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RANDOM WALKS AND THE COLORED JONES FUNCTION

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It can be conjectured that the colored Jones function of a knot can be computed in terms of counting paths on the graph of a planar projection of a knot. On the combinatorial level, the colored Jones function can be replaced by its weight system. We give two curious formulas for the weight system of a colored Jones function: one in terms of the permanent of a matrix associated to a chord diagram, and another in terms of counting paths of intersecting chords.

1. Introduction

1.1. The goal

The paper provides a compact understanding of the weight system of the colored Jones function, using graph-theory considerations rather than Lie algebras, and proves conjectures of Bar-Natan and the second author.

1.2. The colored Jones function

A knot is an embedded circle in 3-space, considered up to ambient isotopy. In 1985, V. Jones discovered a celebrated invariant of knots, the Jones polynomial, [8].

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As soon as the Jones polynomial was discovered, it was compared with the better-understood Alexander polynomial of a knot. The latter can be defined using classical algebraic topology (such as the homology of the infinite cyclic cover of the knot complement), and its skein theory can be understood purely topologically; see for example [9, 14].

On the other hand, the Jones polynomial appears to be difficult to understand topologically, and its combinatorics hide its topological and geometric meaning.

There is a good reason for this, as was explained by Witten, [19]. Namely, the Jones polynomial can be thought of as a partition function of a 3-dimensional quantum field theory (the Chern-Simons theory), and full partition functions are hard to understand in general.

A little down-to-earth, one may ask if the Jones polynomial is stronger than the Alexander polynomial, or vice-versa. It is known that the answer to both questions are negative: in other words there exist distinct knots with the same Jones but different Alexander polynomials and vice-versa.

This is *an accident*, as we will now explain. One may consider the Jones polynomials of a knot and its parallels. This sequence of Jones polynomials is essentially equivalent to the *colored Jones function*.

The latter is a 2-parameter formal power series $\sum_{n,m=0}^{\infty} a_{n,m} h^n \lambda^m$ which determines (and is determined by) the Jones polynomial of a knot and its parallels, [3]. The support of the colored Jones function lies in the triangle $0 \leq m \leq n$.

Similarly, one may consider the Alexander polynomial of a knot and its parallels. In that case, one finds that this data is determined by the Alexander polynomial alone.

About 10 years ago, Melvin–Morton and Rozansky independently conjectured a relation among the diagonal terms $\sum_n a_{n,n}(h\lambda)^n$ of the colored Jones function of a knot and its Alexander polynomial, [13, 15, 16]. D. Bar-Natan and the first author reduced the conjecture about knot invariants to a statement about their *combinatorial weight systems*, and then proved it for all weight systems that come from *semisimple Lie algebras* using combinatorial Lie algebraic methods, [3].

This discussion explains the term 'accident' that we used above.

Over the years, the MMR Conjecture has received attention by many researchers who gave alternative proofs, [4, 10, 11, 17, 18].

The subdiagonal terms $h^k \sum_n a_{n+k,n}(h\lambda)^n$ for a fixed k of the colored Jones function, are (after a suitable parametrization) rational functions whose denominators are powers of the Alexander polynomial. This was first shown by Rozansky in [17], who further conjectured that a similar property should hold for the full Kontsevich integral of a knot. Rozansky's conjecture was recently settled by the first author and Kricker in [6]. This raises the question of understanding each subdiagonal term of the colored Jones function (or the full Kontsevich integral) in topological terms. That said, not much is known about the subdiagonal terms of the colored Jones function. One can conjecture that each subdiagonal term is given in terms of a certain counting of random walks on a planar projection of a knot, see also Lin and Wang, [12]. On a combinatorial level, the colored Jones function may be replaced by its weight system. In [3] formulas for the weight system W_J of the colored Jones function and of its leading order term W_{JJ} were given in terms of the intersection matrix of a chord diagram. In particular, W_{JJ} is equal to the *permanent* of the intersection matrix of the chord diagram. In the last section of [3] it was asked for a better understanding of the W_J weight system, especially one that offers control over the subdiagonal terms in W_J.

The purpose of the paper is to give two curious combinatorial formulas for W_J in Theorems 1 and 2 that answer these questions, and support the conjecture that the colored Jones polynomial is a counting of random walks.

The notion of weight systems which we study here seems to be surrounded by very exciting discrete mathematics, let us mention for instance [2]. We hope our paper will stimulate more research of graph theory community in this area.

1.3. Statement of the results

Although the colored Jones function of a knot is a power series invariant of knots, most proofs of the MMR Conjecture involve not the colored Jones function instead, but its weight system. The reader may think of the weight system as the infinitesimal version of the knot invariant. Weight systems are *purely combinatorial objects.* That is, they involve no knot theory at all. They were introduced by Bar-Natan in [1], and heuristically one may think of them by one of the following ways:

- Either as abstraction of the idea of an *immersed knot.* An abstractly immersed knot is a circle, with a pair of points joined by chords. The chords remember which points will be identified in an embedding of an abstract knot in 3-space.
- Alternatively, one may think of chord diagrams as ways to encode *contractions of tensors*, as is done in representation theory and differential geometry.

- Yet alternatively, one may think of chord diagrams as *Feynman diagrams* of a 3-dimensional quantum field theory.

Algebraically, weight systems are linear functionals on a graded Hopf algebra and they thmeselves form a commutative cocommutative Hopf algebra. One way of producing weight systems is via Lie algebras and their representations. For a thorough discussion on these matters, we refer the reader to [1] and also to [3]. In particular, from now on, we will assume that the reader is vaguely familiar with [1] or [3].

