Sel. math., New ser. 8 (2002) 315 – 339
1022–1824/02/030315–25

Selecta Mathematica, New Series

The Århus integral of rational homology 3-spheres I: A highly non trivial flat connection on S^3

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Abstract. Path integrals do not really exist, but it is very useful to dream that they do and figure out the consequences. Apart from describing much of the physical world as we now know it, these dreams also lead to some highly non-trivial mathematical theorems and theories. We argue that even though non-trivial flat connections on S^3 do not really exist, it is beneficial to dream that one exists (and, in fact, that it comes from the non-existent Chern-Simons path integral). Dreaming the right way, we are led to a rigorous construction of a universal finite-type invariant of rational homology spheres. We show that this invariant is equal (up to a normalization) to the LMO (Le-Murakami-Ohtsuki,[LMO]) invariant and that it recovers the Rozansky and Ohtsuki invariants.

This is part I of a 4-part series, containing the introductions and answers to some frequently asked questions. Theorems are stated but not proved in this part. Part II of this series is titled "Invariance and Universality", part III "The Relation with the Le-Murakami-Ohtsuki Invariant", and part IV "The Relation with the Rozansky and Ohtsuki Invariants".

Mathematics Subject Classification (2000). 57M27.

Key words. Finite type invariants, Gaussian integration, 3-manifolds, Kirby moves, holonomy.

This series has two introductions. The first is philosophical and non-rigorous. We recommend reading it first. The second introduction and the rest of the series are fully rigorous and can be read independently of the first part.

1. Philosophical introduction

1.1. What if there were?

Suppose there was some highly-non-trivial flat connection A on S^3. Well, of course there are no non-trivial flat connections on S^3; in recent years there has been no significant debate over that. But we wrote "highly-non-trivial flat connection", meaning that the object that we are going to talk about is neither a connection nor is it flat (and possibly, does not even exist). But we will see that it is beneficial to assume that such an A does exist, and that it has some fixed good properties. We will make some deductions and guess some formulas, and later on we will prove that they work, with no reference to A.

We claim that given a well-behaved highly-non-trivial flat connection A on S^3, we can construct (under some mild conditions) a link invariant $\mathring{A}_A$ that respects the Kirby moves, and hence an invariant of 3-manifolds. Just constructing a link invariant is easy; simply consider the holonomy $h_A(L)$ of the connection A along a link L. The invariance under small deformations (that do not pass through self-intersections) of an embedding of L follows from the flatness of A. The non-invariance under full homotopy comes from the fact that A is highly-non-trivial. If $h_A(L)$ were invariant under homotopy, it would have been rather dull, for all links of a fixed number of components are homotopic.

The construction of $\mathring{A}_A(L)$ from $h_A(L)$ is extremely simple, and can be summarized by a single catchy motto:

Motto 1. Integrate the Holonomies

Let's try to make sense out of this. Suppose A is a $\mathfrak{g}$-connection for some metrized Lie algebra $\mathfrak{g}$. The holonomy of A along a single path is roughly the product of the values of A seen along the path. Namely, it is a certain long product of elements of $\mathfrak{g}$. So it is naturally valued in $\hat{U}(\mathfrak{g})$, a certain completion of the universal enveloping algebra of $\mathfrak{g}$. By the Poincaré-Birkhoff-Witt theorem, $\hat{U}(\mathfrak{g})$ is isomorphic to $\hat{S}(\mathfrak{g})$, a completion of the symmetric algebra of $\mathfrak{g}$, via the symmetrization map $\beta : \hat{S}(\mathfrak{g}) \to \hat{U}(\mathfrak{g})$ (see e.g. [Di, Paragraph 2.4.10]). The algebra $\hat{S}(\mathfrak{g})$ is the algebra of power series on $\mathfrak{g}^\star$, and those power series that are convergent are called functions and can sometimes be integrated. It is in this sense that one should interpret Motto 1, only that in the case of an X-marked link (a link whose components are in a bijective correspondence with some n-element set of labels $X = \{x_i\}_{i=1}^n$, with a base point on each component, to make the holonomies well defined) the holonomies are in $\hat{U}(\mathfrak{g})^{\otimes n}$, and thus the integration is over n copies of $\mathfrak{g}^\star$, with one integration variable (also denoted x_i) for each component of the link:

Definition 1.1. The Århus integral $\mathring{A}_A$ for the non-trivial flat connection A is the integral

$$\mathring{A}_A(L) = \mathcal{N} \int_{\mathfrak{g}^\star \oplus \ldots \oplus \mathfrak{g}^\star} h_A(L)(x_1, x_2, \ldots, x_n)\, dx_1\, dx_2 \ldots dx_n,$$

where $h_A(L)(x_1, x_2, \ldots, x_n)$ is the holonomy $h_A(L)$ regarded as a function of $(x_1, \ldots, x_n) \in \mathfrak{g}^\star \oplus \ldots \oplus \mathfrak{g}^\star$, the symbol dx denotes the measure on $\mathfrak{g}^\star$ induced by its metric, and $\mathcal{N}$ denotes additional normalizations that may be ignored on the philosophical level.

Let us get to the main point as quickly as possible. Why is $\mathring{A}_A$ invariant under the second and more difficult Kirby move [Ki]? Not worrying too much about the important issue of framing (we'll do that later, in Section 2), the second Kirby move is the operation of 'sliding' one component of the link along a neighboring one. See Figure 1.

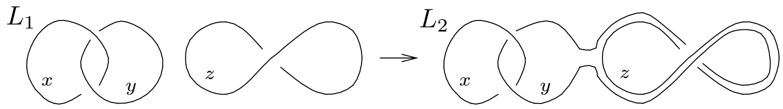


Figure 1. The second Kirby move.

Let us analyze the behavior of $\mathring{A}_A$ under the second Kirby move. The only difference is that the holonomy along the component labeled y changes, and (with an appropriate choice of basepoints) the change is rather simple: it gets multiplied by the holonomy around the component labeled z. In formulas valid in $\hat{U}(\mathfrak{g})^{\otimes 3}$, this amounts to saying that

$$h_A(L_2) = (1 \otimes \times_U \otimes 1)(1 \otimes 1 \otimes \Delta)h_A(L_1), \tag{1}$$

where $\Delta : \hat{U}(\mathfrak{g}) \to \hat{U}(\mathfrak{g}) \otimes \hat{U}(\mathfrak{g})$ is the co-product, which in (1) takes the group-like holonomy of A along z and "doubles" it, and $\times_U : \hat{U}(\mathfrak{g}) \otimes \hat{U}(\mathfrak{g}) \to \hat{U}(\mathfrak{g})$ is the product, which takes one of the copies of the holonomy along z and multiplies it into the holonomy along y.

Truth 1.2. *$\beta : \hat{S}(\mathfrak{g}) \to \hat{U}(\mathfrak{g})$ is a co-algebra map.*

Almost Truth 1.3. *$\beta : \hat{S}(\mathfrak{g}) \to \hat{U}(\mathfrak{g})$ is an algebra map as well.*

Together, 1.2 and 1.3 say that (1) is also valid in $\hat{S}(\mathfrak{g})$. Identifying $\hat{S}(\mathfrak{g})$ with the space $F(\mathfrak{g}^\star)$ of functions on $\mathfrak{g}^\star$, the product $\times_S$ of $\hat{S}(\mathfrak{g})$ becomes the diagonal map $f(x, y) \mapsto f(x, x)$, the co-product Δ becomes the map $f(x) \mapsto f(x + y)$, and (1) becomes

$$h_A(L_2)(x, y, z) = h_A(L_1)(x, y, y + z).$$

But now it is clear why $\mathring{A}_A$ is invariant under the second Kirby move — on holonomies, the second Kirby move is just a change of variables, which does not change the value of the integral!

Let us now see why the "almost" in almost truth 1.3 is harmless in our case.

Definition 1.4. A differential operator is said to be *pure* with respect to some variable y if its coefficients are independent of y and every term in it is of positive order in $\partial/\partial y$. We allow infinite order differential operators, provided they are "convergent" in a sense that will be made precise in the context in which they will be used.

