.

The power series $F_{3_1}(x)$ is the *Kontsevich-Zagier* power series that was studied extensively by Zagier in [Za], and was identified with the Kashaev invariant of the trefoil by Huynh and the second author in [HL]. In [CG1], Costin and the first author gave an explicit formula for the Borel transform of $F_{3_1}(x)$:

Theorem 1 ([CG1]). *If $H_{3_1}(p)$ denotes the Borel transform of $e^{-1/(24x)}F_{3_1}(x)$, then we have*

$$H_{3_1}(p) = 54\sqrt{3}\pi \sum_{n=1}^{\infty} \frac{\chi(n)n}{(-6p + n^2\pi^2)^{5/2}},$$

where

$$\chi(n) = \begin{cases} 1 & \text{if } n \equiv 1, 11 \bmod 12, \\ -1 & \text{if } n \equiv 5, 7 \bmod 12, \\ 0 & \text{otherwise.} \end{cases}$$

Among other things, the above formula implies resurgence of the Borel transform of the series $F_{3_1}(x)$ and locates explicitly the position and shape of its singularities.

In [CG2], Costin and the first author prove by an abstract argument that the Borel transform of the power series $F_{3_1}(x)$ and $F_{4_1}(x)$ are resurgent functions.

The paper is concerned with two formal power series of importance to quantum topology:

- the Kashaev invariant of a knot,
- the LMO invariant of a closed 3-manifold.

Our aim is to prove that these series are Gevrey.

1.3. The Gromov norm of the LMO invariant is Gevrey. Let us give a first impression of the LMO invariant of Le-Murakami-Ohtsuki, [LMO]. It takes values in a (completed graded) vector space $\mathscr{A}(\emptyset)$ of trivalent graphs, modulo some linear AS and IHX relations:

$$Z : \text{3-manifolds} \to \mathscr{A}(\emptyset).$$

The LMO invariant gives a meaningful definition to Chern-Simons perturbation theory near a trivial flat connection. This is explained in detail in [BGRT], Part I. The trivalent graphs are the Feynman diagrams of a ϕ^3-theory (such as the Chern-Simons theory) and their AS and IHX relations are diagrammatic versions of the antisymmetry and the Jacobi identity of the Lie bracket of a metrized Lie algebra.

The vector space $\mathscr{A}(\emptyset)$ has a grading (or degree) defined by half the number of vertices of the trivalent graphs. Let $\mathscr{A}_n(\emptyset)$ denote the subspace of $\mathscr{A}(\emptyset)$ of degree n.

As we discussed above, the LMO invariant takes values in $\mathscr{A}(\emptyset)$. In order to make sense of its Gevrey property, we need to replace $\mathscr{A}(\emptyset)$ by $\mathbb{Q}[[1/x]]$. This is exactly what *weight systems* do: they convert trivalent graphs into numerical constants; see [B-N1]. More precisely, given a *simple Lie algebra* $\mathfrak{g}$, one can define a weight system map (see [B-N1]):

$$W_{\mathfrak{g}} : \mathscr{A}(\emptyset) \to \mathbb{Q}[[1/x]],$$

where each graph of degree n is mapped into a rational number times $1/x^n$. Combining the LMO invariant of a closed 3-manifold M with the weight system of a simple Lie algebra $\mathfrak{g}$, one gets a formal power series:

$$F_{\mathfrak{g},M}(x) = W_{\mathfrak{g}}(Z_M) \in \mathbb{Q}[[1/x]]. \tag{5}$$

This power series is equal to the Ohtsuki series, defined by Ohtsuki [Oh1] for $\mathfrak{g} = sl_2$ and then by the second author for all simple Lie algebras [Le2]. As nice as weight systems are, the power series still depends on Lie algebras; moreover it is known that not all weight systems come from Lie algebras, [Vo].

Ideally, we would like to replace the graph-valued invariant $Z_M \in \mathscr{A}(\emptyset)$ by a single series $|Z_M|_x \in \mathbb{Q}[[1/x]]$ so that:

(a) $|Z_M|_x \in \mathbb{Q}[[1/x]]$ is Gevrey.

(b) The Gevrey property of $|Z_M|$ implies the Gevrey property of $F_{\mathfrak{g},M}(x)$ for all simple Lie algebras.

(c) $|Z_M|_x \neq 1$ iff $Z_M \neq 1$.

Can we accomplish this at once? A simple idea, the Gromov norm, allows us to achieve this.

Definition 1.1. Consider a vector space V with a subset b that spans V. For $v \in V$, define b-norm by

$$|v|_b = \inf \sum_j |c_j|$$

where the infimum is taken over all presentations of the form $v = \sum_j c_j v_j$, $v_j \in b$.

For example, consider $V = \mathbb{Q}[q^{\pm 1}]$—the space of Laurent polynomials in q with rational coefficients, and b the set $\{q^n \mid n \in \mathbb{Z}\}$. In this case the norm of a Laurent polynomial $f(q)$ is known as its l^1-norm, denoted by $\|f(q)\|_1$.

Definition 1.2. (a) For an element $v \in \mathscr{A}(\emptyset)$, with b is the set of *trivalent graphs*, we will denote $|v|_b$ simply by $|v|$. For a detailed discussion, see Section 2.

(b) For an element $v \in \mathscr{A}(\emptyset)$ let $\mathrm{Grad}_n(v)$ be the part of degree n of v. The *Gromov norm* of v is defined as

$$|v|_x = \sum_{n=0}^{\infty} |\mathrm{Grad}_n(v)| \frac{1}{x^n} \in \mathbb{Q}[[1/x]].$$

(c) Let us say that $v \in \mathscr{A}(\emptyset)$ is *Gevrey-s* iff $|v|_x \in \mathbb{Q}[[1/x]]$ is Gevrey-s.

It is easy to see that $|v|_x = 1$ iff $v = 1$. Here, $1 \in \mathscr{A}(\emptyset)$ denotes the element of degree 0 which is 1 times the empty trivalent graph.

Our next theorem explains a Gevrey property of the LMO invariant.

Theorem 2. *For every integral homology sphere M, $|Z_M|_x \in \mathbb{Q}[[1/x]]$ is Gevrey.*

Moreover, $|Z_M|_x = 1$ iff $Z_M = 1$.

Theorem 2 and an easy estimate implies the following:

Theorem 3. *For every closed 3-manifold and every simple Lie algebra $\mathfrak{g}$, the Ohtsuki series $F_{\mathfrak{g},M}(x)$ is Gevrey.*

A key ingredient in the definition of the LMO invariant is the *Kontsevich integral Z_L* of a framed link L in S^3. The Gromov norm of Z_L can be defined in a similar fashion, see Section 2. In fact, Theorem 2 motivates (and even requires) to consider the Gromov norm $|Z_L|_x \in \mathbb{Q}[[1/x]]$ of the Kontsevich integral.

Theorem 4. *For every framed link L in S^3, $|Z_L|_x \in \mathbb{Q}[[1/x]]$ is Gevrey-0.*

Recall that a power series $f(x)$ is Gevrey-0 iff $f(x)$ is a convergent power series for x near ∞.

Theorems 2 and 4 are a special case of the following guiding principle, which we state as Meta-Theorem:

Meta-Theorem 1. *Asymptotic power series that appear in constructive quantum field theory* (*and in particular*, *in* 3-*dimensional Quantum Topology*) *are resurgent functions—and in particular*, *Gevrey of some order* (*usually*, *order* 1).

Let us comment that the factorial growth of power series in perturbative quantum field theory is usually due to the factorial growth of the number of Feynman diagrams; see for example Lemma 2.13. The contribution of each Feynman diagram is growing exponentially only; see for example Lemma 2.12.

1.4. The Habiro ring. So far we discussed how perturbative quantum field theory leads to Gevrey power series (5). Examples of such series (for knots, rather than 3-manifolds) were given in Equations (3) and (4).

