What is a sequence of Nilsson type?

Stavros Garoufalidis

ABSTRACT. Sequences of Nilsson type appear in abundance in Algebraic Geometry, Enumerative Combinatorics, Mathematical Physics and Quantum Topology. We give an elementary introduction on this subject, including the definition of sequences of Nilsson type and the uniqueness, existence, and effective computation of their asymptotic expansion.

1. Sequences of Nilsson type: definition

Sequences of Nilsson type are the ones that are asymptotic to power series in powers of $1/n$ and $\log n$. They appear in abundance Analysis (in asymptotic expansions of integrals), in Mathematical Physics and in Algebraic Geometry (in relation to the Gauss-Manin connection); see for example [**An2, M1, M2, Ph, Sa**]. They also appear in Enumerative Combinatorics (see [**FS, WZ2, Ga2**]) and in Quantum Topology. For instance, the Witten-Reshetikhin-Turaev invariant of a closed 3-manifold is a sequence of complex numbers that depends on the level, and it is expected to be of Nilsson type; see [**Wi, FG, Ga3, Ro, LR, AH**]. In addition, the Kashaev invariant of a knot is expected to be a sequence of Nilsson type; see [**KT, AH, CG**]. The quantum spin network evaluation at a fixed root of unity is known to be a sequence of Nilsson type; see [**GV1, GV2**]. For a general discussion of perturbative and non-perturbative invariants of knotted objects that are expected to be sequences of Nilsson type, see [**Ga1**].

There is a close connection between sequences of Nilsson type and multivalued analytic functions with quasi-unipotent monodromy; see for example Theorem 4.1 below.

Several people familiar with the ideas of Quantum Topology have asked for a self-contained definition of sequences of Nilsson type and their asymptotics, its uniqueness, existence and effective computation.

Asymptotics of sequences is a well-studied subject of analysis that goes back at least to Poincare; see for example [**O, C, M1**]. Since we could not find a reference for sequences of Nilsson type in the existing literature, we decided to write this

introductory article. It concerns the asymptotic expansion of sequences which are relevant in Quantum Topology, and may serve as an elementary introduction to asymptotics. We claim no original results in this survey paper.

In order to define sequences of Nilsson type, we need to introduce Nilsson monomials $h_\omega(n)$ indexed by a well-ordered set Ω, and a finite set Λ of complex numbers of equal magnitude.

For a natural number $d \in \mathbb{N}$, a finite subset S of the rational numbers consider the well-ordered set $\Omega = (S + \mathbb{N}) \times \{0, 1, \ldots, d\}$ indexed by $(\alpha, \beta) < (\alpha', \beta')$ if and only if $\alpha < \alpha'$ or $\alpha = \alpha'$ and $\beta' < \beta$. Ω has the order type of the natural numbers. In particular, for every $\omega \in \Omega$, the set of elements strictly smaller than ω is finite. Consider the Ω-indexed family of monomials of *Nilsson type* given by:

$$(1) \qquad h_\omega(n) = \frac{(\log n)^\beta}{n^\alpha}$$

for $\omega = (\alpha, \beta) \in \Omega$. It is easy to see that $\lim_n h_{\omega'}(n)/h_\omega(n) = 0$ (abbreviated by $h_{\omega'}(n)/h_\omega(n) = o(1)$, and also by $h_\omega(n) \gg h_{\omega'}(n)$) if and only if $\omega < \omega'$. This and all limits below are taken when n goes to infinity.

Fix a finite set Λ of nonzero complex numbers of magnitude $r > 0$. Let $c_{\omega,\lambda}$ be a collection of complex numbers indexed by $\Omega \times \Lambda$.

DEFINITION 1.1. (a) With the above notation, for a complex-valued sequence (a_n) the expression

$$(2) \qquad a_n \sim \sum_{\omega \in \Omega} h_\omega(n) \sum_{\lambda \in \Lambda} c_{\omega,\lambda} \lambda^n$$

means that

- for every $\omega \in \Omega$ we have:

$$(3) \qquad \left(a_n r^{-n} - \sum_{\omega' \leq \omega} h_{\omega'}(n) \sum_{\lambda \in \Lambda} c_{\omega',\lambda} (\lambda r^{-1})^n \right) \frac{1}{h_\omega(n)} = o(1)$$

- $c_{\omega,\lambda} \neq 0$ for some $(\omega, \lambda) \in \Omega \times \Lambda$.

(b) We say that a sequence (a_n) is of Nilsson type if there exist Ω, Λ and $c_{\omega,\lambda}$ such that (2) holds.