If D is pure with respect to y, then the fundamental theorem of calculus shows that the integral of Df with respect to y vanishes (at least when f is appropriately decreasing; the f's we will use below decrease like Gaussians, which is more than enough). With this in mind, the following claim and remark explain why the "almost" in almost truth 1.3 is harmless:

Claim 1.5. *When the native product $\times_S$ of $\hat{S}(\mathfrak{g})$ and the product $\times_U$ it inherits from $\hat{U}(\mathfrak{g})$ via β are considered as products on functions on $\mathfrak{g}^\star$, they differ by some differential operator $D'_{\mathrm{BCH}} : F(\mathfrak{g}^\star) \otimes F(\mathfrak{g}^\star) = F(\mathfrak{g}^\star \oplus \mathfrak{g}^\star) \to F(\mathfrak{g}^\star \oplus \mathfrak{g}^\star)$, composed with the diagonal map $\times_S$:*

$$\times_U - \times_S = \times_S \circ D'_{\mathrm{BCH}}.$$

Proof (sketch). A hint that claim 1.5 should be true is already in the more common formulation of the Poincaré-Birkhoff-Witt theorem (see e.g. [Di, Paragraph 2.3.6]), saying that the symmetric algebra $S(\mathfrak{g})$ is isomorphic *as an algebra* to the associated graded $\operatorname{gr} U(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$. This means that if f and g are homogeneous polynomials, then $f \times_S g$ and $f \times_U g$ differ by lower degree terms, and these lower degree terms come from applying differential operators. As a typical example, let's look at the term of degree one less. If $f = \prod_\alpha a_\alpha \in S(\mathfrak{g})$ and $g = \prod_\beta b_\beta \in S(\mathfrak{g})$, then

$$\begin{aligned} f \times_U g &= f \times_S g + \frac{1}{2} \sum_{\gamma,\delta} [a_\gamma, b_\delta] \prod_{\alpha \neq \gamma} a_\alpha \prod_{\beta \neq \delta} b_\beta \\ &= f \times_S g + \times_S \circ D_1(f, g) \quad \text{(mod lower degrees)}, \end{aligned}$$

where (from this formula) the operator D_1 can be written in terms of a basis $\{l_i\}$ of $\mathfrak{g}$ as $D_1 = \frac{1}{2} \sum_{i,j} (([l_i, l_j] \otimes 1)) \cdot \left(\frac{\partial}{\partial l_i} \otimes \frac{\partial}{\partial l_j} \right)$. Written in full, D'_{BCH} is an infinite sum of operators D_k, where each D_k is of bounded degree and involves k applications of the Lie bracket $[\cdot, \cdot]$. Below it will be justified to think of the bracket as "small", and hence the sum D'_{BCH} is convergent. □

Remark 1.6. The reader may show that $D = (D'_{\mathrm{BCH}} \otimes 1) \circ (1 \otimes \Delta)$ is a pure differential operator on the space $F(\mathfrak{g}^\star) \otimes F(\mathfrak{g}^\star) = F(\mathfrak{g}^\star \oplus \mathfrak{g}^\star)$, with respect to all variables in the second copy of $\mathfrak{g}^\star$.

Remark 1.7. (Compare with [Å-II, Corollary 5.5]). The reader may use the Baker-Campbell-Hausdorff formula (see e.g. [Ja, Section V.5]) to find explicit formulas for the operators D'_{BCH} and D.

Here are some more relevant facts:

Fact 1.8. *The Århus integral $\AA_A$ is invariant under reversal of orientation of any component of the link (better be that way, for the Kirby calculus is about unoriented links).*

Proof. On the level of $F(\mathfrak{g}^\star)$, reversal of the orientation of a component acts by negating the corresponding variable, say, $h_A(L)(x,y,z) \to h_A(L)(x,-y,z)$. This operation does not change the integral of $h_A(L)$. □

Fact 1.9. *$\AA$ is independent of the choice of the base points on the components.*

Proof. Moving the base point on one of the components amounts to acting on the corresponding $\hat{U}(\mathfrak{g})$ by some group element g. This translates to acting on the corresponding variable of $h_A(L)(x,y,\dots)$ by $\operatorname{Ad} g$. But the adjoint action $\operatorname{Ad} g$ acts by a volume preserving transformation, and hence the integral of $h_A(L)$ does not change. □

We leave it for the reader to check that the normalization $\mathcal{N}$ in Definition 1.1 can be chosen so as to make $\AA_A$ invariant under the easier "first" Kirby move, shown in Figure 2. The solution appears in Section 2.1.4.

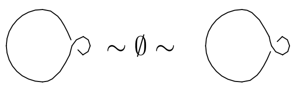


Figure 2. The first Kirby move: an unknotted unlinked component of framing ±1 can be removed.

1.2. There are not, but we can do without

Rather than using a single highly non-trivial flat connection A, we use the average holonomy of many not-necessarily-flat connections, with respect to a non-existent measure, and show that the resulting averaged holonomies have all the right properties. Namely, we replace $h_A(L)$ by $h_{\mathfrak{g},k}(L) := \int h_B(L) d\mu_k(B)$, where $d\mu_k(B)$ is the famed Chern-Simons [Wi] measure on the space of $\mathfrak{g}$-connections, depending on some integer parameter k:

$$d\mu_k(B) = \exp\left(\frac{ik}{4\pi}\int_{S^3} \operatorname{tr} B \wedge dB + \frac{2}{3} B \wedge B \wedge B\right)\mathcal{D}B$$

($\mathcal{D}B$ denotes the path integral measure over the space of $\mathfrak{g}$-connections). In other words, we set

$$\AA_{\mathfrak{g}}(L) = \mathcal{N}\int_{\mathfrak{g}^\star\oplus\dots\oplus\mathfrak{g}^\star}\left(\int h_B(L)(x_1,\dots,x_n) d\mu_k(B)\right) dx_1\dots dx_n.$$

Let us see why $h_{\mathfrak{g},k}(L)$ has all the right properties:

- *Flatness:* We only need to know that $h_{\mathfrak{g},k}(L)$ is a link invariant. Indeed it is, by the usual topological invariance of the Chern-Simons path integral [Wi].

- *Non-triviality:* If $h_{\mathfrak{g},k}(L)$ were a trivial link invariant, so would have been $\AA_{\mathfrak{g}}$. Fortunately the Chern-Simons path integral is not trivial.

We divide the second Kirby move into two steps: first a component is doubled, and then we form the connected sum of one of the copies with some third component. So two more properties are needed:

- *Behavior under doubling:* We need to know that if L_2 is obtained from L_1 by (say) doubling the first component, then $h_{\mathfrak{g},k}(L_2) = (\Delta \otimes 1 \otimes \cdots \otimes 1)h_{\mathfrak{g},k}(L_1)$. This property holds for every individual B, and it is a linear property that survives averaging.
- *Behavior under connected sum:* For notational simplicity, let us restrict to knots. If C_1 and C_2 are two knots and $C_1\#C_2$ is their connected sum, we need to know that $h_{\mathfrak{g},k}(C_1\#C_2) = h_{\mathfrak{g},k}(C_1) \times_U h_{\mathfrak{g},k}(C_2)$. With the proper normalization, this is a well known property of the Chern-Simons path integral, and a typical example of cut-and-paste properties of topological quantum field theories.

We should note that in part II of this series ([Å-II]) we will recombine these last two properties again into one, whose proof, due to Le, H. Murakami, J. Murakami, and Ohtsuki [LMMO], is rather intricate and ingenious.

1.3. The diagrammatic case: formal Gaussian integration

It may or may not be possible to make sense of the ideas outlined in sections 1.1 and 1.2 as stated, per Lie algebra $\mathfrak{g}$ and per integer k. We do not know though we do want to know. Anyway, our approach is different. We do everything in the $k \to \infty$ limit, where the Chern-Simons path integral has an asymptotic expansion in terms of Feynman diagrams. Furthermore, as is becoming a standard practice among topological perturbativites (see e.g. [B-N1], [Ko2], [BT], [Th], [AF]), we divorce the Lie algebras from the Feynman diagrams and work in the universal diagrammatic setting. In this case, the Chern-Simons path integral is valued in the space $\mathcal{A}(\circlearrowleft_X)$ of diagrams as in Figure 3 modulo the STU relations, whose precise form is immaterial here (though it appears in Figure 10).

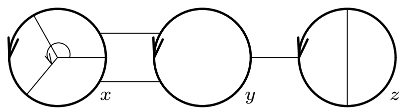


Figure 3. Diagrams in $\mathcal{A}(\circlearrowleft_{\{x,y,z\}})$ are trivalent graphs made of 3 oriented circles labeled x, y, and z, and some number of additional "internal" edges. "Internal vertices", in which three internal edges meet, are "oriented" - a cyclic order on the edges emanating from such a vertex is specified.