In the remaining of this section, we will concentrate with the case of $\mathfrak{g} = \mathfrak{sl}_2$. Our aim is to give a non-perturbative explanation of the Gevrey property of the power series $F_{\mathfrak{sl}_2, M}(x)$, which we will abbreviate by $F_M(x)$ in this section. In fact, we will be dealing with a formal power series invariant of knotted objects:

$$F : \text{knotted objects} \to \mathbb{Q}[[1/x]] \tag{6}$$

where a *knotted object* (denoted in general by $\mathscr{K}$) will be either a knot K in 3-space or an integral homology sphere M. We already discussed the series $F_M(x) := F_{\mathfrak{sl}_2, M}(x)$. In the case of a knot K, the power series $F_K(x)$ will be defined below.

In the absence of a rule (such as a differential equation) for the power series $F_{\mathscr{K}}(x)$, or an explicit formula (in the style of (3) or (4)), how can one prove that our power series are Gevrey? It turns out that the power series $F_{\mathscr{K}}(x)$ have a certain "shape" which explains their Gevrey (and conjectural resurgence) property. Such a shape was discovered by Habiro, who considered the *cyclotomic completion* of the ring of Laurent polynomial (the so-called *Habiro ring*)

$$\hat{\Lambda} = \lim_{\substack{\longleftarrow \\ n}} \mathbb{Z}[q^{\pm 1}]/((q)_n).$$

As a set, it follows that the Habiro ring is

$$\hat{\Lambda} = \left\{ f(q) = \sum_{n=0}^{\infty} f_n(q)(q)_n \,\middle|\, f_n(q) \in \mathbb{Z}[q^{\pm 1}] \right\}.$$

Habiro showed a number of key properties of the ring $\hat{\Lambda}$; see [H2]. For our purposes, it will be important that elements $f(q)$ of the Habiro ring have Taylor series expansions at $q = 1$, and that they are uniquely determined by their Taylor series. In other words, the map from $\hat{\Lambda}$ to $\mathbb{Z}[[q-1]]$, sending $f(q)$ to its Taylor series at 1, is injective. Let

$$T : \hat{\Lambda} \to \mathbb{Q}[[1/x]], \quad (Tf)(x) = f(e^{1/x}).$$

Then Habiro proves that T is injective.

In the case of an integral homology sphere M, Habiro proved that the series $F_M(x)$ comes from a (unique) element $\Phi_M(q)$ of the Habiro ring. In the case of a knot, Huynh and the second author observe in [HL] that the *Kashaev invariant* of a knot K also comes from an element $\Phi_K(q)$ of the Habiro ring. In that case, we define $F_K(x) = (T\Phi_K)(x) = \Phi_K(e^{1/x})$.

In other words, we have a map

$$\Phi : \text{knotted objects} \to \hat{\Lambda}$$

such that

$$F = T \circ \Phi.$$

Thus, instead of writing

$$F_{\mathscr{K}}(x) = \sum_{n=0}^{\infty} a_{\mathscr{K},n} \frac{1}{x^n}$$

for $a_{\mathscr{K},n} \in \mathbb{Q}$, we may write

$$F_{\mathscr{K}}(x) = \sum_{n=0}^{\infty} f_{\mathscr{K},n}(e^{1/x})(e^{1/x})_n$$

for suitable polynomials $f_{\mathscr{K},n}(q) \in \mathbb{Z}[q^{\pm 1}]$. Keep in mind that the polynomials $f_{\mathscr{K},n}(q)$ are not unique. For example, we have the following identity in the Habiro ring:

$$\sum_{n=0}^{\infty} q^{n+1}(q)_n = 1.$$

Most importantly for us, without any additional information about the polynomials $f_{\mathscr{K},n}(q)$ one cannot expect that the series $F_{\mathscr{K}}(x)$ is Gevrey. The information can be formalized by introducing two subrings of $\hat{\Lambda}$. We need an auxiliary definition.

Definition 1.3. (a) We say that a sequence $\big(f_n(q)\big)$ of Laurent polynomials is *q-holonomic* if it satisfies a linear q-difference equation of the form

$$a_d(q^n, q) f_{n+d}(q) + \cdots + a_0(q^n, q) f_n(q) = 0$$

for all $n \in \mathbb{N}$, where $a_j(u, v) \in \mathbb{Z}[u^{\pm 1}, v^{\pm 1}]$ for $j = 0, \ldots, d$ and $a_d \neq 0$.

(b) We say that a sequence $\big(f_n(q)\big)$ of Laurent polynomials is *nicely bounded* if there exist the bounds on their span and coefficients: There are constants $C, C' > 0$ that depend on $\big(f_n(q)\big)$ such that for $n > 0$,

$$\operatorname{span}_q f_n(q) \subset [-C'n^2, C'n^2], \tag{7}$$

$$\|f_n(q)\|_1 \leqq C^n. \tag{8}$$

Now, we may define the following subrings of the Habiro ring.

Definition 1.4. (a) We define:

$$\hat{\Lambda}^{\mathrm{hol}} = \left\{ f(q) = \sum_{n=0}^{\infty} f_n(q)(q)_n \,\middle|\, f_n(q) \in \mathbb{Z}[q^{\pm 1}], \big(f_n(q)\big) \text{ is } q\text{-holonomic} \right\}.$$

(b) We define:

$$\hat{\Lambda}^{b} = \left\{ f(q) = \sum_{n=0}^{\infty} f_n(q)(q)_n \,\middle|\, f_n(q) \in \mathbb{Z}[q^{\pm 1}], \big(f_n(q)\big) \text{ is nicely bounded} \right\}.$$

It is easy to see that $\hat{\Lambda}^{\mathrm{hol}}$ and $\hat{\Lambda}^{b}$ are subrings of $\hat{\Lambda}$. Observe that $\hat{\Lambda}^{\mathrm{hol}}$ is a countable ring, whereas $\hat{\Lambda}$ and $\hat{\Lambda}^{b}$ are not.

It is easy to show that if $\big(f_n(q)\big)$ is a q-holonomic sequence of Laurent polynomials, then it satisfies (7). On the other hand, the authors do not know the answer to the following question.

Question 1. Is it true that $\hat{\Lambda}^{\mathrm{hol}}$ is a subring of $\hat{\Lambda}^{b}$?

1.5. Gevrey series from the Habiro ring. Independently of the answer to the above question, we have:

Theorem 5. *For every knotted object $\mathscr{K}$ we have*

$$\Phi_{\mathscr{K}}(q) \in \hat{\Lambda}^{\mathrm{hol}} \cap \hat{\Lambda}^{b}. \tag{9}$$

Our next theorem relates the ring $\hat{\Lambda}^{b}$ with Gevrey series.

Theorem 6. *If $f(q) \in \hat{\Lambda}^{b}$, then $f(e^{1/x}) \in \mathbb{Q}[[1/x]]$ is Gevrey.*

Theorems 5 and 6 imply the promised result.

Theorem 7. *For every knotted object $\mathscr{K}$, the power series $F_{\mathscr{K}}(x)$ is Gevrey.*

If M is an integral homology sphere, then the above theorem gives an independent proof that the series $F_M(x)$ is Gevrey.

1.6. What next? As was mentioned in Section 1.1, a Gevrey series is the input of a resummation process. In [CG2] we conjecture that the series $F_{\mathscr{K}}(x)$ of every knotted object $\mathscr{K}$ can be resummed. In other words, we conjecture that the Borel transform $G_{\mathscr{K}}(p)$ of $F_{\mathscr{K}}(x)$ is a resurgent function, with singularities given by geometric invariants of the knotted object $\mathscr{K}$. This conjecture is true for the two simplest knots 3_1 and 4_1 and for several elements of $\hat{\Lambda}^{\mathrm{hol}}$; see [CG1] and [CG2]. Based on this partial evidence, we pose the following questions:

Question 2. If $f(q) \in \hat{\Lambda}^{\mathrm{hol}}$, is it true that its Taylor series $(Tf)(x) \in \mathbb{Q}[[1/x]]$ has resurgent Borel transform?

Question 3. Is it true that the Gromov norm $|Z_M|_x \in \mathbb{Q}[[1/x]]$ of the LMO invariant of an integral homology sphere has resurgent Borel transform?

Question 4. Is it true that the Gromov norm $|Z_L|_x \in \mathbb{Q}[[1/x]]$ of the Kontsevich integral of a framed link in S^3 is a resurgent function?

For a detailed discussion on analytic continuation of the power series of our paper, see [G2].