We will say that an asymptotic expansion (2) is $\Omega \times \Lambda$-*minimal* if

- For every $\lambda \in \Lambda$ there exists $\omega \in \Omega$ such that $c_{\omega,\lambda} \neq 0$.
- For every $\omega \in \Omega$ there exists $\lambda \in \Lambda$ such that $c_{\omega,\lambda} \neq 0$.

By considering a subset of Λ or Ω if necessary, it is easy to see that every asymptotic expansion has a minimal representative.

2. Uniqueness

Our first task is to show that a sequence of Nilsson type uniquely determines Ω, Λ and the coefficients $c_{\omega,\lambda}$. The key idea is the following elementary lemma.

LEMMA 2.1. *If Λ is a finite subset of the unit circle and*

$$(4) \qquad \sum_{\lambda \in \Lambda} c_\lambda \lambda^n = o(1)$$

holds for some complex numbers c_λ, then $c_\lambda = 0$ for all $\lambda \in \Lambda$.

PROOF. Divide (4) by λ_1^n for some $\lambda_1 \in \Lambda$. Then we have

$$c_{\lambda_1} + \sum_{\lambda \neq \lambda_1} c_\lambda (\lambda/\lambda_1)^n = o(1)$$

where $\lambda/\lambda_1 \neq 1$. So,

$$\frac{1}{n} \sum_{k=1}^{n} \left(c_{\lambda_1} + \sum_{\lambda \neq \lambda_1} c_\lambda (\lambda/\lambda_1)^k \right) = o(1).$$

By averaging, it follows that

$$c_{\lambda_1} + \frac{1}{n} \sum_{\lambda \neq \lambda_1} c_\lambda \frac{1 - (\lambda/\lambda_1)^{n+1}}{1 - \lambda/\lambda_1} = o(1).$$

Thus, $c_{\lambda_1} = 0$. Since λ_1 was an arbitrary element of Λ, the result follows. $\square$

LEMMA 2.2. *If (a_n) satisfies* (2) *then*

(5)
$$\limsup_n |a_n|^{1/n} = r.$$

PROOF. Since $c_{\omega,\lambda} \neq 0$ for some $(\omega, \lambda) \in \Omega \times \Lambda$, without loss of generality assume that $c_{\omega_0,\lambda} \neq 0$ for some $\lambda \in \Lambda$ where ω_0 is the smallest element of Ω. Equation (2) for $\omega = \omega_0$ gives that

(6)
$$\frac{a_n r^{-n}}{h_{\omega_0}(n)} - \sum_{\lambda \in \Lambda} c_{\omega_0,\lambda} (\lambda r^{-1})^n = o(1)$$

Now λr^{-1} are on the unit circle. It follows that

$$\left| \frac{a_n r^{-n}}{h_{\omega_0}(n)} \right| < C$$

for some $C > 0$. Since $\lim_n h_\omega(n)^{1/n} = 1$ for all $\omega \in \Omega$, it follows that

$$\limsup_n |a_n|^{1/n} \leq r$$

Since some $c_{\omega_0,\lambda}$ is nonzero and λr^{-1} are on the unit circle, Lemma 2.1 implies that $\lim_n \sum_{\lambda \in \Lambda} c_{\omega_0,\lambda} (\lambda r^{-1})^n \neq 0$. Since the sequence is bounded, it follows that there exists a subsequence n_k such that

$$\lim_{n_k} \sum_{\lambda \in \Lambda} c_{\omega_0,\lambda} (\lambda r^{-1})^{n_k} = C' \neq 0$$

Combined with Equation (6), it follows that

$$\lim_{n_k} |a_{n_k}|^{1/n_k} = r$$

The result follows. $\square$

Lemma 2.2 implies that sequences of Nilsson type satisfy $\limsup_n |a_n|^{1/n} > 0$.

PROPOSITION 2.1. *Suppose that*

(7)
$$a_n \sim \sum_{\omega \in \Omega} h_\omega(n) \sum_{\lambda \in \Lambda} c_{\omega,\lambda} \lambda^n$$

and

(8)
$$a_n \sim \sum_{\omega' \in \Omega} h_{\omega'}(n) \sum_{\lambda' \in \Lambda} c'_{\omega',\lambda'} \lambda'^n$$

are $\Omega \times \Lambda$-*minimal and* $\Omega' \times \Lambda'$-*minimal asymptotic expansions. Then* $\Omega = \Omega'$, $\Lambda = \Lambda'$. *Moreover, for all* $(\omega, \lambda) \in \Omega \times \Lambda$ *we have* $c_{\omega,\lambda} = c'_{\omega,\lambda}$.