Choosing a base point on each oriented circle (akin to our choice of a base point on each link component, in Section 1.1) and cutting the circles open, we get a sum of diagrams, deserving of the name $h_\infty(L)$, in the space $\mathcal{A}(\uparrow_X)$. A typical element of that space appears in Figure 4.

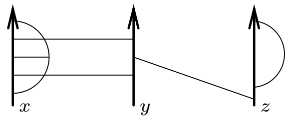


Figure 4. A diagram in $\mathcal{A}(\uparrow_{\{x,y,z\}})$. Here and below, internal trivalent vertices are always oriented counterclockwise, and quadrivalent vertices are just artifacts of the planar projection. This diagram is a cut-open of the diagram in Figure 3.

At this point, we are in very good shape. There is a well known parallelism between spaces of diagrams such as $\mathcal{A}(\uparrow_X)$ and various spaces that pertain to Lie algebras, such as $U(\mathfrak{g})^{\otimes n}$. We have a "holonomy" living in the former space, and a technique (Motto 1) living in the latter. All we have to do is to imitate the Lie-level technique on the diagram level. To do that, let us first summarize the technique of Section 1.1 in one line:

$$h_A(L) \in U(\mathfrak{g})^{\otimes n} \xrightarrow{PBW} S(\mathfrak{g})^{\otimes n} \xrightarrow{\int} \mathbb{C}$$

The parallelism: (The primary reference for points 1 to 4 below is [B-N2]. See also [B-N3], [LM].)

(1) The parallel of $U(\mathfrak{g})^{\otimes n}$ is, as already noted, $\mathcal{A}(\uparrow_X)$. Their affinity is first seen in the existence of a structure-respecting map $\mathcal{T}_\mathfrak{g} : \mathcal{A}(\uparrow_X) \to U(\mathfrak{g})^{\otimes n}$. Very briefly, $\mathcal{T}_\mathfrak{g}$ is defined by mapping all internal vertices to copies of the structure constants tensor, all internal edges the metric of $\mathfrak{g}$, and the vertical arrows to the ordered products of the Lie algebra elements seen along them, namely to elements of $U(\mathfrak{g})$.

Notice that the parallel of the Lie bracket $[\cdot,\cdot]$ is a vertex. Thus iterated brackets correspond to high-degree diagrams, which are "small" in the sense of the completed space $\mathcal{A}(\uparrow_X)$. This justifies the last sentence in the proof of claim 1.5.

(2) The parallel of $S(\mathfrak{g})^{\otimes n}$ is the space $\mathcal{B}(X)$ of "X-marked uni-trivalent diagrams"[1], the space of uni-trivalent graphs whose trivalent vertices are oriented and whose univalent vertices are marked by the symbols in the set X (possibly with omissions and/or repetitions), modulo the AS and IHX relations, whose precise form is immaterial here (though it appears in Figure 9). An example appears in Figure 5. There is a structure-respecting

[1] Called "Chinese characters" in the culturally insensitive [B-N2].

map $\mathcal{T}_{\mathfrak{g}} : \mathcal{B}(X) \to S(\mathfrak{g})^{\otimes n}$; it maps uni-trivalent diagrams with k external legs to degree k elements of $S(\mathfrak{g})^{\otimes n}$.

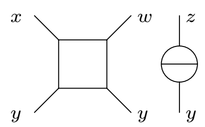


Figure 5. An $\{x, y, z, w\}$-marked uni-trivalent diagram.

(3) The parallel of the Poincaré-Birkhoff-Witt isomorphism $U(\mathfrak{g})^{\otimes n} \to S(\mathfrak{g})^{\otimes n}$ is a map $\sigma : \mathcal{A}(\uparrow_X) \to \mathcal{B}(X)$, which is more easily described through its inverse χ. If $C \in \mathcal{B}(X)$ is an X-marked uni-trivalent diagram with k_x legs marked x for $x \in X$, then $\chi(C) \in \mathcal{A}(\uparrow_X)$ is the average of the $\prod_x k_x!$ ways of attaching the legs of C to n labeled vertical arrows (labeled by the elements of X), attaching legs marked by x only to the x-labeled arrow, for all $x \in X$.

(4) The parallel of $\mathbb{C}$ is the set of uni-trivalent diagrams that $\mathcal{T}_{\mathfrak{g}}$ maps to degree 0 elements of $S(\mathfrak{g})^{\otimes n}$ — namely, it is the space called $\mathcal{A}(\emptyset)$ of "manifold diagrams" — uni-trivalent diagrams with no external legs. An example appears in Figure 6.

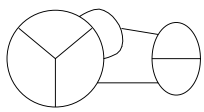


Figure 6. A connected manifold diagram. All vertices are oriented counterclockwise.

(5) Last, we need a parallel $\int^{FG} : \mathcal{B}(X) \to \mathcal{A}(\emptyset)$ for the partially defined integration map $\int : S(\mathfrak{g})^{\otimes n} \to \mathbb{C}$. This is a new and (hopefully) amusing ingredient, so let us say a bit more.

The new $\int^{FG}$ better be defined on $\sigma h_\infty(L)$. To ensure this, we first need some knowledge about the structure of $\sigma h_\infty(L)$, and the following easy claim suffices:

Claim 1.10. *If (l_{xy}) is the $n \times n$ linking matrix of L (that is, l_{xy} is the linking number of the component x with the component y, and l_{xx} is the self-linking of the component x — the linking number of that component with its framing), then*

$$\underbrace{\sigma h_\infty(L) = \exp_{\cup}\left(\frac{1}{2}\sum_{x,y} l_{xy}\; {}^{x}\!\frown^{y} + \begin{pmatrix}\textit{connected uni-trivalent diagrams}\\ \textit{that have trivalent vertices}\end{pmatrix}\right)}_{\textit{I'm Gaussian!}}. \quad (2)$$

Here $\exp_\cup$ *denotes power-series exponentiation using the disjoint union product on* $\mathcal{B}(X)$*, and* ${}^x\frown^y$ *denotes the only connected uni-trivalent diagrams that have no trivalent vertices — solitary edges whose univalent ends are marked* i *and* j*.*

Proof (sketch). There is a co-product $\Box : \mathcal{B}(X) \to \mathcal{B}(X) \otimes \mathcal{B}(X)$, mapping every uni-trivalent diagram to the sum of all possible ways of splitting it by its connected components. A simple argument shows that $Z = \sigma h_\infty(L)$ satisfies $\Box Z = Z \otimes Z$, and thus $Z = \exp(\Box\text{-primitives})$. The $\Box$-primitives of $\mathcal{B}(X)$ are the connected uni-trivalent diagrams, and hence $Z = \exp(\text{connected characters})$. (This is a variant of a standard argument from quantum field theory, saying that the logarithm of the partition function can be computed using connected Feynman diagrams). All that is left is to determine the coefficients of the simplest possible connected uni-trivalent diagrams, the ${}^x\frown^y$'s. These correspond to degree 1 Vassiliev invariants, namely, to linking numbers. $\Box$

Equation (2) says "I'm Gaussian!".Remembering that $\mathcal{T}_\mathfrak{g}$ maps leg-count to degree and thinking of trivalent vertices as "small" and hence of the second term in (2) as a perturbation, we see that $\mathcal{T}_\mathfrak{g}\sigma h_\infty(L)$ is indeed a Gaussian! Thus the standard Gaussian integration technique of Feynman diagrams applies (a refresher is in Section 4). But this is a diagrammatic technique, and hence it can be applied straight at the diagrammatic level of $\sigma h_\infty(L)$. It takes two steps:

(1) Splitting the quadratic part out, negating and inverting it, getting

$$\exp_\cup\left(-\frac{1}{2}\sum_{x,y} l^{xy}\, {}_{\partial_x}\smile_{\partial_y}\right),$$

where (l^{xy}) is the inverse matrix of (l_{xy}) and we've introduced a new set of "dual" labels $\partial_X = \{\partial_x : x \in X\}$.

(2) Putting the inverted quadratic part back in and gluing its legs to all other legs in all possible ways, making sure that the markings match.

These steps together with all previous steps are summarized in the commutative diagram in Figure 7 and in a more graphical form in Figure 8.