1.7. Plan of the proof. Since Gevrey series are not familiar objects in quantum topology, we have made an effort to motivate their appearance and usefulness in quantum topology. For the analyst, we would like to point out that our Gevrey series (and their expected resurgence properties) are not expected to be solutions of differential equations (linear or not) with polynomial coefficients. Thus, our results are new from this perspective.

We have also separated into different sections results from quantum topology and from asymptotics.

In Section 2 we discuss in detail the LMO invariant, starting from the necessary discussion of the Kontsevich integral of a framed link in 3-space. Basically, the LMO invariant of a 3-manifold is obtained by the (suitably normalized) Kontsevich integral of a surgery presentation link, after we glue all legs. We will use combinatorial counting arguments to bound the number of unitrivalent graphs, as well as the original definition of the Kontsevich integral to estimate the coefficients of these graphs, before and after the gluing of the legs. In addition in Section 2.9 we show that various analytic reparametrizations of the LMO invariant (such as the Ohtsuki series) are Gevrey. This ends the perturbative quantum field theory discussion of the paper.

In Section 4 we give a nonperturbative explanation of the Gevrey property of our power series for the simple Lie algebra $\mathfrak{sl}_2$. In that case, the Kontsevich integral is replaced by the colored Jones function of a link. The latter is a multisequence of Laurent polynomials. We discuss two key properties of the colored Jones function: q-holonomicity (introduced in [GL1]) and integrality, introduced by Habiro in [H1], [H2]. Together with Habiro's definition of $\Phi_M(q)$ (given in terms of a surgery presentation of M), q-holonomicity implies that $\Phi_M(q) \in \hat{\Lambda}^{\mathrm{hol}}$, and integrality implies that $\Phi_M(q) \in \hat{\Lambda}^b$. Combined together with Theorem 6 (shown in the next section), they give a proof of Theorem 7.

Finally, in Section 3 we use elementary estimates to give a proof of Theorem 6.

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2. The LMO invariant is Gevrey

In this section we will give a proof of Theorem 2.

Omitting technical details, the *Aarhus version* of the LMO invariant ([BGRT], Part II and III), is defined as follows. Suppose an integral homology M is obtained from S^3 by surgery on a framed link L.

• Consider a presentation of L as the closure of a framed string link T.

• Consider the suitably normalized *Kontsevich integral* $\check{Z}_T$ of the string link T. It takes values in a completed $\mathbb{Q}$-vector space of vertex-oriented unitrivalent graphs (Jacobi diagrams) with legs colored by the components of L.

• Separating out the strut part from $\check{Z}_T$ and closing we get the formal Gaussian integral $\int \check{Z}_T$, which takes values in the algebra $\mathscr{A}(\emptyset)$ of Jacobi diagrams without legs.

• Finally, normalize $\int \check{Z}_T$ in a minor way to get the LMO invariant Z_M.

The precise definition will be recalled later. To prove Theorem 2 we will need to have an estimate

(a) for the norm of the Kontsevich integral and

(b) for the norms of the maps appearing in the definition of the LMO invariant.

To get the desired estimates it will be simpler to exclude Jacobi diagrams with tree components. This is guaranteed when L is a boundary link. And it suffices since every integral homology sphere can be obtained by surgery along a unit-framed boundary link.

2.1. Jacobi diagrams. We quickly recall the basic definitions and properties here, referring the details to [B-N1], [BLT].

An *open Jacobi diagram* is a vertex-oriented uni-trivalent graph, i.e., a graph with univalent and trivalent vertices together with a cyclic ordering of the edges incident to the trivalent vertices. A univalent vertex is called *a leg*, and a trivalent vertex is also called an *internal vertex*. The *degree* of an open Jacobi diagram is half the number of vertices (trivalent and univalent). The i-degree is the number of internal vertices, and the e-degree is the number of legs.

Suppose X is a compact oriented 1-manifold (possibly with boundary) and Y a finite set. A *Jacobi diagram based on* $X \cup Y$ is a graph D together with a decomposition $D = X \cup \Gamma$, where Γ is an open Jacobi diagram with some legs labeled by elements of Y, such that D is the result of gluing all the non-labeled legs of Γ to distinct interior points of X. Note that repetition of labels is allowed. The *degree* of D, by definition, is the degree of Γ.

The space $\mathscr{A}^f(X, Y)$ is the vector space over $\mathbb{Q}$ spanned by Jacobi diagrams based on $X \cup Y$ modulo the usual AS, IHX and STU relations (see [B-N1]). The completion of $\mathscr{A}^f(X, Y)$ with respect to degree is denoted by $\mathscr{A}(X, Y)$.

Of special interest are the following flavors of Jacobi diagrams:

(a) $(X, Y) = (\circlearrowleft_m, \emptyset)$, where $X = \circlearrowleft_m$ the union of m numbered, oriented circles. Then, $\mathscr{A}(X, Y) = \mathscr{A}(\circlearrowleft_m)$ is the space where the Kontsevich integral of m-component framed link lies.

(b) $(X, Y) = (\uparrow_m, \emptyset)$, where $X = \uparrow_m$, the union of m numbered, oriented intervals. Then, $\mathscr{A}(X, Y) = \mathscr{A}(\uparrow_m)$ is the space where the Kontsevich integral of m-component framed braid (or a dotted Morse link) lies.

(c) $(X, Y) = (\emptyset, \{1, 2, \dots, m\})$. Then, we denote $\mathscr{A}(X, Y)$ by $\mathscr{A}(\star_m)$

(d) $(X, Y) = (\circlearrowleft_r \cup \uparrow_s, \emptyset)$. Then $A(X, Y)$ is the space where the Kontsevich integral of a tangle T lies, where r is the number of interval components of T and s is the number of circle components of T.

The spaces $\mathscr{A}(\circlearrowleft_m)$, $\mathscr{A}(\uparrow_m)$ are related with an obvious projection map:

$$p : \mathscr{A}(\uparrow_m) \to \mathscr{A}(\circlearrowleft_m) \tag{10}$$

which identifies the two end points of each interval in $\uparrow_m$.

The spaces $\mathscr{A}(\uparrow_m)$ and $\mathscr{A}(\star_m)$ are also related with a *symmetrization map*

$$\chi : \mathscr{A}(\star_m) \to \mathscr{A}(\uparrow_m) \tag{11}$$

which is a linear map defined on a diagram Γ by taking the average over all possible ways of ordering the legs labeled by j, $1 \leqq j \leqq m$, and attach them to the j-th oriented interval. It is known that χ is a vector space isomorphism [B-N1].

Remark 2.1. Note that $\mathscr{A}(\uparrow_m)$ is an algebra, where the product of two Jacobi diagrams is obtained by placing (or stacking) the first on top of the second. $\mathscr{A}(\star_m)$ is also an algebra, where the product of two diagrams is their disjoint union. However, the map χ, which is a vector space isomorphism, is not an algebra isomorphism. To get an algebra isomorphism one needs the wheeling map, see [BLT]. To avoid confusion we use $\#$ to denote the product in $\mathscr{A}(\uparrow_m)$ and $\sqcup$ the product in $\mathscr{A}(\star_m)$.

The diagonal map $\Delta^{(m)} : \mathscr{A}(\star_1) \to \mathscr{A}(\star_m)$ is a linear map defined on a Jacobi diagram $\Gamma \in \mathscr{A}(\star_1)$ by taking the sum of all possible Jacobi diagrams $\Gamma' \in \mathscr{A}(\star_m)$ such that if we switch all the labels in Γ' to 1, then from Γ' we get Γ. It is clear that if Γ has k legs, then there are m^k such Γ'.

Suppose $X = Y = \emptyset$. The space $\mathscr{A}(\emptyset)$ is the space in which lie the values of the LMO invariants of 3-manifolds [LMO]. With disjoint union as the product, $\mathscr{A}(\emptyset)$ becomes a commutative algebra, and all other $\mathscr{A}(X, Y)$ have a natural $\mathscr{A}(\emptyset)$-module structure.

An open Jacobi diagram is *tree-less* if none of its connected components is a tree. Let $\mathscr{A}^{\mathrm{tl}}(\star_m)$ be the subspace of $\mathscr{A}(\star_m)$ spanned by treeless Jacobi diagrams. The following is obvious but useful later.