Consider the 0-*framed colored Jones weight system*

$$W_J : \mathcal{A} \to \mathbb{Q}[\lambda]$$

where $\mathcal{A}$ is the vector space over $\mathbb{Q}$ spanned by *chord diagrams* on an oriented line, modulo the 4-term and 1-term relations, see [3] and also below. We will normalize W_J to equal 1 on the chord diagram with no chords (in [3] the value of the empty chord diagram was λ+1 instead). With this normalization, it turns out that for a chord diagram D, $W_J(D)$ is a polynomial of λ of degree the number of chords of D. $W_{JJ}(D)$ is defined to be the coefficient of λ^{deg} in W_J.

Given a chord diagram D, its chords are ordered (from left to right) and we can consider its intersection matrix $\mathrm{IM}(D)$ as in [3, Definition 3.4] of size the number of chords of D defined by

$$\mathrm{IM}(D)_{ij} = \begin{cases} \mathrm{sign}(i-j) & \text{if the chords } i \text{ and } j \text{ of } D \text{ intersect} \\ 0 & \text{otherwise.} \end{cases}$$

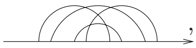
We will consider a blown-up variant IM_J of the intersection matrix, of size 3 times the number of chords of D, composed of blocks of 3 by 3 matrices as follows:

$$\mathrm{IM}_J(D)_{ij} = \begin{cases} A_{\mathrm{sign}(i-j)} & \text{if the distinct chords } i \text{ and } j \text{ of } D \text{ intersect} \\ A_0 & \text{if } i = j \\ A_c & \text{if chords } i, j \text{ do not intersect} \\ & \quad \text{and } i \text{ is completely contained in } j \\ 0 & \text{otherwise,} \end{cases}$$

where

$$A_0 = \begin{bmatrix} \lambda+2 & 0 & 0 \\ 0 & \lambda+2 & 1 \\ \lambda & -\lambda-2 & 1 \end{bmatrix}, \quad A_- = \begin{bmatrix} 0 & 0 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_+ = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_c = \begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Example 1.1.

$D =$  , $\quad \mathrm{IM}(D) = \begin{bmatrix} 0 & -1 & -1 & -1 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad \mathrm{IM}_J(D) = \begin{bmatrix} A_0 & A_- & A_- & A_- \\ A_+ & A_0 & 0 & A_- \\ A_+ & A_c & A_0 & A_- \\ A_+ & A_+ & A_+ & A_0 \end{bmatrix}.$

Theorem 1. *We have*

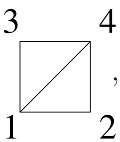
$$W_J = \mathrm{Per}(\mathrm{IM}_J)$$

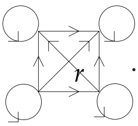
where $\mathrm{Per}(A)$ *denotes the permanent of a square matrix* A*.*

There is an alternative (and equivalent) formula of W_J in terms of counting cycles. In order to state it, given a chord diagram D consider its *labeled intersection graph* $\mathrm{LIG}(D)$ as in [3, Definition 3.4]. The vertices of $\mathrm{LIG}(D)$ correspond to the chords of D (thus, are ordered) and the edges of $\mathrm{LIG}(D)$ correspond to the intersection of the chords of D.

We will use a variation $\mathrm{LID}(D)$, the *labeled intersection digraph* of D defined as follows. Orient each edge from the smaller vertex to the larger and add an oriented loop on each vertex. The oriented loops are *leaving* the vertices. Next add directed edges (ij) for each pair of chords i, j such that i is completely contained in j. In addition we color these new arcs *red* (and we draw them as $\xrightarrow{r}$) to distinguish them from the original arcs.

Example 1.2. For the chord diagram D of Example 1.1, we have

$\mathrm{LIG}(D) =$ 

, $\quad \mathrm{LID}(D) =$ 

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A bit more generally, consider a *digraph* $G=(V, A)$ (i.e., a directed graph) where V is the set of vertices and A is the set of arcs. If e is an arc of A with initial vertex u and terminal vertex v then we write $e=(u, v)$. We assume that there is one loop at each vertex and a loop at a vertex is considered as an arc *leaving* that vertex, and in addition some arcs which are not loops are *red*. We will consider the arcs with variables associated with them: the variable of an arc e is denoted by x_e. We will need the following notion of *acceptable* object, given G:

Definition 1.3. A collection K of arcs together with a *thickening* of one end of each of the arcs of K is called *acceptable* for G if the following properties are satisfied:

- If (ij) is a red arc of G then both arcs (ij) and (ji) may appear in K, but they must always be thickened at i. If (ij) is an uncolored arc of G then (ij) with any end thickened may appear in K, but (ji) may not.
- Each vertex of V is incident with 0, 2 or 4 thickened arcs of K. If a loop belongs to K then we assume it contributes 2 to the degree of the corresponding vertex. Moreover, a loop is always thickened at its initial segment, i.e., in agreement with its orientation.
- Exactly half of the arcs incident with a vertex are thickened at the vertex.
- If there are two arcs thickened at a vertex, then one of them enters and the other one leaves.

We will study the following partition function

$$J(G) = \sum_{K \text{ acceptable}} 2^{\deg_4(K)}(\lambda+2)^{|V|-\deg_4(K)} x_K (-1)^{a(K)}$$

of a digraph G, where $x_K = \prod_{e \in K} x_e$, $\deg_4(K)$ denotes the number of vertices of K incident with 4 arcs of K and $a(K)$ is the number of arcs of K with initial segment thickened, i.e., directed in agreement with the thickening.

The motivation for $J(G)$ comes from the case of the intersection digraph $\mathrm{LID}(D)$ of a chord diagram and the following:

Theorem 2. *For a chord diagram D, we have*

$$W_J(D) = J(\mathrm{LID}(D))|_{x_e=1}.$$

Corollary 1.4. *After a change of variables $d=\lambda+2$, let $W_{JJ^{(n)}}$ denote the coefficient of $d^{\deg -n}$ in W_J. Then,*

$$W_{JJ^{(n)}}(D) = 2^n \sum_K (-1)^{a(K)}$$

where the sum is over all acceptable K such that $\deg_4(K)=n$.