One last comment is in order. While the Chern-Simons h_∞ is perfectly good on the philosophical level, it is a bit difficult to work with. Thus we replace it by a substitute whose properties we understand better, a variant $\check{Z}$ due to [LMMO] of the Kontsevich integral [Ko1], [B-N2] (which by itself is a holonomy, of the Knizhnik-Zamolodchikov connection). It is conjectured that the original Kontsevich integral is equal to h_∞.

$$\begin{array}{ccccc}
h_\infty(L)\in\mathcal{A}(\uparrow_X) & \xrightarrow{\sigma} & \mathcal{B}(X) & \xrightarrow[\text{(partially defined)}]{\int^{FG}} & \mathcal{A}(\emptyset)\ni \mathring{A}(L) \\
\big(\text{or } \check{Z}(L)\big) & & & & \\
\mathcal{T}_{\mathfrak{g}}\downarrow & & \mathcal{T}_{\mathfrak{g}}\downarrow & & \mathcal{T}_{\mathfrak{g}}\downarrow \\
U(\mathfrak{g})^{\otimes n} & \xrightarrow{PBW} & S(\mathfrak{g})^{\otimes n} & \xrightarrow[\text{(partially defined)}]{\int} & \mathbb{C}
\end{array}$$

Figure 7. The bottom row is where the nonsensical Section 1.1 lives. It is also were Section 1.2 lives, both in the finite k case (which we do not consider below) and in the $k\to\infty$ limit. The top row leads to a well defined and interesting invariant, and is the main focus of the rest of this series of papers.

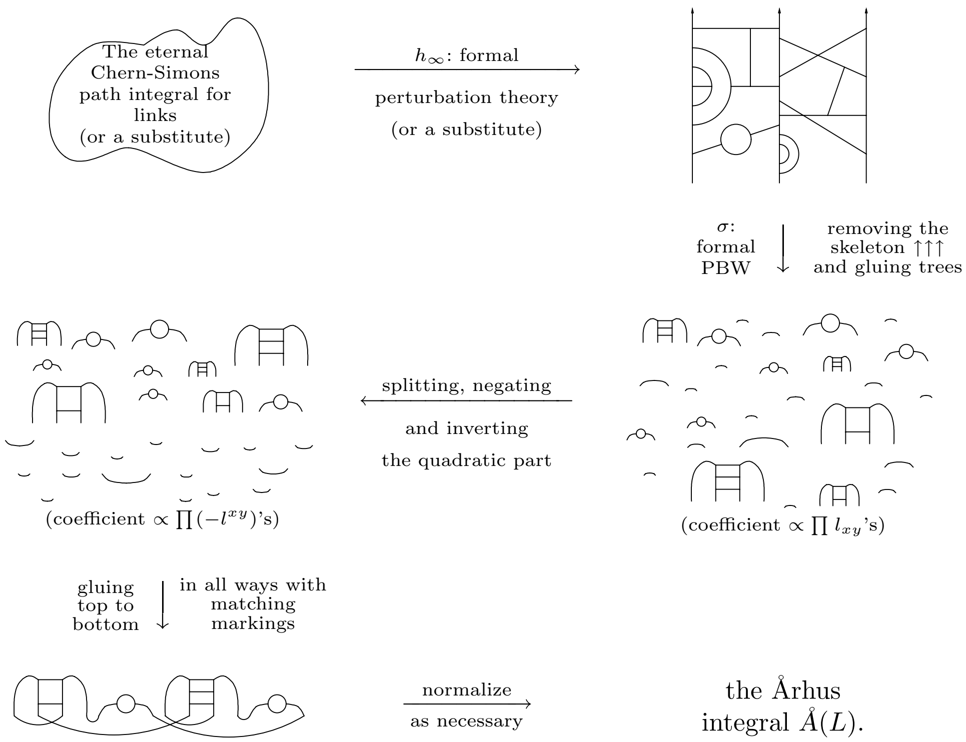


Figure 8. Introduction for the graphically oriented. All diagrams here are representatives of big sums of diagrams.

2. Rigorous introduction

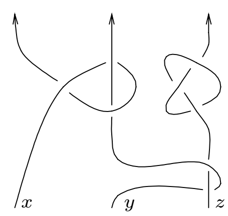
In this section, our goal is very modest: to give the precise definition of an invariant $\AA$ of rational homology spheres, and to state its main properties. The proofs of these properties of $\AA$ (and even the fact that it is well defined), though mostly natural and conceptual, will be postponed to the later parts of this series.

2.1. Definition of the Århus integral

All (3-dimensional) rational homology spheres are surgeries on algebraically regular framed links in S^3 ("regular links" throughout this series, precise definition below), and all regular links are closures of algebraically regular framed pure tangles ("regular pure tangles", precise definition below). All that we do in this section is to define a certain invariant $\AA$ of regular pure tangles, following the philosophical ideas of Section 1. In part II of this series ([Å-II]) we follow the same philosophical ideas and show that these ideas lead to simple proofs that $\AA$ descends to an invariant of regular links and then to an invariant of rational homology spheres.

2.1.1. Domain and target spaces. Let us start with the definitions of the domain and target spaces of $\AA$:

Definition 2.1. An (X-marked) pure tangle (also called "string link") is an embedding T of n copies of the unit interval, $I\times\{1\},\dots,I\times\{n\}$ into $I\times\mathbb{C}$, so that $T((\epsilon,i))=(\epsilon,i)$ for all $\epsilon\in\{0,1\}$ and $1\leq i\leq n$, considered up to endpoint-preserving isotopies. We assume that the components of the pure tangle are labeled by labels in some n-element label set X. An example is on the right.



Similarly to many other knotted objects, pure tangles can be "framed" (we omit the precise definition), and similarly to braids, framed pure tangles can be closed to form a framed link. By pullback, this allows one to define linking numbers and self-linking numbers for framed pure tangles.

Definition 2.2. A framed link is called "algebraically regular" ("regular link", in short) if its linking matrix (with self-linkings on the diagonal) is invertible. A framed tangle is called "algebraically regular" ("regular pure tangle", in short), if the same condition holds, or, alternatively, if its closure is a regular link. Let RPT be the set of all (X-marked) regular pure tangles.

This completes the definition of the domain space of $\AA$. Let's turn to the target space:

Definition 2.3. A "manifold diagram" is a trivalent graph with oriented vertices (i.e., a cyclic order is specified on the edges emanating from each vertex), such as the one in Figure 6. The "degree" of a manifold diagram is half the number of vertices it has. The target space of $\mathring{A}$, called $\mathcal{A}(\emptyset)$, is the graded completion of the linear space spanned by all manifold diagrams modulo the IHX and AS (antisymmetry) relations displayed in Figure 9.

Figure 9. The IHX and AS relations. Each equality here represents a whole family of relations, obtained by completing each of the diagram-stubs shown to a full diagram in all possible ways, but in the same way within each equality.

2.1.2. Intermediate spaces. Our map $\mathring{A} : RPT \to \mathcal{A}(\emptyset)$ is a composition of several maps. Let us now define the intermediate spaces we pass through:

Definition 2.4. An "X-marked pure tangle diagram" is a graph D made of the following types of edges and vertices:

- Edges: n upward-pointing vertical directed lines marked by the elements of X (whose union is "the skeleton of D"), and some number of undirected edges, sometimes called "chords" or "internal edges".
- Vertices: the endpoints of the skeleton, vertices in which an internal edge ends on the skeleton, and oriented trivalent vertices in which three internal edges meet.

The graph D should be "connected modulo its skeleton". Namely, if the skeleton of D is collapsed to a single point, the resulting graph should be connected. An example appears in Figure 4. The "degree" of D is half the number of trivalent vertices it has. The graded completion of the space of all X-marked pure tangle diagram modulo the STU relations displayed in Figure 10 is denoted by $\mathcal{A}(\uparrow_X)$. If $X = \{x\}$ is a singleton, we set $\mathcal{A} = \mathcal{A}(\uparrow_x) = \mathcal{A}(\uparrow_X)$. We note that the STU relations implies the IHX and AS relations, see [B-N2].

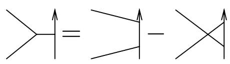


Figure 10. The STU family of relations with only diagram-stubs shown.

Definition 2.5. An "X-marked uni-trivalent diagram" is a graph C made of undirected edges and two types of vertices: oriented trivalent vertices ("internal vertices") and univalent vertices marked by elements of the label set X (the "legs" of C). The graph C should be connected modulo its univalent vertices. Namely, if the univalent vertices of C are all joined, the resulting graph should be connected. An example appears in Figure 5. The "degree" of D is half the total number of vertices it has. The graded completion of the space of all X-marked uni-trivalent diagrams modulo the IHX and AS relations of Figure 9 is denoted by $\mathcal{B}(X)$. The space $\mathcal{B}(X)$ is an algebra with the bilinear extension of the disjoint union operation $\cup$ as a product.