Lemma 2.2. *If $v \in \mathscr{A}^{\mathrm{tl}}(\star_m)$ has i-degree n, then the e-degree of v is less than or equal to n.*

2.2. Norm of Jacobi diagrams. The set of Jacobi diagrams based on $X \cup Y$ clearly spans the space $\mathscr{A}(X, Y)$. The norm of $v \in \mathscr{A}_n(X, Y)$ with respect to this spanning subset is denoted by $|v|$. The *Gromov norm* of $v \in A(X, Y)$ with respect to the set of Jacobi diagrams is defined by

$$|v|_x = \sum_{n=0}^{\infty} |\mathrm{Grad}_n(v)| \frac{1}{x^n} \in \mathbb{Q}[[1/x]]. \tag{12}$$

We will say that $v \in \mathscr{A}(X, Y)$ is Gevrey-s iff $|v|_x \in \mathbb{Q}[[1/x]]$ is Gevrey-s.

It is clear that if the product vu can be defined, then $|vu| \leqq |v|\,|u|$. Since the product of two Gevrey-s power series is Gevrey-s, and the inverse of a Gevrey-s series with nonzero constant term is Gevrey-s (see for example, [Ba], Exer. 6, 7, p. 5), it follows that:

Lemma 2.3. (a) *If $v, u \in A(X, Y)$ are Gevrey-s and the product vu can be defined, then vu is also Gevrey-s.*

(b) *If $v \in \mathscr{A}(\emptyset)$ has non-zero constant term and is Gevrey-s, then $1/v \in \mathscr{A}(\emptyset)$ is Gevrey-s.*

For an element $v \in \mathscr{A}^{\mathrm{tl}}(\star_m) \subset \mathscr{A}(\star_m)$, in addition to the above norm, there is another one defined using the spanning set of *treeless Jacobi diagrams*.

Lemma 2.4. *The above two norms are equal.*

Proof. The lemma follows at once from the fact that the subspace spanned by Jacobi diagrams other than treeless ones intersects $\mathscr{A}^{\mathrm{tl}}(\star_m)$ only by the zero vector. □

Recall the symmetrization map χ from (11). The next proposition estimates the norm of χ and χ^{-1}.

Proposition 2.5. (a) *For every $v \in \mathscr{A}(\star_m)$, one has $|\chi(v)| \leqq |v|$. In other words, the operator χ has norm less than or equal to* 1.

(b) *Suppose $x \in \mathscr{A}(\uparrow_m)$ has e-degree $k \geqq 1$, then $|\chi^{-1}(v)| \leqq 2k|v|$.*

(c) *For any $v \in \mathscr{A}(\star_1)$ of e-degree k, one has $|\Delta^{(m)}(v)| \leqq m^k|v|$.*

Proof. (a) and (c) follow immediately from the definition. We give here the proof of (b).

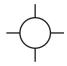
We use induction. Suppose the statement holds true when v has e-degree $< k$. It is enough to prove for the case when $v = \Gamma \cup \uparrow_m$, where Γ is an open Jacobi diagram with k legs. Using the STU relation, one can see that $u := \chi(\Gamma) - v$ has e-degree $< k$. One has

$$|u| = |\chi(\Gamma) - v| \leqq |\chi(\Gamma)| + |v| \leqq 2.$$

By induction, $|\chi^{-1}(u)| < 2(k-1)$. Since $v = u + \chi(\Gamma)$, we have $\chi^{-1}(v) = \chi^{-1}(u) + \Gamma$, and hence

$$|\chi^{-1}(v)| \leqq |\chi^{-1}(u)| + |\Gamma| \leqq 2(k-1) + 1 < 2k. \quad \square$$

2.3. The unknot. Let $w_{2n} \in \mathscr{A}(\star_1)$ be the *wheel* with $2n$ legs. It is the open Jacobi diagram consisting of a circle and $2n$ intervals attached to it. For example,

$w_4 =$  .

Define

$$v = \exp\left(\sum_{n=1}^{\infty} b_{2n}\omega_{2n}\right) \in \mathscr{A}(\star_1) \quad \text{and} \quad \sqrt{v} = \exp\left(\frac{1}{2}\sum_{n=1}^{\infty} b_{2n}\omega_{2n}\right) \in \mathscr{A}(\star_1),$$

where the *modified Bernoulli numbers* b_{2n} are defined by the power series expansion

$$\sum_{n=0}^{\infty} b_{2n}x^{2n} = \frac{1}{2}\log\frac{\sinh x/2}{x/2}. \tag{13}$$

Notice that

$$\frac{1}{2}\log\frac{\sinh x/2}{x/2} = \frac{1}{48}x^2 - \frac{1}{5760}x^4 + \frac{1}{362880}x^6 + \cdots.$$

The modified Bernoulli numbers are related to the zeta function $\zeta(n) := \sum_{k=1}^{\infty} k^{-n}$ by

$$b_{2n} = (-1)^{n-1}\frac{\zeta(2n)}{2n(2\pi)^{2n}}. \tag{14}$$

In [BLT] it was shown that $\chi^{-1}(v)$ is the Kontsevich integral of the unknot.

Proposition 2.6. *The series v and $\sqrt{v}$ are Gevrey*-0.

Proof. Since $|w_{2n}| \leqq 1$, it is enough to to show that the series $\exp\left(\sum_n |b_{2n}|x^{-2n}\right)$ is convergent for large enough x. Since $|b_{2n}| = (-1)^{n-1}b_{2n}$, it follows that

$$\begin{aligned}
\exp\left(\sum_n |b_{2n}|x^{-2n}\right) &= \exp\left(-\sum_n b_{2n}(1/x)^{2n}\right) \\
&= \exp\left(-\frac{1}{2}\log\frac{\sin\big(1/(2x)\big)}{1/(2x)}\right) \\
&= \sqrt{\frac{1/(2x)}{\sin\big(1/(2x)\big)}}.
\end{aligned}$$

The latter converges for $|x| > 1/(4\pi)$. $\square$

Question 5. Is it true that

$$|v|_x = \sqrt{\frac{1/(2x)}{\sin\big(1/(2x)\big)}}?$$

Since $\mathrm{Grad}_n v$, the part of degree n of v, has e-degree $2n$, Proposition 2.5(c) implies that

Corollary 2.7. *For every positive integer m the series $\Delta^{(m)}(v) \in \mathscr{A}(\star_m)$ is Gevrey-*0.

2.4. The Kontsevich integral. The *framed* Kontsevich integral of a *framed* tangle T takes value in $\mathscr{A}(T)$, see for example [LM], [B-N2], [BLT]. This is a slight modification of the original integral defined by Kontsevich [Ko]. The framed Kontsevich integral depends on the positions of the boundary points. To get rid of this dependence one has to choose standard positions for the boundary points. It turns out that the best positions are in a limit, when all the boundary points go to one fixed point. In addition one has to regularize the Kontsevich integral in the limit. In the limit one has to keep track of the order in which the boundary points go to the fixed point. This leads to the notion of *parenthesized framed tangle*. The latter were called *q-tangle* in [LM] and *non-commutative tangles* in [B-N2]. For details, see [LM] and [B-N2].

In all framed tangles in this paper, we assume that a non-associative structure is fixed. Theorem 4 is a special case of the following theorem.

Theorem 8. *For every framed tangle T the Kontsevich integral Z_T is Gevrey-*0.

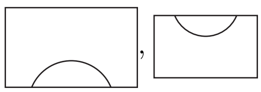
Proof. This follows from the facts that:

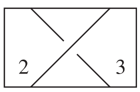
(a) The Kontsevich integral satisfies a locality property. In other words, a framed tangle is the assembly (i.e., the product) of elementary blocks of three kinds: local extrema, crossings, and change of parenthetization; for a computerized example, see [B-N2], Sec. 1.2. The Kontsevich integral is the corresponding product of the invariants of the elementary blocks.

(b) The invariants of each blocks are Gevrey-0.

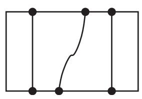
(c) The product of Gevrey-0 series is Gevrey-0 by Lemma 2.3.