Corollary 1.5 ([3]). *We have:*

$$W_{JJ} = \mathrm{Per}(\mathrm{IM}).$$

How fast can one compute permanents?

Corollary 1.6. *For general matrices of size n we need $n!$ steps. However, a theorem of A. Galluccio [5] and the second author implies that W_J can be computed in 4^g steps, where g is the* genus of $\mathrm{LIG}(D)$, *that is the smallest genus of a surface that* $\mathrm{LIG}(D)$ *embeds.*

1.4. Plan of the proof

In Section 2, we review the weight system of the colored Jones function, and reduce Theorem 2 to Theorem 3; the latter concerns digraphs. Section 3, is devoted to the proof of Theorem 3 using a trip to combinatorics. In Section 4, we translate our results using the language of permanents, and deduce Theorem 1. In the final Section 5 we prove the corollaries that follow Theorem 2.

1.5. Computer code

A computer code for the formula of Theorem 1 is given in an appendix separate from the paper, as was suggested by the journal.

1.6. Acknowledgement

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2. A review of the W_J weight system

The goal of this section is to reduce Theorem 2 to Theorem 3 stated below; this will be achieved by a careful examination of the W_J weight system. Recall from [3, Section 4.2] that W_J can be computed as follows:

Step 1. Color each chord of a chord diagram D by the following operator:

$$\begin{aligned}\hat{B}(v_k \otimes v_{k'}) &= (k+1)(\lambda - k' + 1) v_{k+1} \otimes v_{k'-1} \\ &+ (\lambda - k + 1)(k'+1) v_{k-1} \otimes v_{k'+1} \\ &+ 1/2((\lambda - 2k)(\lambda - 2k') - \lambda(\lambda+2)) v_k \otimes v_{k'}\end{aligned}$$

from [3, p. 121].

The key calculation is the following elementary rearrangement of $\hat{B}$, easily checked:

Lemma 2.1.

$$\hat{B}(v_k \otimes v_{k'}) = ((\lambda+2)I + B^+ + B^-)(v_k \otimes v_{k'})$$

where

$$B^+ = \sum_{\epsilon=0,1} (-1)^\epsilon B^+_\epsilon, \qquad B^+_\epsilon = -(1+k)(\lambda+1-k')v_{k+\epsilon} \otimes v_{k'-\epsilon}$$

$$B^- = \sum_{\epsilon=0,1} (-1)^\epsilon B^-_\epsilon, \qquad B^-_\epsilon = -(1+k')(\lambda+1-k)v_{k-\epsilon} \otimes v_{k'+\epsilon}.$$

This done, the coloring of chords of D may be viewed as a function $\rho:\text{chords}(D) \to \{I, B^+_0, B^+_1, B^-_0, B^-_1\}$.

Step 2. The end-points of the n chords of D partition the base line into $2n+1$ segments $s_0, \ldots, s_{2n}$ listed from left to right. We associate number $m(s_i)$ with each of these segments as follows:

1. Let $m(s_0)=0$.
2. If $i \geq 0$ and last point of s_i is left end-vertex of chord v then $m(s_{i+1})$ is computed from $m(s_i)$ and $\rho(v)$ using 2.1:
   - If $\rho(v) \in \{I, B^+_0, B^-_0\}$ then $m(s_{i+1}) = m(s_i)$,
   - If $\rho(v) = B^+_1$ then $m(s_{i+1}) = m(s_i)+1$,
   - If $\rho(v) = B^-_1$ then $m(s_{i+1}) = m(s_i)-1$.
3. If $i \geq 0$ and last point of s_i is right end-vertex of chord v then $m(s_{i+1})$ is computed from $m(s_i)$ and $\rho(v)$ using 2.1:
   - If $\rho(v) \in \{I, B^+_0, B^-_0\}$ then $m(s_{i+1}) = m(s_i)$,
   - If $\rho(v) = B^+_1$ then $m(s_{i+1}) = m(s_i)-1$,
   - If $\rho(v) = B^-_1$ then $m(s_{i+1}) = m(s_i)+1$.

Step 3. We let

$$W_J(D) = \sum_\rho \prod_{\text{chords}\, v} \omega_\rho(v)$$

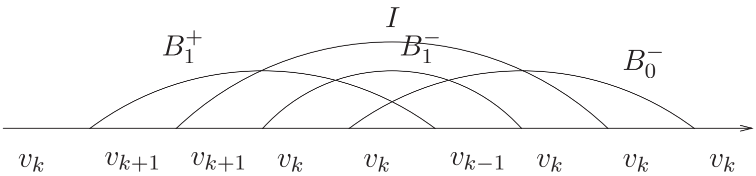
where ρ is a coloring of the chords of D and $\omega_\rho(v)$ is a specific weight that is computed using Lemma 2.1 again:

- $\omega_\rho(v) = \lambda+2$ if $\rho(v)=I$,
- $\omega_\rho(v) = -(-1)^\epsilon(1+k)(\lambda+1-k')$ if $\rho(v)=B^+_\epsilon$,
- $\omega_\rho(v) = -(-1)^\epsilon(1+k')(\lambda+1-k)$ if $\rho(v)=B^-_\epsilon$,

where $k=m(s_L(v))$, $k'=m(s_R(v))$, $s_L(v)$ is the segment ending at the left end-point of chord v and $s_R(v)$ is the segment ending at the right end-point of chord v.

The reader is urged to look at [3, Chapter 4] for an explanation of the above algorithm in terms of the representation theory of the $\mathfrak{sl}_2$ Lie algebra.