2.1.3. Maps. The pre-normalized Århus integral $\mathring{A}_0$ is the following composition:

$$\mathring{A}_0 : \left\{\begin{array}{c}\text{regular}\\ \text{pure}\\ \text{tangles}\end{array}\right\} = \tag{3}$$
$$= RPT \xrightarrow[\substack{\text{the [LMMO]}\\ \text{version of the}\\ \text{Kontsevich integral}}]{\check{Z}} \mathcal{A}(\uparrow_X) \xrightarrow[\substack{\text{formal}\\ \text{PBW}}]{\sigma} \mathcal{B}(X) \xrightarrow[\substack{\text{formal}\\ \text{Gaussian}\\ \text{integration}}]{\int^{FG}} \mathcal{A}(\emptyset).$$

We just have to recall the definitions of the maps $\check{Z}$ and σ, and define the map $\int^{FG}$.

Definition 2.6. The map $\check{Z}$ was defined by Le, H. Murakami, J. Murakami, and Ohtsuki in [LMMO]. It is the usual framed version of the Kontsevich integral Z (see [LM], or a simpler definition in [BGRT, Section 2.2][2]), normalized in a funny way. Namely, let $\nu = Z(\bigcirc) \in \mathcal{A}$ be the Kontsevich integral of the unknot[3], and let $\Delta_X : \mathcal{A} \to \mathcal{A}(\uparrow_X)$ be the "X-cabling" map that replaces the single directed line in $\mathcal{A}$ by n directed lines labeled by the elements of X, and sums over all possible ways of lifting each vertex on the directed line to its n clones (see e.g. [B-N4], [LM]). Set

$$\check{Z}(T) = \nu^{\otimes n} \cdot \Delta_X(\nu) \cdot Z(L) \tag{4}$$

for any X-marked framed pure tangle L, using the action of $\mathcal{A}^{\otimes n}$ on $\mathcal{A}(\uparrow_X)$ defined by sticking any n diagrams in $\mathcal{A}$ on the n components of the skeleton of a diagram in $\mathcal{A}(\uparrow_X)$. In (4) the factor of $\Delta_X(\nu)$ is easily explained; it accounts for the difference between the Kontsevich integral of pure tangles and of their closures. The other factor, $\nu^{\otimes n}$, is the surprising discovery of [LMMO]: it makes $\check{Z}$ better behaved relative to the second Kirby move.

Definition 2.7. The map σ, first defined in [B-N3], is a simple generalization of the formal PBW map $\sigma : \mathcal{A} \to \mathcal{B}$ of [B-N2]. It is more easily described through its inverse χ. If $C \in \mathcal{B}(X)$ is an X-marked uni-trivalent diagram with k_x legs

[2] Strictly speaking, [BGRT] deals only with knots. But the generalization to links is obvious.
[3] See [BGRT] for the conjectured value of this invariant, and [BLT] for the proof.

marked x for any $x \in X$, then $\chi(C) \in \mathcal{A}(\uparrow_X)$ is the average[4] of the $\prod_x k_x!$ ways of attaching the legs of C to n labeled vertical arrows (labeled by the elements of X), attaching legs marked by x only to the x-labeled arrow, for all $x \in X$.

The new map, $\int^{FG} : \mathcal{B}(X) \to \mathcal{A}(\emptyset)$ is only partially defined. Its domain is the set $\mathcal{B}^{FG}(X)$ of "non-degenerate perturbed Gaussians":

Definition 2.8. An element $G \in \mathcal{B}(X)$ is a "non-degenerate perturbed Gaussian" if it is of the form

$$G = PG \cup \exp_{\cup}\left(\frac{1}{2}\sum_{x,y} l_{xy}\, {}^{x}\frown^{y}\right)$$

for some invertible matrix (l_{xy}), where $P : \mathcal{B}(X) \to \mathcal{B}^{+}(X) \subset \mathcal{B}(X)$ is the natural projection onto the space $\mathcal{B}^{+}(X)$ of uni-trivalent diagrams that have at least one trivalent vertex on each connected component, and ${}^{x}\frown^{y}$ denotes the only connected uni-trivalent diagrams that have no trivalent vertices. It is clear that the matrix (l_{xy}) is determined by G if G is of that form. Call (l_{xy}) the "covariance matrix" of G.

The last name is explained by claim 1.10. That claim also says that the following definition applies to $\sigma\check{Z}(T) \in \mathcal{B}^{FG}(X)$ for any regular pure tangle L:

Definition 2.9. Let G be a non-degenerate perturbed Gaussian with covariance matrix (l_{xy}), and let (l^{xy}) be the inverse covariance matrix. Set

$$\int^{FG} G = \left\langle \exp_{\cup}\left(-\frac{1}{2}\sum_{x,y} l^{xy}\, {}_{\partial_x}\smile_{\partial_y}\right), PG \right\rangle.$$

Here the pairing $\langle \cdot,\cdot \rangle : \mathcal{B}(\partial_X) \otimes \mathcal{B}^{+}(X) \to \mathcal{A}(\emptyset)$ (where $\partial_X = \{\partial_x : x \in X\}$ is a set of "dual" variables) is defined by

$$\langle C_1, C_2 \rangle = \begin{pmatrix} \text{sum of all ways of gluing the } \partial_x\text{-marked legs of } C_1 \text{ to} \\ \text{the } x\text{-marked legs of } C_2, \text{ for all } x \in X \end{pmatrix}.$$

This sum is of course 0 if the numbers of x-marked legs do not match. If the numbers of legs do match and each diagram has k_x legs marked by x for $x \in X$, the sum is a sum of $\prod_x k_x!$ terms.

2.1.4. Normalization. In part II of this series we will show how the philosophy of Section 1.1 can be made rigorous, implying the following proposition:

[4] Notice that the normalization is different than in [B-N2], [B-N3] where by careless design, a sum was used instead of an average.

Proposition 2.10 (Proof in [Å-II, Section 3])**.** *regular pure tangle invariant* $\AA_0$ *descends to an invariant of regular links and as such it is invariant under the second Kirby move (Figure 1).*

If we want an invariant of rational homology spheres, we still have to fix $\AA_0$ to satisfy the first Kirby move (Figure 2). This is done in a standard way, similar to the way the Kauffman bracket is tweaked to satisfy the first Reidemeister move. The trick is to multiply the relatively complicated $\AA_0$ by a much simpler invariant of regular links, that has an opposite behavior under the first Kirby move and is otherwise uninteresting. The result is invariant under both Kirby moves, and by conservation of interest, it is as interesting as the original $\AA_0$:

Definition 2.11. Let U_+ be the unknot with framing $+1$, and let U_- be the unknot with framing -1. Let L be a regular link, and let σ_+ (σ_-) be the number of positive (negative) eigenvalues of the linking matrix of L (note that $\sigma_\pm$ are invariant under the second Kirby move, which acts on the linking matrix by a similarity transformation). Let the Århus integral $\AA(L)$ of L be

$$\AA(L) = \AA_0(U_+)^{-\sigma_+}\AA_0(U_-)^{-\sigma_-}\AA_0(L),$$

with all products and powers taken using the disjoint union product of $\mathcal{A}(\emptyset)$.

Theorem 1. (Proof in [Å-II, Section 3.3])**.** $\AA$ *is invariant under the first Kirby move as well, and hence it is an invariant of rational homology spheres.*

2.2. Main properties of the Århus integral

2.2.1. First property of $\AA$**: it has a conceptual foundation.** This property was already proven in Section 1. It is the main reason why the proofs of all other properties (and of Proposition 2.10) are relatively simple.

2.2.2. Second property of $\AA$**: it is universal.** Like there are finite-type (Vassiliev) invariants of knots (see e.g. [B-N2], [Bi], [Go1], [Go2], [Ko1], [Va1], [Va2] and [B-N6]), so there are finite-type (Ohtsuki) invariants of integer homology spheres (see e.g. [Oh3], [GO], [LMO], [Le1] and [B-N6]). These invariants have a rather simple definition, and just as in the case of knots, they seem to be rather powerful, though precisely how powerful they are we still do not know. We argue that the Århus integral $\AA$ plays in the theory of Ohtsuki invariants the same role as the Kontsevich integral plays in the theory of Vassiliev invariants. Namely, that it is a "universal Ohtsuki invariant". However, manifold invariants are somewhat more subtle than knot invariants, and the proper definition of universality is less transparent (see [Oh3], [GO], [Le1]):

Definition 2.12. An invariant U of integer homology spheres with values in $\mathcal{A}(\emptyset)$ is a "universal Ohtsuki invariant" if

(1) The degree m part $U^{(m)}$ of U is of Ohtsuki type $3m$ ([Oh3]).