More precisely, the Kontsevich integral of the elementary blocks is given by:

 $\to \sqrt{v}$,



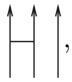
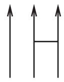
$\to \exp\left(\frac{1}{2}\,{}^{1}\frown^{2}\right)$,

 $\to \Phi$,

where the *strut* ${}^{1}\frown^{2}$ is the only open Jacobi diagram homeomorphic to an interval, and Φ is any *associator*. For a definition of an associator, see [Dr] and also [B-N2], [B-N3]. Proposition 2.6 implies that $\sqrt{v}$ is Gevrey-0. In [LM], J. Murakami and the second author gave an explicit formula for the KZ-*associator* Φ^{KZ}:

$$
\begin{aligned}
\Phi^{\mathrm{KZ}} &= 1 + \sum_{l=1}^{\infty} \sum_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{p},\boldsymbol{q}} (-1)^{|\boldsymbol{b}|+|\boldsymbol{p}|} \eta(\boldsymbol{a}+\boldsymbol{p}, \boldsymbol{b}+\boldsymbol{q}) \binom{\boldsymbol{a}+\boldsymbol{p}}{\boldsymbol{b}+\boldsymbol{q}} B^{|\boldsymbol{q}|} (A,B)^{(\boldsymbol{a},\boldsymbol{b})} A^{|\boldsymbol{p}|} \\
&= 1 + \frac{1}{24}[A,B] - \frac{\zeta(3)}{(2\pi i)^3}[A,[A,B]] + \cdots
\end{aligned} \tag{15}
$$

where

$A =$ , $B =$ ,

$[X, Y] = XY - XY,$

$$
\zeta(a_1, a_2, \ldots, a_k) = \sum_{n_1 < n_2 < \cdots < n_k \in \mathbb{N}} n_1^{-a_1} n_2^{-a_2} \ldots n_k^{-a_k} \tag{16}
$$

are the multiple zeta numbers and for $\boldsymbol{a} = (a_1, \ldots, a_l)$ and $\boldsymbol{b} = (b_1, \ldots, b_l)$ we put

$$
\eta(\boldsymbol{a}, \boldsymbol{b}) = \zeta(\underbrace{1, 1, \ldots, 1}_{a_1 - 1}, b_1 + 1, \underbrace{1, 1, \ldots, 1}_{a_2 - 1}, b_2 + 1, \ldots, \underbrace{1, 1, \ldots, 1}_{a_l - 1}, b_l + 1),
$$

$$
\begin{aligned}
|\boldsymbol{a}| &= a_1 + a_2 + \cdots + a_l, \\
\binom{\boldsymbol{a}}{\boldsymbol{b}} &= \binom{a_1}{b_1}\binom{a_2}{b_2} \cdots \binom{a_l}{b_l}, \\
(A,B)^{(\boldsymbol{a},\boldsymbol{b})} &= A^{a_1} B^{b_1} \ldots A^{a_l} B^{b_l}.
\end{aligned}
$$

Equation (15) implies that the KZ associator Φ^{KZ} is Gevrey-0. □

Remark 2.8. It is not true that every associator $\Phi \in \mathscr{A}(\uparrow_3)$ is Gevrey-0. In fact, it is not even true that the twist of a Gevrey-0 associator is Gevrey-0, since the twist may have arbitrarily large coefficients.

Remark 2.9. There is an alternative proof of Theorem 8 that does not use associators. First decompose T into smaller tangles, where each smaller one is either elementary of type 1, or a braid. By deformation we can assume that in any braid X, the horizontal distance between any 2 strands is bigger than 1. Then the very Kontsevich integral formula of $Z(X)$, see [Ko] and [B-N1], Sec. 4.3 (and also [B-N1], Fig. 13), is regular and easily seen to be Gevrey-0.

2.5. The LMO invariant. In this section we review the Aarhus version of the LMO invariant from [BGRT], Part II. For an equality of the Aarhus integral with the LMO invariant, see [BGRT], Part III.

We define a bilinear map

$$
\langle \cdot, \cdot \rangle : \mathscr{A}(\star_m) \otimes \mathscr{A}^{\mathrm{tl}}(\star_m) \to \mathscr{A}(\emptyset) \tag{17}
$$

as follows. Suppose $\Gamma_1 \in \mathscr{A}(\star_m)$ and $\Gamma_2 \in \mathscr{A}^{\mathrm{tl}}(\star_m)$ are Jacobi diagrams with respectively k_j, l_j legs of label j, $j = 1, 2, \ldots, m$. If there is a j such that $k_j \neq l_j$, let $\langle \Gamma_1, \Gamma_2 \rangle = 0$, otherwise let $\langle \Gamma_1, \Gamma_2 \rangle$ be the sum of all possible ways to glue legs of label j in Γ_1 to legs of the same labels in Γ_2. Note that there are $\prod_{j=1}^{m} (k_j)!$ terms in the sum. If $v \in \mathscr{A}(\star_m)$ and $u \in \mathscr{A}^{\mathrm{tl}}(\star_m)$ have k_j legs of label j, then

$$|\langle v, u \rangle| \leqq \left(\prod_{j=1}^{m} k_j! \right) |v|\,|u|. \tag{18}$$

It is known that the integral homology sphere M can be obtained from S^3 by surgery along a *boundary link* L, where the framing $\varepsilon_1, \ldots, \varepsilon_m$ of the link components are ± 1. Suppose furthermore L is the closure of a framed boundary string link T. It is known ([LM]) that

$$Z_L = p\big([Z_T] \mathbin{\#} \big[\chi^{-1}\big(\Delta^{(m)}(v)\big)\big]\big).$$

Let us introduce some convenient notation. For Jacobi diagrams $\Gamma_j \in \mathscr{A}(\star_1)$, $j = 1, 2, \ldots, m$ let $\Gamma_1 \otimes \cdots \otimes \Gamma_m \in \mathscr{A}(\star_m)$ be the union of all Γ_j, with the legs of Γ_j relabeled by j. Using linearity we can define $v_1 \otimes \cdots \otimes v_m \in \mathscr{A}(\star_m)$ for $v_j \in \mathscr{A}(\star_1)$. With the above notation, let us define

$$\begin{aligned} \check{Z}_T &:= [Z_T] \mathbin{\#} \big[\chi^{-1}\big(\Delta^{(m)}(v)\big)\big] \mathbin{\#} [\chi^{-1}(v^{\otimes m})], \\ \tilde{Z}_T &:= \chi(\check{Z}_T) \sqcup E, \\ E = E(\varepsilon_1, \ldots, \varepsilon_m) &:= \exp\left(-\frac{1}{2} \sum_{j=1}^{m} \epsilon_j \, {}_j\!\frown_j \right) \in \mathscr{A}(\star_m). \end{aligned}$$

Notice that $\tilde{Z}_T$ has no struts. Since T is a boundary framed link, it follows from [HM] (see also [GL0]) that $\tilde{Z}_T$ is treeless. One can define

$$\int T := \langle E, \tilde{Z}_T \rangle \in \mathscr{A}(\emptyset).$$

Note that our $\int T$ is equal to $\int^{FG} \check{Z}_L$ in [BGRT].

Suppose $U_\pm$ are the trivial string knots with framing ± 1. Suppose among $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m$ there are s_+ positive numbers and s_- negative numbers. Then the LMO invariant of M can be calculated by

$$Z_M = \frac{\int T}{(\int U_+)^{s_+} (\int U_-)^{s_-}}.$$

2.6. Proof of Theorem 2. Using the multiplicative property of Gevrey-1 series, see Lemma 2.3, to prove that Z_M is Gevrey-1 it is enough to prove the following lemma.

Lemma 2.10. *For every boundary string link T with framing $\varepsilon_1, \ldots, \varepsilon_m \in \{\pm 1\}$, the series $\int T$ is Gevrey*-1.