Example 2.2. For the following coloring of the chord diagram of Example 1.1



we have (assuming $m(s_0)=k$) that

$$\omega_\rho(1) = (1+k)(\lambda+1-k), \qquad \omega_\rho(2) = \lambda+2$$
$$\omega_\rho(3) = (1+(k-1))(\lambda+1-(k+1)), \quad \omega_\rho(4) = -(1+k)(\lambda+1-k).$$

Each coloring ρ of the chords is determined by a subset V' of chords such that $\rho(v)=I$ for $v \notin V'$ and by a *coloring* c where $c(v)$ is an assignment of an element $(\epsilon_v, \delta_v) \in \{0,1\} \times \{+,-\}$ for each $v \in V'$. Hence we can write

$$W_J(D) = J'(\mathrm{LID}(D))|_{x_e=1}$$

where

$$J'(G) = \sum_{V' \subset V} (\lambda+2)^{|V-V'|} \sum_{c \text{ col of } V'} \prod_{v \in V'} \omega'_c(v).$$

An important observation is that $\omega'_c(v)$ can be computed in terms of the *local structure* of the labeled intersection digraph $\mathrm{LID}(D)$. The next lemma describes this.

Lemma 2.3. *Let* $\mathrm{LID}(D)=(V,A)$. *Then*

$$\omega'_c(v) = -(-1)^{\epsilon_v}\left(1 + \sum_{e \in A,\, v \in e} z_v(e)\right)\left(\lambda + 1 - \sum_{e \in A,\, v \in e} \bar{z}_v(e)\right)$$

where $z_v, \bar{z}_v$ *are defined as follows:*

- *If* e *uncolored then*
- *If* $c(v)=(+,\epsilon)$ *and* $e=(w,v)$ *then* $z_v(e)=\delta_w \epsilon_w x_e$,
- *If* $c(v)=(+,\epsilon)$ *and* $e=(v,w)$ *then* $\bar{z}_v(e)=\delta_w \epsilon_w x_e$,
- *If* $c(v)=(-,\epsilon)$ *and* $e=(w,v)$ *then* $\bar{z}_v(e)=\delta_w \epsilon_w x_e$,
- *If* $c(v)=(-,\epsilon)$ *and* $e=(v,w)$ *then* $z_v(e)=\delta_w \epsilon_w x_e$,
- *If* e *is red and* $e=(v,w)$ *then* $z_v(e)=\bar{z}_v(e)=\delta_w \epsilon_w x_e$,
- *and* $z_v(e)=\bar{z}_v(e)=0$ *otherwise.*

Proof. If $c(v)=(+,\epsilon)$ then $c(v)$ corresponds to the operator B^+_ϵ and hence $\omega'_c(v)|_{x_e=1} = -(-1)^{\epsilon_v}(1+k)(\lambda+1-k')$. A moment's thought reveals that $k=k_1-k_2+k_3-k_4$ and $k'=k'_1-k'_2+k_3-k_4$ where

$$k_1 = |\{e=(w,v)\,\text{uncolored};\, c(w)=(+,1)\}|, \quad k_2 = |\{e=(w,v)\,\text{uncolored};\, c(w)=(-,1)\}|$$

$$k'_1 = |\{e=(v,w)\,\text{uncolored};\, c(w)=(+,1)\}|, \quad k'_2 = |\{e=(v,w)\,\text{uncolored};\, c(w)=(-,1)\}|$$

$$k_3 = |\{e=(v,w)\,\text{red};\, c(w)=(+,1)\}|, \quad k_4 = |\{e=(v,w)\,\text{red};\, c(w)=(-,1)\}|.$$

The reasoning is analogous for $c(v)=(-,\epsilon)$. ∎

Thus, Theorem 2 follows from the following

Theorem 3. *For all digraphs G with one loop at each vertex, we have*

$$J'(G) = J(G).$$

3. Understanding the state sums $J(G)$ and $J'(G)$

In this section we prove Theorem 3, via a trip to combinatorics with curious cancellations caused by applications of the binomial theorem.

Let us begin by rewriting $J'(G)$. Let $\epsilon_{V'}=\sum_{v\in V'}\epsilon_v$. Then,

$$J'(G) = \sum_{V'\subset V}(-1)^{|V'|}(\lambda+2)^{|V-V'|}\sum_{c\ \text{col of}\ V'}(-1)^{\epsilon_{V'}} \sum_{V_1\subset V',\, V_2\subset V'}(-1)^{|V_2|}(\lambda+1)^{|V'-V_2|}\prod_{v\in V_1}\left(\sum_{v\in e} z_v(e)\right)\prod_{v\in V_2}\left(\sum_{v\in e}\bar{z}_v(e)\right)$$

where V_1 and V_2 are possibly overlapping subsets of V.

Note that

$$\prod_{v\in V_1}\left(\sum_{v\in e} z_v(e)\right) = \sum_f \prod_{v\in V_1} z_v(e(f,v))$$

where $f:V_1\to A$ maps v to the arc denoted by $e(f,v)$ such that $v\in e(f,v)$ and moreover if $e(f,v)$ red then $e=(v,.)$, i.e. e starts in v. In other words, f associates with each vertex v of V_1 an arc incident with it. Similarly, we can rewrite $\prod_{v\in V_2}(\sum_{v\in e}\bar{z}_v(e))$. Hence,

$$J'(G) = \sum_{V'\subset V, V_1\subset V', V_2\subset V'}(-1)^{|V'|}(\lambda+2)^{|V-V'|}(\lambda+1)^{|V'-V_2|}(-1)^{|V_2|} \sum_{f:V_1\to A, g:V_2\to A}\ \sum_{c\ \text{col of}\ V'}(-1)^{\epsilon_{V'}}\prod_{v\in V_1}(z_v(e(f,v)))\prod_{v\in V_2}(\bar{z}_v(e(g,v)))\,.$$

Let us rewrite the formula further. We fix $W_1 = V_1 \cap V_2$, $W_2 = V_1 \cup V_2$ and we let h to be disjoint union of f and g.