(2) If OGL denotes the Ohtsuki-Garoufalidis-Le map[5], defined in Figure 11, from manifold diagrams to formal linear combinations of unit framed algebraically split links in S^3, and S denotes the surgery map from such links to integer homology spheres, then

$$(U \circ S \circ OGL)(D) = D + (\text{higher degree diagrams}) \qquad (\text{in } \mathcal{A}(\emptyset))$$

whenever D is a manifold diagram (we implicitly linearly extend S and U, to make this a meaningful equation).

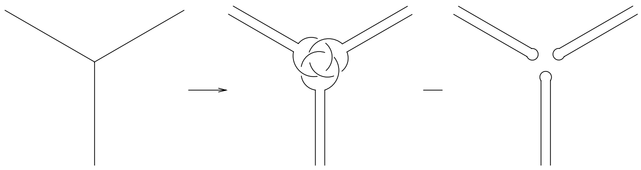


Figure 11. The OGL map: Take a manifold diagram D, embed it in S^3 in some fixed way of your preference, double every edge, replace every vertex by a difference of two local pictures as shown here, and put a $+1$ framing on each link component you get. The result is a certain alternating sum of 2^v links with e components each, where v and e are the numbers of vertices and edges of D, respectively.

Theorem 2 (Proof in [Å-II, Section 4.2]). *Restricted to integer homology spheres, Å is a universal Ohtsuki invariant.*

Corollary 2.13.

(1) *Å is onto all degree-homogeneous subspaces of $\mathcal{A}(\emptyset)$.*
(2) *All Ohtsuki invariants factor through the map Å.*
(3) *The dual of $\mathcal{A}(\emptyset)$ is the associated graded of the space of Ohtsuki invariants (with degrees divided by* 3*; recall from* [GO] *that the associated graded of the space of Ohtsuki invariants vanishes in degrees not divisible by* 3*.).*

In view of Le [Le1], this theorem and corollary follow from the fact (discussed below) that the Århus integral computes the LMO invariant, and from the universality of the LMO invariant ([Le1]). But as the definitions of the two invariants are different, it is nice to have independent proofs of the main properties.

[5] Nomenclatorial justification: Ohtsuki [Oh3] implicitly considered a map similar to OGL, with alternating summation over $2^{\{\text{edges}\}}$ rather than over $2^{\{\text{vertices}\}}$. Later, Garoufalidis and Ohtsuki [GO] introduced "white vertices", which amount to an alternating summation over $2^{\{\text{edges}\}} \times 2^{\{\text{vertices}\}}$. Finally, Le [Le1, Lemma 5.1] noticed that in this context the alternating summation over $2^{\{\text{edges}\}}$ is superfluous, leaving us with the definition presented here.

From Corollary 2.13 and the computation of the low degree parts of $\mathcal{A}(\emptyset)$ in [B-N5] and [Kn] it follows that the low-degree dimensions of the associated graded of the space of Ohtsuki invariants are given by the table below. The last row of this table lists the dimensions of "primitives" — multiplicative generators of the algebra of Ohtsuki invariants.

degree ($3m$)	0	3	6	9	12	15	18	21	24	27	30	33
dimension	1	1	2	3	6	9	16	25	42	50	90	146
dimension of primitives	0	1	1	1	2	2	3	4	5	6	8	9

2.2.3. Third property of $\AA$**.** It computes the Le-Murakami-Ohtsuki (LMO) invariant.

Theorem 3 (Proof in [Å-III])**.** *Å and the invariant* $\widehat{LMO}$ *defined in* [LMO] *are equal.*[6]

2.2.4. Fourth property of $\AA$**.** It recovers the Rozansky and Ohtsuki invariants. In [Ro1], [Ro3], Rozansky shows how to construct a "perturbative" invariant of rational homology spheres, valued in the space of power series in some formal parameter, which we call $\hbar$. His construction is associated with the Lie algebra $sl(2)$, but it can be easily extended (see [Å-IV]) for other semi-simple Lie algebras $\mathfrak{g}$. We call the resulting Rozansky invariant $R_{\mathfrak{g}}$.

Theorem 4 (Proof in [Å-IV])**.** *For any semi-simple Lie algebra* $\mathfrak{g}$*,*

$$R_{\mathfrak{g}} = \mathcal{T}_{\mathfrak{g}} \circ \hbar^{\deg} \circ \AA$$

where $\mathcal{T}_{\mathfrak{g}} : \mathcal{A}(\emptyset) \to \mathbb{C}$ *is the operation of replacing the vertices and edges of a trivalent graph by the structure constants of* $\mathfrak{g}$ *and the metric of* $\mathfrak{g}$*, as in Section 1 and as in* [B-N2]*, and* $\hbar^{\deg}$ *is the operator that multiplies each diagram* D *in* $\mathcal{A}(\emptyset)$ *by* $\hbar$ *raised to the degree of* D*.*

Corollary 2.14. *LMO recovers* $R_{\mathfrak{g}}$ *for any* $\mathfrak{g}$*. In particular, by* [Ro2]*, LMO recovers the "p-adic" invariants of* [Oh1], [Oh2].

The last statement was proved in the case of $\mathfrak{g} = sl(2)$ by Ohtsuki [Oh4].

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[6] We refer to the invariant $\hat{\Omega}$ of [[]Section 6.2]LMO. Le, Murakami, and Ohtsuki also consider in [LMO] an invariant Ω, which we usually denote by LMO. The invariant LMO is defined for general 3-manifolds, and on rational homology spheres it differs from $\widehat{\text{LMO}}$ only by a normalization.

3. Frequently asked questions

Let us answer some frequently asked questions. More detailed answers to most of these questions will be given in the the later parts of this series.

Question 3.1. Your construction uses the Chern-Simons path integral (at least ideologically). In what way is your construction different than the original construction of 3-manifold invariants by Witten [Wi]?

Answer. Our path integral is over connections on S^3, rather than over connections on an arbitrary 3-manifold. This means that we can replace the path integral by any well-behaved universal Vassiliev invariant, and get a rigorous result.

Question 3.2. What is the relation between your construction and Witten's?

Answer. One answer is that the Århus integral should somehow be related to the $k \to \infty$ asymptotics of the Witten invariants. A more precise statement is Theorem 4; recall that Rozansky conjectures (and demonstrates in some cases) that his invariant is related to the trivial connection contribution to the $k \to \infty$ asymptotics of the Witten invariants (in their Reshetikhin-Turaev guise, see [Ro1], [Ro3]). But perhaps a more fair answer is *we do not know.* There ought to be a direct path-integral way to see the relation between integration over all connections on some 3-manifold, and integration over connections on S^3 followed by "integration of the holonomies" in the sense of Section 1.1. But we do not know this way.

Question 3.3. Which one is more general?

Answer. Neither one. Assuming all relevant conjectures, $\AA$ only sees the $k \to \infty$ limit, and only "in the vicinity of the trivial connection". But this means it sees a splitting (trivial vs. other flat connections) that the Witten invariants do not see. Also, by Vogel [Vo], we know that $\mathcal{A}(\emptyset)$ "sees" more than all semi-simple super Lie algebras see, while the $k \to \infty$ limit of the Witten invariants is practically limited to semi-simple super Lie algebras. On the other hand, there are Witten-like theories with finite gauge groups, see e.g. [FQ], which have no parallel in the $\AA$ world.

Question 3.4. What is the relation between the Århus integral and the Axelrod-Singer perturbative 3-manifold invariants [AS1], [AS2] and/or the Kontsevich "configuration space integrals" [Ko2]?

Answer. We expect the Århus integral to be the same as Kontsevich's configuration space integrals and as the formal (no-Lie-algebra) version of the Axelrod-Singer invariants, perhaps modulo some minor corrections (in both cases).

Question 3.5. What is the relation between the Århus integral and the LMO invariant of Le, Murakami, and Ohtsuki [LMO]?

Answer. They are the same up to a normalization whenever $\AA$ is defined. See Theorem 3.

Question 3.6. What did you add?

Answer. A conceptual construction, and (thus) a better understanding of the relation between the LMO invariant/Århus integral and the Rozansky and Ohtsuki invariants. See Section 2.2.4.

Question 3.7. Does it justify a new name?