Proof. By definition, we have

$$E = \sum_{k_1,\dots,k_m \geqq 0} E_{k_1,\dots,k_m},$$

where

$$E_{k_1,\dots,k_m} := \prod_{j=1}^{m} \left(\frac{(-\varepsilon_j/2)^{k_j}}{k_j!} \right) \left({}_1\frown{}_1\right)^{k_1} \left({}_2\frown{}_2\right)^{k_2} \dots \left({}_m\frown{}_m\right)^{k_m}. \tag{19}$$

Let $\tilde{Z}_T^{(2n,2k)}$ be the part of $\tilde{Z}_T$ of i-degree $2n$ and e-degree $2k$. Since $\tilde{Z}_T$ is treeless, by Lemma 2.2, we have $k \leqq n$, and hence the degree of $\tilde{Z}_T^{(2n,2k)}$ is less than or equal to $2n$. By Theorem 8, Corollary 2.7, Proposition 2.5(b), and the multiplicative property of Gevrey-0 series, $\tilde{Z}_T$ is Gevrey-0. Thus, there is a constant C such that for every $n \geqq 1$ we have

$$|\tilde{Z}_T^{(2n,2k)}| < C^n.$$

Since E consists of struts only, $\langle E, v \rangle$ has degree equal half the i-degree of v. Hence

$$\mathrm{Grad}_n \textstyle\int T = \displaystyle\sum_{k=0}^{n} \langle E, \tilde{Z}_T^{(2n,2k)} \rangle. \tag{20}$$

Recall that $E = \sum E_{k_1,\dots,k_m}$, and $E_{k_1,\dots,k_m}$ has $2k_j$ legs of label j. For fixed k, the inner product $\langle E_{k_1,\dots,k_m}, \tilde{Z}_T^{(2n,2k)} \rangle$ is non-zero only when $k_1 + \dots + k_m = k$. Using (18) and (19) we have

$$\begin{aligned}
|\langle E_{k_1,\dots,k_m}, \tilde{Z}_T^{(2n,2k)} \rangle| &< C^n \prod_{j=1}^{m} \frac{(2k_j)!}{k_j!} \\
&< C^n \prod_{j=1}^{m} 2^{k_j} k_j! \quad \left(\text{since } \frac{(2k)!}{k!} \leqq 2^k k!\right) \\
&< C^n 2^n n! \quad (\text{since } k_1 + \dots + k_m = k \leqq n).
\end{aligned}$$

It follows that the norm of

$$\langle E, \tilde{Z}_T^{(2n,2k)} \rangle = \sum_{k_1+\dots+k_m=k} \langle E_{k_1,\dots,k_m}, \tilde{Z}_T^{(2n,2k)} \rangle$$

can be estimated by

$$|\langle E, \tilde{Z}_T^{(2n,2k)} \rangle| < C^n 2^n n! \left(\sum_{k_1+\dots+k_m=k} 1 \right) = C^n 2^n n! \binom{k+m-1}{m} \leqq C^n 2^n n! 2^{n+m}.$$

Using (20), we get

$$|\mathrm{Grad}_n \textstyle\int T| < n C^n 2^n n! 2^{n+m} < n! C'^n$$

for an appropriate constant $C' = C'_T$. This concludes the proof of Theorem 2. □

Remark 2.11. Without doubt, Theorem 2 holds for rational homology spheres as well. This requires a technical modification of the proof that allows one to deal with Jacobi diagrams with tree components. This is possible, but it requires another layer of technicalities that we will not present here.

2.7. Proof of Theorem 3. Theorem 3 follows immediately from Theorem 2 and the following Lemma 2.12.

Lemma 2.12. *For every simple Lie algebra* $\mathfrak{g}$ *there is a constant* C *such that for any Jacobi diagram* $\Gamma \in \mathcal{A}(\emptyset)$ *of degree* $n > 0$ *we have*

$$|W_{\mathfrak{g}}(\Gamma)| \leqq C^n.$$

Proof. Γ is obtained from a cloud of $2n$ Y graphs by a complete pairing of their legs. By the definition of the weight system, it follows that $W_{\mathfrak{g}}(\Gamma)$ is obtained by the contraction of the indices of a tensor in $\bigoplus_{2n}(\mathfrak{g}^{\otimes 3})$. The result follows. $\square$

2.8. On the dimension of the space of Feynman diagrams. Although our proof of Theorem 2 is completed, in this section we will give some estimates for the space $\mathcal{D}_n(\emptyset)$ and $\mathcal{A}_n(\emptyset)$ of Feynman diagrams, introduced in Section 1.3. Our crude estimates explain the Gevrey nature of the Gromov norm of the LMO invariant.

Let $\mathcal{D}_n(\emptyset)$ denote the (finite dimensional) vector space with basis the set of Jacobi diagrams with no legs of degree n, and $\mathcal{A}_n(\emptyset)$ is its quotient by the AS and IHX relations.

Let us say that a Jacobi diagram is *normalized* if it is made out of a number of disjoint circles together with a number of chords that each begin and end on the same circle. Let $\mathcal{S}$ (resp. $\mathcal{S}_n$) denote the set of normalized graphs (resp. of degree n).

Lemma 2.13. (a) $\dim \mathcal{D}_n(\emptyset) \leqq n!^3 C'^n$ *for some* C'.

(b) $\mathcal{S}_n$ *is a spanning set for* $\mathcal{A}_n(\emptyset)$.

(c) $\dim \mathcal{A}_n(\emptyset) \leqq n! C^n$ *for some* C.

Proof. If Γ is a trivalent graph of degree n, then we can cut it along each of its edges. We obtain a cloud of $2n$ Y graphs. Γ can be reconstructed by matching the legs of the Y graphs. There are $6n$ legs, and they can be matched in $(6n)!!$ ways. Using Stirling's formula [F], p. 50–53,

$$\sqrt{2\pi} n^{n+1/2} e^{-n+1/(12n+1)} < n! \sqrt{2\pi} n^{n+1/2} e^{-n+1/(12n)}, \tag{21}$$

the result follows. This proves (a). The above bound may feel a little crude, since we did not take into account automorphisms of the graphs. Nevertheless, it seems to be asymptotically optimal; see also [Bo], p. 55. Cor. 2.17.

For parts (b), (c), we need to understand what we gain by the AS and IHX relation. If Γ is a connected trivalent graph of degree n, choose a cycle in it. Then, using the IHX relation repeatedly, write Γ as a linear combination of "chord diagrams" on that cycle. Since there are at most $(2n)!!$ chord diagrams with n chords on a circle, the result follows for connected graphs. Applying the above reasoning to each connected component of a trivalent graph implies the result in general. $\square$

2.9. An integral version of the Ohtsuki series. In quantum topology, there are two commonly used Taylor series expansions of an element $f(q)$ of the Habiro ring; namely setting $q = e^{1/x}$ or setting $q = 1 + 1/x$. So far we have worked with $q = e^{1/x}$. The other substitution $q = 1 + 1/x$ leads to another map

$$T^{\mathbb{Z}} : \hat{\Lambda} \to \mathbb{Z}[[1/x]], \quad (T^{\mathbb{Z}} f)(x) = f(1 + 1/x),$$

which is also injective. We may also consider a map

$$F^{\mathbb{Z}} : \text{knotted objects} \to \mathbb{Z}[[1/x]], \quad F^{\mathbb{Z}} = T^{\mathbb{Z}} \circ \Phi.$$

The interest in the latter formal power series lies in the fact that it has integer coefficients. In fact, the original definition of the Ohtsuki series is in this form, see [Oh1].

From the point of view of analysis, the series $F_{\mathscr{K}}(x)$ and $F^{\mathbb{Z}}_{\mathscr{K}}(x)$ are simple reparametrizations of one another, by an analytic change of variables. Our next lemma shows that the notion of a Gevrey series is independent of an analytic change of variables.