Remark 3.1. What exactly is h? Answer: h is a function that assigns to each vertex of V' zero, one or two arcs incident with it (if the arc is red then it must start in that vertex). Hence $|h(v)| \leq 2$ for all $v \in V'$ and if $e \in h(v)$ then $v \in e$. Here we slowly move towards the formalism of acceptable objects. If $e \in h(v)$ then thicken the end of e containing v. Hence h becomes a system of thickened arcs of G so that there are at most two arcs in the system that are thickened at each vertex of V', and red arcs are thickened always at the start. ∎

If we have such an h, then $W_1 = \{v : |h(v)| = 2\}$ and $W_2 = \{v : |h(v)| \geq 1\}$. Hence, h determines the sets W_1 and W_2. Fix an h as above, consider its corresponding sets W_1, W_2, and let $h(W_2)$ denote the system of thickened arcs determined by h. We have

$$J'(G) \;=\; \sum_{V' \subset V} (-1)^{|V'|} (\lambda + 2)^{|V - V'|} \sum_h A(V', h)$$

where

$$A(V', h) \;=\; \sum_{c \text{ col of } V'} \;\; \sum_{V_2' \subset W_2 - W_1} \;\; \sum_{g : W_1 \cup V_2' \to h(W_2) : g(v) \in h(v)} B$$

and

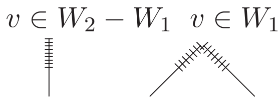
$$B = (-1)^{\epsilon_{V'}} (-1)^{|V_2' \cup W_1|} (\lambda + 1)^{|V' - (W_1 \cup V_2')|} \prod_{v \in W_1 \cup V_2'} \bar{z}_v(e(g, v)) \prod_{v \in W_1 \cup V_1'} z_v(e(f, v))$$

where $V_1' = W_2 - (W_1 \cup V_2')$ and $f : W_1 \cup V_1' \to h(W_2)$ is such that the disjoint union of f and g is h.

The next two lemmas restrict the possible configurations of h that contribute non-zero $A(V', h)$.

Lemma 3.2. *Let $v \in W_2$. If the only arcs of $h(W_2)$ incident with v are the arcs of $h(v)$, then $A(V', h) = 0$.*

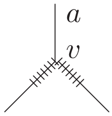
Proof.



Fix V_2', g and $c(V' - \{v\})$. If $B \neq 0$, then the color $\delta_v \in \{-, +\}$ of v may be determined by g and the orientation of the arcs of $h(v)$. However, there is still the choice $\epsilon_v = 0$ or 1. This influences only $(-1)^{\epsilon_{V'}}$, hence the lemma follows. ∎

Lemma 3.3. *Let $v \in W_1$. If both arcs of $h(v)$ are uncolored and oriented in the same way with respect to v then $A(V',h)=0$. If there are exactly three arcs of $h(W_2)$ incident with v then $A(V',h)=0$.*

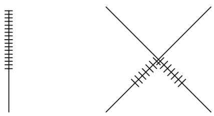
Proof.



Since $v \in W_1$ we have $|h(v)| = 2$. For the first part, if both arcs of $h(v)$ are uncolored and oriented in the same way, then there is no way to choose color $\delta_v \in \{-,+\}$ so that $B \neq 0$. For the second part, if there are exactly three arcs of $h(W_2)$ incident with v then there is exactly one arc, say a, which is incident with v and belongs to $h(W_2)-h(v)$. Fix V_2' and consider pairs g_1, g_2 of functions g which differ only on v. If $B \neq 0$ and at least one of the arcs of $h(v)$ is uncolored, then the color $\delta_v \in \{-,+\}$ of v is determined by the orientation of the arcs of $h(v)$ and the choice between g_1 and g_2. This color is opposite for g_1 and g_2, and so the edge a is counted with different signs for g_1 and g_2 while all the rest remains the same. Hence the total contribution is 0. If both arcs of $h(v)$ are red then both colors $\delta_v \in \{-,+\}$ of v are possible for $B \neq 0$ and both g_1 and g_2. Hence again the edge a contributes twice $+1$ and twice -1 and the total contribution is 0. ∎

Note that the second property of the above lemma assures that the system $h(W_2)$ of thickened *uncolored* edges is a set for each h which contributes a non-zero term to $J'(G)$.

Corollary 3.4. (a) *If $e \in h(W_2)$, then both vertices of e belong to W_2.*
(b) *Each vertex of W_2 has degree (i.e., valency)* 2 *or* 4 *in $h(W_2)$ and if $N(v)$ is the set of edges of $h(W_2)$ incident with v then $|N(v)| = 2|h(v)|$. In other words, the allowed configurations are*



Proof. It follows from Lemmas 3.2 and 3.3 that $|N(v)| \geq 2|h(v)|$ at each vertex v. On the other hand, each edge of $h(W_2)$ has one *thick end* and one *thin end*, and so there cannot be more thin ends than thick ends. Hence $|N(v)| = 2|h(v)|$ and the corollary follows. ∎

Corollary 3.5. *If $W_2 \neq V'$, then $A(V',h)=0$.*

Proof. We write

$$A(V',h) = \sum_{c \text{ col of } V'-W_2} (-1)^{\epsilon_{V'-W_2}} \text{ (rest)}$$

where the 'rest' is not influenced by the colorings in $V'-W_2$. Hence,

$$A(V',h) = \text{(rest)} \sum_{C \subset V'-W_2} (-1)^{|C|} 2^{|V'-W_2|}$$

which vanishes unless $V'=W_2$. ∎

Summarizing, a function h such that $A(V',h) \neq 0$ determines a collection of thickened arcs that is almost an acceptable object:

- each vertex has degree 2 or 4 in V' and 0 in $V-V'$,
- exactly half of the arcs incident with a vertex are thickened at that vertex,
- if there are two uncolored arcs thickened at a vertex then they have opposite orientation with respect to the vertex,
- the red arcs are always thickened at the start.