Answer. It is rather common in mathematics that different names are used to describe the same thing, or almost the same thing, depending on the context or the specific construction. See for example the Kauffman bracket and the Jones polynomial, the Reidemeister and the Ray-Singer torsions, and the Čech-de-Rham-singular-simplicial cohomology. Perhaps the name "Århus integral" should only be used when the construction is explicitly relevant, with the names "Axelrod-Singer invariants", "Kontsevich's configuration space integrals", and "LMO invariant" marking the other constructions. When only the functionality (i.e., Theorem 2) matters, the name "LMO invariant" seems most appropriate, as Theorem 2 was first considered and proved in the LMO context.

Question 3.8. Did not Reshetikhin once consider a construction similar to yours?

Answer. Yes he did, but he never completed his work. Our work was done independently of his, though in some twisted way it was initiated by his. Indeed, it was Reshetikhin's ideas that led one of us (Rozansky) to study what he called "the Reshetikhin formula", and that led him to discover his "trivial connection contribution to the Reshetikhin-Turaev invariants" (the Rozansky invariants, see Section 2.2.4). The Århus integral was discovered by "reverse engineering" starting from the Rozansky invariants — we first found a diagram-valued invariant that satisfies Theorem 4 (working with the ideas we discussed in [BGRT]), and only then we realized that our invariant has the simple interpretation discussed in Section 1. The result, the Århus integral, still carries some affinity to Reshetikhin's construction.

Question 3.9. What is the relation between the Århus integral and the p-adic 3-manifold invariants considered by Ohtsuki [Oh1]?

Answer. See Corollary 2.14.

Question 3.10. How powerful is $\AA$?

Answer. It is a "universal Ohtsuki invariant" (see Section 2.2.2). In particular, as the Casson invariant is Ohtsuki-finite-type (see [Oh3]), $\AA$ is stronger than the Casson invariant. In terms of the table following Corollary 2.13, the Casson invariant is just the degree 3 primitive, and there are many more. But how powerful $\AA$ *really* is, how useful it can really be, we do not know. Can anybody answer that question for the Jones polynomial?

Question 3.11. Can you say anything about 3-manifolds with embedded links?

Answer. Everything works in that case too. See [Å-II, Section 5.1].

4. Appendix: Gaussian integration: a quick refresher

Whether or not you've seen perturbed Gaussian integration before, you surely do not want to waste much energy on calculus tricks. Hence we include here a refresher that you can swallow whole without wasting any time (and insert all the right factors of $\pm i$, $2^{\pm 1}$, $\pi^{\pm 1/2}$, etc. if you do have some time to spare).

Let V be a vector space and dv a Lebesgue measure on V. A perturbed Gaussian integral is an integral of the form $I_T = \int_V e^T dv$, where T is a polynomial (or a power series) on V, of some specific form — T must be a sum $T = \frac{1}{2}Q + P$, where Q is a non-degenerate 'big' quadratic, and P is some (possibly) higher degree 'perturbation', which in some sense should be 'small' relative to Q. If the perturbation P is missing, then I_T is a simple un-perturbed Gaussian integral.

The first step is to expand the e^P part to a power series,

$$I_T = \int_V e^T dv = \sum_{m=0}^{\infty} \frac{1}{m!} \int_V P(v)^m e^{Q(v)/2} dv.$$

Then, we use some Fourier analysis. Recall that the Fourier transform takes integration to evaluation at 0, takes multiplication by a polynomial to multiple differentiation, and takes a Gaussian to another Gaussian, with the negative-inverse quadratic form. All and all, we find that

$$I_T \sim \sum_{m=0}^{\infty} \frac{1}{m!}\, \partial_{P^m} \exp\left(-\frac{1}{2} Q^{-1}(v^\star)\right)\bigg|_{v^\star=0}. \tag{5}$$

The notation here means:

- $\sim$ means equality modulo 2's, π's, i's, and their likes.
- $Q^{-1}(v^\star)$ is the inverse of $Q(v)$. It is a quadratic form on $V^\star$, and it is evaluated on some $v^\star \in V^\star$.
- ∂_{P^m} is P^m regarded as a differential operator acting on functions on $V^\star$. To see how this works, recall that polynomials on V are elements of the symmetric algebra $S^\star(V^\star)$ of $V^\star$, and they act on $S^\star(V)$, namely on polynomials (and hence functions) on $V^\star$, via the standard 'contract as much as you can' action $S^k(V^\star) \otimes S^l(V) \to S^{l-k}(V)$.

Here are two ways to look at the result, equation (5):

(1) We can re-pack the sum as an exponential and get

$$I_T \sim \left\langle e^P, e^{-Q^{-1}/2} \right\rangle, \tag{6}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual pairing $S^\star(V^\star) \otimes S^\star(V) \to \mathbb{C}$.

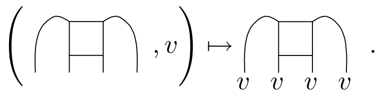
(2) We can expand $e^{-Q^{-1}/2}$ as a power series and get

$$I_T \sim \left\langle \sum_{m=0}^{\infty} \frac{P^m}{m!}, \sum_{n=0}^{\infty} \frac{(-Q^{-1}/2)^n}{n!} \right\rangle. \tag{7}$$

This last equation has a clear combinatorial interpretation: Take an arbitrary number of unordered copies of P and an arbitrary number of unordered copies of Q^{-1}. If the total degrees happen to be the same, you can contract them and get a number. Sum all the numbers you thus get, and you've finished computing I_T.

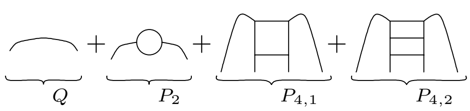
For the sake of concreteness, let us play with the example $T = \frac{1}{2}Q + P_2 + P_{4,1} + P_{4,2}$, where Q is the big quadratic, P_2 is another quadratic which is regarded as a perturbation, and $P_{4,1}$ and $P_{4,2}$ are two additional quartic perturbation terms. It is natural to represent each of these terms by a picture of an animal (usually connected) with as many legs as its degree. This is because $P_{4,1}$ (say) is in $S^4(V^\star)$. That is, it is a 4-legged animal which has to be fed with 4 copies of some vector v to produce the number $P_{4,1}(v)$:

$(P_{4,1}, v) \mapsto P_{4,1}(v)$ is

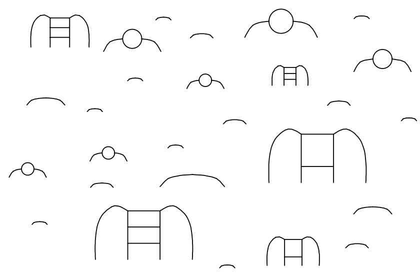


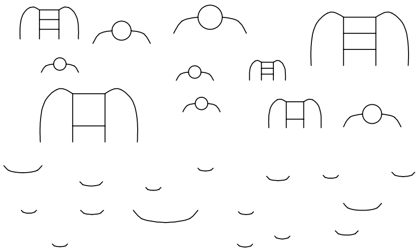
With this in mind, T is represented by a sum of such connected animals:

$T = \frac{1}{2}$



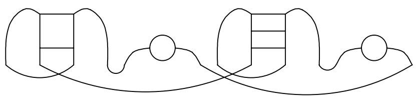
Exponentiation is done using power series. Each term in $\exp T$ is some power of T and hence an element of some symmetric power $S^k(V^\star)$. As such, it is represented by some sum of pictures, each of which is some disjoint union of the connected animals making T. Namely, $\exp T$ is some sum of 'clouds' right.





The next step, as seen from equation (7), is to separate Q from the rest, negate it, and invert it. If we think of legs pointing down as legs in $V^\star$ and legs pointing up as legs in V, the result is a sum of 'split coulds' like the one on the left.

The final step according to equation (7) is to contract the $(-Q^{-1})$'s with the P's, whenever the degrees allow that. In pictures, we just connect the down-pointing legs to the up-pointing legs in all possible ways, and the result is a big sum of diagrams that look like this:



Notice that in these diagrams (commonly referred to as "Feynman diagrams") there are no 'free legs' left. Therefore they represent complete contractions, that is, scalars.

Acknowledgement

The seeds leading to this work were planted when the four of us (as well as Le, Murakami (H&J), Ohtsuki, and many other like-minded people) were visiting Århus, Denmark, for a special semester on geometry and physics, in August 1995. We wish to thank the organizers, J. Dupont, H. Pedersen, A. Swann and especially J. Andersen for their hospitality and for the stimulating atmosphere they created. We also wish to thank N. Habegger, A. Haviv, M. Hutchings, T. Q. T. Le, J. Lieberum, A. Referee, N. Reshetikhin and S. Yitzchaik for additional remarks and suggestions, the Center for Discrete Mathematics and Theoretical Computer Science at the Hebrew University for financial support, and the Volkswagen-Stiftung (RiP-program in Oberwolfach) for their hospitality and financial support.