Lemma 2.14. *Consider a formal power series $f(x) \in \mathbb{C}[[1/x]]$ and let $g(x) = f(e^{1/x} - 1) \in \mathbb{C}[[x]]$. Then $f(x)$ is Gevrey iff $g(x)$ is Gevrey.*

Proof. Let

$$f(x) = \sum_{k=0}^{\infty} a_k \frac{1}{x^k}, \tag{22}$$

$$f(e^{1/x} - 1) = \sum_{k=0}^{\infty} b_k \frac{1}{x^k}. \tag{23}$$

Then the sequences (a_n) and (b_n) are related by an upper-triangular matrix with 1 on the diagonal. The asymptotic behavior of the entries of this matrix makes the lemma possible. For a thorough discussion on that subject, see also Hardy's book [Ha]. The entries of the matrix are given by Stirling numbers. The *Stirling numbers $s_{n,k}$* of the first kind satisfy

$$\frac{1}{x^k} = k! \sum_{n=k}^{\infty} \frac{s_{n,k}}{n!} (1 - e^{-1/x})^n.$$

Substituting for $1/x^k$ from the above identity into (23) and rearranging, it follows that

$$a_n = \sum_{k=0}^{n} b_k (-1)^{n-k} \frac{(n-k)!}{n!} s_{n,n-k}.$$

Suppose now that (b_n) is Gevrey:

$$|b_n| \leqq n! C' C^n.$$

On the other hand, we have

$$s_{n,n-k} = \frac{n^{2k}}{2^k k!}\left(1 + \frac{c_1(k)}{n} + \frac{c_2(k)}{n^2} + \cdots\right),$$

where $c_1(k) = -k(2k+1)/3, \ldots$ which are polynomials in k.

Since $s_{n,n-k} \geqq 0$ for all n and k with $k \leqq n$, and

$$\left(\frac{(n-k)!}{n!}\right)^2 n^{2k} = \left(\frac{n^k}{(n-k+1)\ldots n}\right)^2 \leqq 1,$$

it follows that

$$\begin{aligned}
|a_n| &= \left|\sum_{k=0}^{n} b_{n-k}(-1)^k \frac{(n-k)!}{n!} s_{n,n-k}\right| \\
&\leqq n! C' C^n \sum_{k=0}^{n} C^{-k} \left(\frac{(n-k)!}{n!}\right)^2 \frac{n^{2k}}{2^k k!}\left(1 + \frac{c_1(k)}{n} + \frac{c_2(k)}{n^2} + \cdots\right) \\
&= n! C' C^n \sum_{k=0}^{\infty} (2C)^{-k} \frac{1}{k!}\left(1 + \frac{c_1(k)}{n} + \frac{c_2(k)}{n^2} + \cdots\right) \\
&= n! C' C^n e^{-(2C)^{-1}}\left(1 + \frac{d_1(k)}{n} + \frac{d_2(k)}{n^2} + \cdots\right),
\end{aligned}$$

where $d_1(k), d_2(k), \ldots$ are polynomials in k. This concludes one half of the theorem. The other half is similar. □

Remark 2.15. In [CG3] a more general statement is shown. Namely, suppose that $f(x) \in \mathbb{C}[[1/x]]$ is a power series and $\tau(x)$ is analytic in a neighborhood of infinity and small (i.e., $\tau(\infty) = 0$). Consider the power series $f^\tau(x) = f\big(1/x + \tau(x)\big)$. Then, $f(x)$ is Gevrey iff $f^\tau(x)$ is Gevrey.

Theorem 6 and Lemma 2.14 imply that:

Corollary 2.16. *For every knotted object $\mathcal{K}$, the integral Ohtsuki series $F^{\mathbb{Z}}_{\mathcal{K}}(x) \in \mathbb{Z}[[1/x]]$ is Gevrey.*

3. Proof of Theorem 6

In this section we give a proof of Theorem 6. To simplify notation, let $\langle f(x)\rangle_k$ denote the coefficient of $1/x^k$ in a formal power series $f(x)$.

Lemma 3.1. (a) *If two sequences $\big(f_n(q)\big)$ and $\big(g_n(q)\big)$ are nicely bounded, so is their product.*

(b) *If a sequence* $(f_n(q))$ *is nicely bounded then there exist constants* C, C' *such that for every* k *and every* n *we have*

$$|\langle f_k(e^{1/x})\rangle_n| \leqq \frac{1}{n!}k^{2n}e^{C'n+Ck}.$$

In particular, there exist constants C'' *such that for every* n *and every* k *with* $0 \leqq k \leqq n$ *we have*

$$|\langle f_k(e^{1/x})\rangle_n| \leqq n!e^{C''n}.$$

(c) *The sequence* $((q)_n)$ *is nicely bounded.*

Proof. Part (a) is easy.

For part (b), we may write

$$f_k(q) = \sum_j a_{k,j}q^j,$$

$j \in [C'k^2 + c', C''k^2 + c'']$, and $|a_{k,j}| \leqq e^{Ck}$ for all such j. It follows that

$$\begin{aligned}
|\langle f_k(e^{1/x})\rangle_n| &= \left|\sum_j a_{k,j}\langle e^{j/x}\rangle_n\right| \\
&= \frac{1}{n!}\left|\sum_j a_{k,j}j^n\right| \\
&\leqq \frac{1}{n!}e^{Ck}\sum_j j^n \\
&= \frac{1}{n!}e^{Ck}k^{2n}e^{C'n}.
\end{aligned}$$

If in addition $k \leqq n$, then Stirling's formula (21) and the above implies that

$$\begin{aligned}
|\langle f_k(e^{1/x})\rangle_n| &\leqq \frac{1}{n!}e^{Ck}k^{2n}e^{C'n} \\
&\leqq \frac{1}{n!}e^{Cn}n^{2n}e^{C'n} \\
&\leqq n!e^{C''n}.
\end{aligned}$$

For part (c), it is easy to see that

$$\operatorname{span}_q(q)_n = [0, n(n+1)/2],$$
$$\|(q)_n\|_1 \leqq 2^n. \quad \square$$

Proof of Theorem 6. Let us fix an element

$$f(q) = \sum_{n=0}^{\infty} f_n(q)(q)_n$$

of $\hat{\Lambda}^b$ where $(f_n(q))$ is nicely bounded. Since $\langle (e^{1/x})_k \rangle_n = 0$ for $k > n$, it follows that

$$\langle f(e^{1/x}) \rangle_n = \sum_{k=0}^{n} \langle f_k(e^{1/x})(e^{1/x})_k \rangle_n. \tag{24}$$

Lemma 3.1 (a) and (c) implies that the sequence $(f_n(q)(q)_n)$ is nicely bounded, and therefore by (b) there exists a constant so that for every n and every k with $0 \leqq k \leqq n$ we have

$$|\langle f_k(e^{1/x})(e^{1/x}; e^{1/x})_k \rangle_n| \leqq n! C''^n.$$

Using Equation (24), the result follows. □

4. Proof of Theorem 5

In this section we give a proof of Theorem 5 using two properties of the colored Jones polynomial: q-holonomicity (see [GL1]), and integrality, due to Habiro [H1].

We call a *virtual sl_2-module* any $\mathbb{Q}(q^{1/4})$-linear combination of finite-dimensional sl_2-modules. Suppose L is a framed oriented link with m numbered components, and $U_1, \dots, U_m$ are virtual sl_2-modules, then there is defined the colored Jones polynomial (rather rational function)

$$J_L(U_1, \dots, U_r) \in \mathbb{Q}(q^{\pm/4})$$

normalized by the quantum dimension for the unknot; see [Tu]. Habiro introduced two important sequences of virtual sl_2, denoted by P_n' and P_n'', see [H1].

4.1. Proof of Theorem 5 for knots. Let us fix a knot K with framing 0 in 3-space. [HL] give the following formula for the Kashaev invariant $\Phi_K(q)$:

$$\Phi_K(q) = \sum_{n=0}^{\infty} J_K(P_n'')(q)(q)_n(q^{-1})_n \in \hat{\Lambda}, \tag{25}$$

where, according to Habiro we have $J_K(P_n'')(q) \in \mathbb{Z}[q^{\pm 1}]$ for all n; see [H2].

In [GL1] we proved that the sequence $(J_K(P_n''))$ is q-holonomic and in [GL2] we proved that it is also nicely bounded. It follows that the sequence $(J_K(P_n'')(q^{-1})_n)$ is q-holonomic and nicely bounded (by Lemma 3.1). Thus, $\Phi_K(q)$ lies in $\hat{\Lambda}^{\mathrm{hol}} \cap \hat{\Lambda}^b$. This concludes the proof of Theorem 5 for knots.

Although we will not need it here, let us mention that [HL] prove that when $q = e^{2\pi i/N}$, then

$$\Phi_K(e^{2\pi i/N}) = \langle K \rangle_N,$$

where $\langle K \rangle_N$ is the well-known *Kashaev invariant* of a knot K; see [Ka] and [MM].