Let us call such object *good* on V'. Note also that for each coloring of a good object on V' which contributes non-zero to B we must have $\epsilon_v = 1$ for each $v \in V'$ and hence $(-1)^{\epsilon_{V'}} = (-1)^{|V'|}$. Collecting our efforts so far, we have

$$J'(G) = \sum_{V' \subset V} (\lambda+2)^{|V-V'|} \sum_{K \text{ good on } V'} A'(V',K) \tag{1}$$

where

$$A'(V',K) = \sum_{V_2' \subset V'-W_1} (\lambda+1)^{|V'-(W_1 \cup V_2')|} C(V_2',V',K),$$

W_1 is the set of vertices of V' of degree 4 in K and

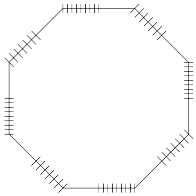
$$C(V_2',V',K) = \sum_{g':W_1 \to K} (-1)^{|W_1|} (-1)^{|V_2'|} \sum_{c \text{ col}} \prod_{v \in W_1 \cup V_2'} \bar{z}_v(e(g,v)) \prod_{v \in V'-V_2'} z_v(e(f,v)),$$

and g', g, f have the following properties:

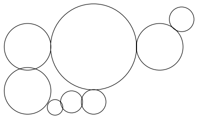
- if $p \in \{g', g, f\}$ then $p(v)$ is an arc of K incident with v and thickened at v,
- $g: W_1 \cup V_2' \to K$ is unique such function extending g',
- $f: V'-V_2' \to K$ is unique such function with $f \cup g = K$.

Lemma 3.6. *$C(X,V',K) = C(Y,V',K)$ for arbitrary X,Y subsets of $V'-W_1$.*

Proof. Exactly half of the edges of K incident with each vertex are thickened at that vertex and hence K may be regarded as a union of cycles $Z_1,\dots,Z_m$ such that each Z_i has the form



and such that each vertex of K lies in at most two of the cycles Z_i. In other words, we may think that K is pictured schematically as follows



(where for simplicity, we have drawn the cycles Z_i as circles). As we observed above,

$$C(X,V',K) = \sum_{g':W_1\to K} (-1)^{|W_1|}(-1)^{|X|} \sum_{c \text{ col}} \prod_{v\in W_1\cup X} \bar{z}_v(e(g,v)) \prod_{v\in V'-X} z_v(e(f,v))$$

It suffices to show the following

Claim.

$$C(X,V',K) = \sum_{K'} (-1)^{a(K')} x_K$$

where K' is any collection of thickened arcs obtained from K by changing orientation of some (possibly none) red arcs of K so that K' is an acceptable object (i.e. if two arcs are thickened at a vertex then they are oppositely oriented with respect to that vertex). Moreover, $a(K')$ is the number of arcs of K' directed in agreement with the thickening.

Proof of the Claim. We can write

$$C(X,V',K) = \sum_{g':W_1\to K} D$$

and

$$D=(-1)^{|W_1|}(-1)^{|X|}\sum_{c\ \mathrm{col}}\prod_{v\in W_1\cup X}\bar{z}_v(e(g,v))\prod_{v\in V'-X}z_v(e(f,v)).$$

First we observe that the Claim is true when K has a vertex of degree 2 where the thickened edge is red. Indeed, in this case both sides of the formula in the Claim equal 0. Hence let K donot have such a vertex. Let S be the set of vertices of K where two red arcs are thickened. Observe that exactly $2^{|S|}$ colorings c contribute a non-zero term to D: we observed before that necessarily $\epsilon(v)=1$ for each $v\in V'$. Moreover, each color $\delta(v)$ contributing non-zero to D is uniquely determined for each v where at least one uncolored arc is thickened: explicitly, let v be a vertex of Z_i and let e be the unique arc of Z_i thickened at v, and let e be uncolored. Then $\delta(v)$ depends on the orientation of e and whether e belongs to g or not. Hence there is at most one coloring $\delta(v)$ which contributes a non-zero term to D. Also observe that the unique 'non-zero coloring' of these vertices of each Z_i compose well together: this follows from the fact that if v is a vertex of degree 4 in K then exactly 2 arcs are thickened at v, and if both are uncolored then they have different orientation with respect to v and one belongs to g and the other belongs to f. Finally observe that for vertex v where two red arcs are thickened, any $\delta(v)$ contributes a non-zero.

Next we will observe that the contribution of each of these $2^{|S|}$ colorings to D is the same and equals $(-1)^{a(K')}x_K$ where K' is any collection of thickened arcs obtained from K by changing orientation of some (possibly none) red arcs of K so that K' is an acceptable object (i.e., if two arcs are thickened at a vertex then they are opositely oriented with respect to that vertex). This proves the Claim since the number of objects K' equals $2^{|S|}$. Hence it remains to confirm the contribution of each of the allowed colorings to D.