References

[AS1] S. Axelrod and I. M. Singer. Chern-Simons Perturbation Theory. Proc. XXth DGM Conference 1991 (S. Catto and A. Rocha, eds.). World Scientific, New York, 1992, 3–45.

[AS2] S. Axelrod and I. M. Singer. Chern-Simons Perturbation Theory II. *Jour. Diff. Geom.* **39** (1994), 173–213.

[AF] D. Altschuler and L. Freidel. Vassiliev knot invariants and Chern-Simons perturbation theory to all orders. *Comm. Math. Phys.* **187** (1997), 261–287, arXiv:q-alg/9603010.

[B-N1] D. Bar-Natan. *Perturbative Aspects of the Chern-Simons Topological Quantum Field Theory.* Ph.D. thesis, Princeton Univ. Dep. of Mathematics, June 1991.

[B-N2] D. Bar-Natan. On the Vassiliev knot invariants. *Topology* **34** (1995), 423–472.

[B-N3] D. Bar-Natan. Vassiliev homotopy string link invariants. *Jour. of Knot Theory and its Ramifications* **4** (1995), 13–32.

[B-N4] D. Bar-Natan. Non-associative tangles. in *Geometric topology* (proceedings of the Georgia international topology conference) (W. H. Kazez, ed.). Amer. Math. Soc. and International Press, Providence, 1997, 139–183.

[B-N5] D. Bar-Natan. Some computations related to Vassiliev invariants. electronic publication, `http://www.ma.huji.ac.il/~drorbn`.

[B-N6] D. Bar-Natan. Bibliography of Vassiliev Invariants. electronic publication, ibid.

[BGRT] D. Bar-Natan, S. Garoufalidis, L. Rozansky and D. P. Thurston. Wheels, wheeling, and the Kontsevich integral of the unknot. *Israel Journal of Mathematics* **119** (2000), 217–237, arXiv:q-alg/9703025.

[BLT] D. Bar-Natan, T. Q. T. Le and D. P. Thurston. Two applications of elementary knot theory to Lie algebras and Vassiliev invariants. Hebrew University, SUNY at Buffalo and Harvard University preprint, April 2002, arXiv:math.QA/0204311.

[Bi] J. S. Birman. New points of view in knot theory. *Bull. Amer. Math. Soc.* **28** (1993), 253–287.

[BL] J. S. Birman and X-S. Lin. Knot polynomials and Vassiliev's invariants Inv. Math. **111** (1993), 225–270.

[BT] R. Bott and C. Taubes. On the self-linking of knots. *Jour. Math. Phys.* **35** (1994).

[Di] J. Dixmier. *Enveloping Algebras.* North-Holland, Amsterdam, 1977.

[FQ] D. Freed and F. Quinn. Chern-Simons theory with finite gauge group. *Comm. Math. Phys.* **156** (1993), 435–472.

[GO] S. Garoufalidis and T. Ohtsuki. On finite type 3-manifold invariants III: manifold weight systems. *Topology* **37/2** (1998), arXiv:q-alg/9705004.

[Go1] M. Goussarov. A new form of the Conway-Jones polynomial of oriented links. in Topology of manifolds and varieties (O. Viro, ed.). Amer. Math. Soc., Providence, 1994, 167–172.

[Go2] M. Goussarov. On n-equivalence of knots and invariants of finite degree. in Topology of manifolds and varieties (O. Viro, ed.). Amer. Math. Soc., Providence, 1994, 173–192.

[Ha] N. Habegger. A computation of the universal quantum 3-manifold invariant for manifolds of rank greater than 2. Nantes University preprint, 1996.

[Ja] N. Jacobson. *Lie algebras.* Dover, New-York, 1979.

[Ki] R. Kirby. A calculus of framed links in S^3. *Invent. Math.* **45** (1978), 35–56.

[Kn] J. A. Kneissler. The number of primitive Vassiliev invariants up to degree twelve. University of Bonn preprint, June 1997, arXiv:q-alg/9706022.

[Ko1] M. Kontsevich. Vassiliev's knot invariants. *Adv. in Sov. Math.,* **16** (1993), 137–150.

[Ko2] M. Kontsevich. Feynman diagrams and low-dimensional topology. First European Congress of Mathematics II. Birkhäuser, Basel, 1994, 97–121.

[Le1] T. Q. T. Le. An invariant of integral homology 3-spheres which is universal for all finite type invariants. in Solitons, geometry and topology: on the crossroad (V. Buchstaber and S. Novikov, eds.). AMS Translations Series 2, Providence.

[LMMO] T. Q. T. Le, H. Murakami, J. Murakami and T. Ohtsuki. A three-manifold invariant via the Kontsevich integral. Max-Planck-Institut Bonn preprint, 1995.

[LM] T. Q. T. Le and J. Murakami. The universal Vassiliev-Kontsevich invariant for framed oriented links. *Compositio Math.* **102** (1996), 41–64, arXiv:hep-th/9401016.

[LMO] T. Q. T. Le and J. Murakami and T. Ohtsuki. On a universal perturbative invariant of 3-manifolds. *Topology* **37** (1998), arXiv:q-alg/9512002.

[Oh1] T. Ohtsuki. A polynomial invariant of integral homology 3-spheres. *Math. Proc. Camb. Phi. Soc.* **117** (1995), 83–112.

[Oh2] T. Ohtsuki. A polynomial invariant of rational homology 3-spheres. *Invent. Math.* **123** (1996), 241–257.

[Oh3] T. Ohtsuki. Finite type invariants of integral homology 3-spheres. *Jour. of Knot Theory and its Ramifications* **5** (1996), 101–115.

[Oh4] T. Ohtsuki. Combinatorial quantum method in 3-dimensional topology. Math. Soc. of Japan Memoirs **3**, Tokyo, 1999.

[Ro1] L. Rozansky. The trivial connection contribution to Witten's invariant and finite type invariants of rational homology spheres. *Comm. Math. Phys.* **183** (1997), 23–54, arXiv:q-alg/9503011.

[Ro2] L. Rozansky. Witten's invariants of rational homology spheres at prime values of K and the trivial connection contribution. *Comm. Math. Phys.* **180** (1996), 297–324, arXiv:q-alg/9504015.

[Ro3] L. Rozansky. On finite type invariants of links and rational homology spheres derived from the Jones polynomial and Witten-Reshetikhin-Turaev invariant. in Proc. of the Århus Conf. *Geometry and physics* Lecture notes in pure and applied mathematics (J. E. Andersen, J. Dupont, H. Pedersen and A. Swann, eds.), vol. 184. Marcel Dekker, New-York, 1997, 379–397, arXiv:q-alg/9511025.

[Th] D. Thurston. Integral expressions for the Vassiliev knot invariants. Harvard University senior thesis, April 1995, arXiv:math.QA/9901110.

[Va1] V. A. Vassiliev. *Cohomology of knot spaces, Theory of Singularities and its Applications (Providence)* (V. I. Arnold, ed.). Amer. Math. Soc. Providence, 1990.

[Va2] V. A. Vassiliev. Complements of discriminants of smooth maps: topology and applications. Trans. of Math. Mono., vol. 98. Amer. Math. Soc., Providence, 1992.

[Vo] P. Vogel. Algebraic structures on modules of diagrams. Université Paris VII preprint, July 1995.

[Wi] E. Witten. Quantum field theory and the Jones polynomial. *Comm. Math. Phys.* **121** (1989), 351–399.

[Å-II] D. Bar-Natan, S. Garoufalidis, L. Rozansky and D. P. Thurston. The Århus integral of rational homology 3-spheres II: Invariance and universality. *Selecta Math.*, arXiv:math/9801049 (to appear).

[Å-III] D. Bar-Natan, S. Garoufalidis, L. Rozansky and D. P. Thurston. The Århus integral of rational homology 3-spheres III: The relation with the Le-Murakami-Ohtsuki invariant. Hebrew University, Harvard University, University of Illinois at Chicago and University of California at Berkeley preprint, August 1998, arXiv:math/9808013.

[Å-IV] D. Bar-Natan, S. Garoufalidis, L. Rozansky and D. P. Thurston. The Århus integral of rational homology 3-spheres IV: The relation with the Rozansky and Ohtsuki invariants, in preparation.

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