4.2. Proof of Theorem 5 for integral homology spheres.

Proof of Theorem 5. The proof will use integrality and holonomicity properties of the colored Jones function of a link in 3-space.

Consider an integral homology sphere M. We can find surgery presentation $M = S^3_{L,f}$ where L is an algebraically split link L in S^3 of r ordered components, with framing $f = (f_1, \dots, f_r)$.

Habiro considers the following series:

$$\Phi_M(q) = \sum_{k_1,\dots,k_r=0}^{\infty} J_L(P'_{k_1}, \dots, P'_{k_r}) \prod_{i=1}^{r} (-f_i q)^{-f_i k_i (k_3+3)/4}, \tag{26}$$

where J_L is the colored Jones function of the 0-framed link L. Habiro proves that:

- For all $k_1, \dots, k_r \in \mathbb{N}$, we have

$$J_L(P'_{k_1}, \dots, P'_{k_r}) \in \frac{\{2m+1\}!}{\{m\}!\{1\}} \mathbb{Z}[q^{\pm 1/2}], \tag{27}$$

where $m = \max\{k_1, \dots, k_r\}$, and

$$\{a\}! := \prod_{j=1}^{a} (q^{a/2} - q^{-a/2}) = (-1)^a q^{-a(a+1)/2} (q)_a. \tag{28}$$

Moreover,

$$J_L(P'_{k_1}, \dots, P'_{k_r}) \prod_{i=1}^{r} (-f_i q)^{-f_i k_i (k_3+3)/4} \in \mathbb{Z}[q^{\pm 1}] \tag{29}$$

for all $k_1, \dots, k_r \in \mathbb{N}$. Thus, $\Phi_M(q) \in \hat{\Lambda}$ is a convergent series.

- The right-hand side of Equation (26) is independent of the surgery presentation $M = S^3_{L,f}$, and depends on M alone.

To simplify notation, let us define:

$$\alpha(k_1, \dots, k_r) = \frac{1}{(q)_m} J_L(P'_{k_1}, \dots, P'_{k_r}) \prod_{i=1}^{r} (-f_i q)^{-f_i k_i (k_3+3)/4}, \tag{30}$$

$$\beta(m) = \sum_{k_1,\dots,k_r;\, \max\{k_1,\dots,k_r\}=m} \alpha(k_1, \dots, k_r). \tag{31}$$

Since $\dfrac{\{2m+1\}!}{\{m\}!\{1\}}$ is divisible by $\{m\}!$, it follows that for all $k_1, \dots, k_r$, we have $\alpha(k_1, \dots, k_r) \in \mathbb{Z}[q^{\pm 1}]$, and consequently, for all m we have $\beta(m) \in \mathbb{Z}[q^{\pm 1}]$. Moreover,

$$\Phi_M(q) = \sum_{m=0}^{\infty} \beta(m)(q)_m. \tag{32}$$

Theorem 5 for integral homology shperes follows from

Theorem 9. *For every unit-framed algebraically split link (L, f), the sequence $(\alpha(m))$ is q-holonomic and nicely bounded.*

Proof. It suffices to show that $(\beta(m))$ is q-holonomic, and nicely bounded.

Let us first recall that the class of q-holonomic functions in several variables is closed under the operations of

(P1) sum,

(P2) product,

(P3) specialization,

(P4) definite summation,

(P5) contains the proper q-hypergeometric functions.

For a proof, see [Ze].

Without loss of generality, let us assume that $r = 2$ (the general case follows from inclusion-exclusion). Then, we have

$$\beta(m) = \sum_{k_1=0}^{m} \alpha(k_1, m) + \sum_{k_2=0}^{m} \alpha(m, k_2) - \alpha(m, m). \tag{33}$$

Changing basis from $\{P'_k\}$ to $\{V_l\}$ it follows that

$$\{k_1\}!\{k_2\}! J_L(P'_{k_1}, P'_{k_2}) = \sum_{l_1=0}^{k_1} \sum_{l_2=0}^{k_2} P_{k_1,k_2}^{l_1,l_2} J_L(V_{l_1}, V_{l_2}),$$

where $P_{k_1,k_2}^{l_1,l_2} \in \mathbb{Z}[q^{\pm/2}]$ are explicit Laurent polynomials which are proper q-hypergeometric; see [GL2], Sec. 4. This, together with Equation (30) implies that

$$\{m\}!\{k_1\}!\{k_2\}!\alpha(k_1, k_2) = \sum_{l_1=0}^{k_1} \sum_{l_2=0}^{k_2} R_{k_1,k_2}^{l_1,l_2} J_L(V_{l_1}, V_{l_2}), \tag{34}$$

where $R_{k_1,k_2}^{l_1,l_2}(q) \in \mathbb{Z}[q^{\pm/2}]$ are proper q-hypergeometric Laurent polynomials, and $m = \max\{k_1, k_2\}$.

Now $J_L(V_{l_1}, V_{l_2})$ can be written as a multisum:

$$J_L(V_{l_1}, V_{l_2}) = \sum_{j_1=0}^{l_1} \sum_{j_2=0}^{l_2} F_{l_1,l_2,j_1,j_2}, \tag{35}$$

where $F_{l_1,l_2,j_1,j_2} \in \mathbb{Z}[q^{\pm/2}]$ is a proper q-hypergeometric summand; see [GL1], Sec. 3.

Equations (34), (35) and Properties (P4), (P5) imply that $\alpha(k_1, k_2)$ is q-holonomic in both variables (k_1, k_2). Together with Property (P4), it follows that $\sum_{k_1=0}^{r} \alpha(k_1, s)$ is q-holonomic in both variables (r, s), and (by Property (P3)) $\sum_{k_1=0}^{m} \alpha(k_1, m)$ is q-holonomic in m. Alternatively, the WZ algorithm of [WZ] and Equations (34) and (35) imply directly (and constructively) that $\sum_{k_1=0}^{m} \alpha(k_1, m)$ is q-holonomic in m.

Likewise, $\sum_{k_2=0}^{m} \alpha(m, k_2)$ is q-holonomic in m, and $\alpha(m, m)$ is q-holonomic in m. Property (P1) and Equation (33) imply that $\beta(m)$ is q-holonomic in m.

Alternatively, we could have used the identity

$$\beta(m) = \sum_{0 \leqq k_1, \ldots, k_r \leqq m} \alpha(k_1, \ldots, k_r) - \sum_{0 \leqq k_1, \ldots, k_r \leqq m-1} \alpha(k_1, \ldots, k_r) \tag{36}$$

and the q-holonomicity of $\alpha(k_1, \ldots, k_r)$ (as follows by the WZ algorithm) to deduce the q-holonomicity of $\beta(m)$.

It remains to show that $\beta(m)$ is nicely bounded. Let us say that a multi-indexed sequence $\big(f(r_1, r_2, \ldots)\big)$ of Laurent polynomials is *nicely bounded* if it satisfies (7) and (8) for all $r_1, r_2, \ldots$ with $r_1, r_2, \ldots \leqq n$.

It is easy to see that the class of nicely bounded functions satisfies properties (P1)–(P4) and contains the proper q-hypergeometric terms that are Laurent polynomials.

Repeating our previous steps, Equations (35) and (34) imply that $\{m\}!\{k_1\}!\{k_2\}!\alpha(k_1, k_2)$ is nicely bounded as a function of both variables (k_1, k_2). Lemma 4.1 below (communicated to us by D. Boyd), implies that $\alpha(k_1, k_2)$ is nicely bounded as a function of both variables (k_1, k_2).

Our previous steps (or Equation (36)) now imply that $\beta(m)$ is nicely bounded. □

This concludes the proof of Theorem 5 for integer homology spheres. □

Lemma 4.1 (Boyd). *If $\big(f_n(q)\big)$ is a sequence of Laurent polynomials such that $\big((q)_n f_n(q)\big)$ is nicely bounded, then $\big(f_n(q)\big)$ is nicely bounded, too.*

For a proof, see [GL2], Sec. 7.

Remark 4.2. In the special case where M is obtained by ± 1 surgery on a knot in 3-space, Lawerence-Ron have shown independently that the formal power series $F_M(x)$ is Gevrey; see [LR].

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