First observe that it is true when $X=\emptyset$. Next, let us put a vertex v of Z_i-W_1 into X and let e be the arc of Z_i thickened at v. Then e is uncolored by our assumption and we need to change $\delta(v)$ in order to have a nonzero contribution, hence the product of signs along Z_i changes but $(-1)^{|X|}$ also changes and so the final total sign is the same. ∎

We let

$$\begin{aligned}C(V',K) &= \sum_{g':W_1\to K}\sum_{K'}(-1)^{a(K')}x_K\\ &= 2^{\deg_4(K)}\sum_{K'}(-1)^{a(K')}x_K.\end{aligned}$$

Equation (1) together with Lemma 3.6 implies that

$$\begin{aligned} J'(G) &= \sum_{V'\subset V} (\lambda+2)^{|V-V'|} \sum_{K \text{ good on } V'} C(V',K) \sum_{A\subset V'-W_1} (\lambda+1)^{|A|} \\ &= \sum_{V'\subset V} (\lambda+2)^{|V-V'|} \sum_{K \text{ good on } V'} C(V',K)(\lambda+2)^{|V'-W_1|} \\ &= \sum_{K \text{ good on } V'} (\lambda+2)^{|V-W_1|} C(V',K) \\ &= \sum_{K \text{ acceptable}} (\lambda+2)^{|V|-\deg_4(K)} 2^{\deg_4(K)} (-1)^{a(K)} x_K \\ &= J(G) \end{aligned}$$

which concludes the proof of Theorem 3. ∎

4. Converting to Permanents

The goal of this section is to convert the state sum $J(D)$ into a permanent, in the following way:

Theorem 4.

$$J(\mathrm{LID}(D))|_{x_e=1} = \mathrm{Per}(\mathrm{IM}_J(D))$$

Note that Theorem 1 follows from Theorems 2 and 4.

We will achieve the conversion of $J(\mathrm{LID}(D))$ into a permanent by a local modification (we can say, a blow-up) of each of the vertices of the original digraph $\mathrm{LID}(D)$. It turns out that the modification triples each of the vertices of G. Why triple? Because in a sense $J(\mathrm{LID}(D))$ has to do with the $\mathfrak{sl}_2$-Lie algebra. Thus, we are back to Lie algebras, this time through a common blow-up trick of combinatorics.

Proof of Theorem 4. We have that

$$J(\mathrm{LID}(D))|_{x_e=1} = \sum_{K \text{ acceptable}} (\lambda+2)^{|V|-\deg_4(K)} 2^{\deg_4(K)} (-1)^{a(K)},$$

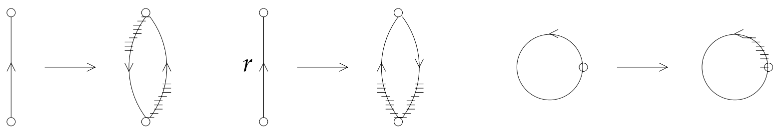
where $a(K)$ equals the number of arcs of K thickened in agreement with their orientation. A single loop of K is always leaving its vertex, and it is directed in agreement with its thickening. Hence each single loop contributes '(-1)' to $a(K)$. Let us now get rid of these single loops: an acceptable object without single loops will be called *connected* and a connected object where each vertex of V has degree at least 2 will be *super*. We have

$$\begin{aligned} J(\mathrm{LID}(D))|_{x_e=1} &= \sum_{K \text{ acceptable}} (\lambda+2)^{|V|-\deg_4(K)} 2^{\deg_4(K)} (-1)^{a(K)} \\ &= \sum_{K \text{ connected}} (\lambda+2)^{|V|-\deg_4(K)} 2^{\deg_4(K)} (-1)^{a(K)} \sum_{U \subset V-K} (-1)^{|U|} \\ &= \sum_{K \text{ super}} (\lambda+2)^{|V|-\deg_4(K)} 2^{\deg_4(K)} (-1)^{a(K)}. \end{aligned}$$

We will use a variation TID(D)), the *thickened intersection digraph* of D defined as follows. Double each of the arcs of LID(D) and:

1. if the arc is uncolored then thicken such pair at opposite ends,
2. if the arc is red then thicken each arc of the pair at the start, and then change the orientation of one of them,
3. thicken each loop at its initial segment, i.e., in agreement with its orientation.

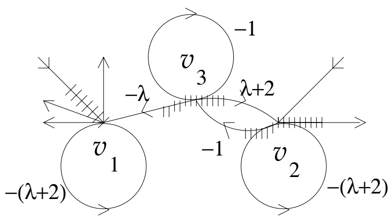
In pictures, the thickening of LID is the substitution



Now we can write

$$J(\mathrm{LID}(D))|_{x_e=1} = \sum_{K \text{ super subobject of } \mathrm{TID}(D)} (\lambda+2)^{|V|-\deg_4(K)} 2^{\deg_4(K)} (-1)^{a(K)}.$$

Let us describe now how a thickened digraph $D(\mathrm{IM}_J)$ may be constructed from TID(D). The construction easily follows from the definition of matrix IM_J: it consists in replacing each vertex of TID(D) by a 'gadget' on three vertices, as follows:



The construction goes as follows:

1. For each vertex v of $\mathrm{TID}(D)$ introduce three vertices v_1, v_2, v_3 for $D(\mathrm{IM}_J)$.
2. Define the thickened arcs and their weights among each triple v_1, v_2, v_3 as follows:
   - add loop l_i at each v_i and let $w(l_1) = w(l_2) = -(\lambda+2)$ and $w(l_3) = -1$,
   - add arc (v_3, v_1) thickened at v_3 with weight $-\lambda$,
   - add arc (v_3, v_2) thickened at v_3 with weight $\lambda+2$,
   - add arc (v_2, v_3) thickened at v_2 with weight -1.
3. For each thickened arc (u, w) of $\mathrm{TID}(D)$ do the following:
   - If (u, w) is thickened at u then add $(u_2, w_1), (u_2, w_2)$ thickened at u_2 with weights equal to 1,
   - If (u, w) is thickened at w then add $(u_1, w_1), (u_2, w_1)$ thickened at w_1 with weights equal to 1.

It follows directly from the definition of the permanent that

$$\mathrm{Per}(\mathrm{IM}_J) = \sum_{L \in \mathcal{L}} (-1)^{a(L)} w_L,$$

where $\mathcal{L}$ is the set of all acceptable subobjects of $D(\mathrm{IM}_J)$ where each degree equals 2.

We need to show that

$$\sum_{\substack{L \text{ with each degree } 2 \\ \text{acceptable subobject of} D(\mathrm{IM}_J)}} (-1)^{a(L)} w_L = \sum_{\substack{K \text{ super subobject} \\ \text{of } \mathrm{TID}(D)}} (\lambda+2)^{|V| - \deg_4(K)} 2^{\deg_4(K)} (-1)^{a(K)}.$$

We will prove it by constructing a partition of acceptable subobjects of $D(\mathrm{IM}_J)$ where each vertex has degree 2, and associating each partition class which contributes non-zero to $\mathrm{Per}(D(\mathrm{IM}_J))$ with uniquely determined super subobject of $\mathrm{TID}(D)$.

Let L be an acceptable subobject of $D(\mathrm{IM}_J)$ where each vertex has degree 2. Denote by OL the set of all thickened arcs of type $(u_i w_j)$, $u \neq w$, and let $IL = L - OL$. Note that if we forget the lower indices at vertices, OL naturally corresponds to a set OK of thickened arcs of $\mathrm{TID}(D)$. Note that no arc in OK is a loop and $a(OL) = a(OK)$. It remains to be seen what to do with the thickened arcs of IL. We may consider each triple of vertices v_1, v_2, v_3 separately. Let ILv denote the set of thickened arcs of L among v_1, v_2, v_3. We distinguish four cases.

- ILv consists of all three loops or the loop at v_1 and the arcs $(v_2, v_3), (v_3, v_2)$. Let C_0 be the class of all L which in at least one triple v_1, v_2, v_3 behave in this way. Note that the total contribution of C_0 to $\mathrm{Per}(D(\mathrm{IM}_J))$ is 0, and so we may assume that this case never happens (it corresponds to single loop at v in $\mathrm{TID}(D)$ which is not allowed for supersubobjects).

- ILv consists of loop at v_1 or loop at v_2 (but not both), and loop at v_3. Then let $IKv=\emptyset$. Hence v will have degree 2 in K and contribute $(\lambda+2)$ to $J(\mathrm{LID}(D))$, which exactly equals to the contribution of ILv to $\mathrm{Per}(D(\mathrm{IM}_J))$.
- ILv consists of $\{(v_2,v_3),(v_3,v_2)\}$ or $\{(v_3,v_1),(v_2,v_3)\}$. Then v has degree 2 in OK and the edge of OK incident and thickened at v is entering v. In this case let IKv consist of the loop at v. Note that the total contribution of ILv to $\mathrm{Per}(D(\mathrm{IM}_J))$ is $-(\lambda+2)+\lambda=-2$ and $2(-1)$ is also the contribution of IKv to $J(\mathrm{LID}(D))$.
- $ILv=\emptyset$. Then v has degree 4 in OK and we let $IKv=\emptyset$. In this case each v_i, $i=1,2$ is incident with one arc of L not thickened at v_i. Let L' be obtained from L by exchanging the incidence of non-thickened arcs between v_1 and v_2. The total contribution of ILv and $IL'v$ to $\mathrm{Per}(D(\mathrm{IM}_J))$ is 2 and 2 is also the contribution of IKv to $J(\mathrm{LID}(D))$.

It is easy to check that we have indeed partitioned the set of all acceptable subobjects of $D(\mathrm{IM}_J)$ where each vertex has degree 2 and that we exhausted all super subobjects of $\mathrm{TID}(D)$. This finishes the proof of Theorem 4. ∎

5. Proof of the corollaries

Corollary 1.4 is immediate. For Corollary 1.5, observe that $W_{JJ}=W_{JJ^{(0)}}$ is given by Corollary 1.4 for $n=0$. We claim that this formula equals to $\mathrm{Per}(\mathrm{IM})$. This may be observed as follows: assume $i<j$. Associate $\mathrm{IM}(D)_{ij}$ with the arc (i,j) of $\mathrm{LID}(D)$ thickened at i and $\mathrm{IM}(D)_{ji}$ with arc (i,j) thickened at j. This associates, with each term of the expansion of $\mathrm{Per}(\mathrm{IM})$, an acceptable object of uncolored arcs only, with each degree equal to 2 and no loops. Denote the set of such acceptable objects by $\mathcal{K}_1$. It is straightforward to check that

$$\mathrm{Per}(\mathrm{IM}) = \sum_{K\in\mathcal{K}_1} (-1)^{a(K)}.$$

On the other hand, $W_{JJ^{(0)}} = \sum_{K\in\mathcal{K}_2}(-1)^{a(K)}$, where $\mathcal{K}_2$ is the set of all acceptable objects where each degree is 0 or 2 (loops are allowed and they contribute 2 to the degree). First observe that the contribution of the acceptable objects of $\mathcal{K}_2$ that contain a red arc cancels out since we can change the orientation of a red arc in such an object, and get again an object of $\mathcal{K}_2$, with oposite contribution. Hence assume $\mathcal{K}_2$ has no objects with red arcs. If $K\in\mathcal{K}_2$ then let $L(K)$ denote the acceptable subobject of K obtained from K by deleting all loops. If $L=L(K)$ let $V(L)$ denote the set of vertices of

L of non-zero degree and let $\mathcal{E}(L)=\{K\in\mathcal{K}_2; L(K)=L\}$. By the binomial theorem,

$$\sum_{K\in\mathcal{E}(L)}(-1)^{a(K)}=(-1)^{a(L)}\sum_{W\subset(V-V(L))}(-1)^{|W|}=0$$

whenever $V(L)\neq V$. This proves the corollary.

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A. Numerical examples

Example codes (along with the informations required for running it) are available from SpringerLink at http://dx.doi.org/10.1007/s00493-005-0041-3.

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