PROOF. Let ω_0 and ω'_0 denote the smallest elements of Ω and Ω'. Lemma 2.2 implies that $r = r'$ where r and r' are the magnitudes of the elements of Λ and Λ' respectively. Equation (3) for ω_0 and ω'_0 implies that

$$(9) \quad \frac{a_n r^{-n}}{h_{\omega_0}(n)} - \sum_{\lambda \in \Lambda} c_{\omega_0,\lambda}(\lambda r^{-1})^n = o(1), \qquad \frac{a_n r^{-n}}{h_{\omega'_0}(n)} - \sum_{\lambda' \in \Lambda'} c'_{\omega'_0,\lambda'}(\lambda' r^{-1})^n = o(1)$$

If $\omega_0 \neq \omega'_0$, we may assume that $\omega_0 < \omega'_0$. In that case, observe that $h_{\omega'_0}(n)/h_{\omega_0}(n) = o(1)$. Multiply the second equation above by $h_{\omega'_0}(n)/h_{\omega_0}(n)$ and subtract from the first. It follows that

$$-\sum_{\lambda \in \Lambda} c_{\omega_0,\lambda}(\lambda r^{-1})^n + \frac{h_{\omega'_0}(n)}{h_{\omega_0}(n)} \sum_{\lambda' \in \Lambda'} c'_{\omega'_0,\lambda'}(\lambda' r^{-1})^n = o(1)$$

Since $h_{\omega'_0}(n)/h_{\omega_0}(n) = o(1)$, it follows that

$$\sum_{\lambda \in \Lambda} c_{\omega_0,\lambda}(\lambda r^{-1})^n = o(1)$$

Lemma 2.1 implies that $c_{\omega_0,\lambda} = 0$ for all λ contrary to our minimality assumption of (7). It follows that $\omega_0 = \omega'_0$. Subtracting, Equation (9) implies that

$$-\sum_{\lambda \in \Lambda} c_{\omega_0,\lambda}(\lambda r^{-1})^n + \sum_{\lambda' \in \Lambda'} c'_{\omega'_0,\lambda'}(\lambda' r^{-1})^n = o(1)$$

Lemma 2.1 implies that if $c_{\omega_0,\lambda} \neq 0$ for some $\lambda \in \Lambda$, then $\lambda \in \Lambda'$ and moreover $c_{\omega_0,\lambda} = c'_{\omega_0,\lambda'}$.

An easy induction on $\omega \in \Omega$ proves the following statement. For every $\omega \in \Omega$, the following holds. If $c_{\omega,\lambda} \neq 0$ for some $\lambda \in \Lambda$, then $\lambda \in \Lambda'$ and $\omega \in \Omega'$ and $c_{\omega,\lambda} = c'_{\omega,\lambda}$.

The minimality assumption and the above statement implies that $\Omega = \Omega'$ and $\Lambda = \Lambda'$ and $c_{\lambda,\omega} = c'_{\lambda,\omega}$ for all $(\omega, \lambda) \in \Omega \times \Lambda$. $\qquad \square$

REMARK 2.2. *Proposition 2.1 proves uniqueness in a non-effective way. We will come back to the problem of computing* $c_{\omega,\lambda}$ *later on.*

3. Alternative expression for sequences of Nilsson type

If (a_n) is a sequence of Nilsson type, we can write (2) in the form:

$$(10) \qquad a_n \sim \sum_{\lambda,\alpha,\beta} \lambda^n n^\alpha (\log n)^\beta S_{\lambda,\alpha,\beta} g_{\lambda,\alpha,\beta}(1/n)$$

where

- the summation in (10) is over a finite set,
- the *growth rates* λ are complex numbers numbers of equal magnitude,
- the *exponents* α are rational numbers and the *nilpotency exponents* β are natural numbers,
- the *Stokes constants* $S_{\lambda,\alpha,\beta}$ are complex numbers,
- $g_{\lambda,\alpha,\beta}(x) \in \mathbb{C}[[x]]$ are formal power series in x with complex coefficients and leading term 1.

REMARK 3.1. *In the definition of a sequence of Nilsson type, we may additionally require that*

- Λ *is a set of algebraic numbers,*
- *the formal power series $g_{\lambda,\alpha,\beta}(x)$ is Gevrey-1, i.e., that the coefficient of x^k in $g_{\lambda,\alpha,\beta}(x)$ is bounded by $C^k k!$ for all k, where C depends on $g_{\lambda,\alpha,\beta}$,*
- *the coefficients of the formal power series $g_{\lambda,\alpha,\beta}(x)$ lie in a fixed number field K,*

These additional requirements hold for the evaluations of classical spin networks, see [**GV1**], *as well as Sections 4 and 6.1 below.*

EXAMPLE 3.1. For example, if $d = 1$ and $S = \{1, 3/2\}$, then $\Omega = (1 + \mathbb{N}) \cup (3/2 + \mathbb{N})$ and we have:

$$\frac{\log n}{n} \gg \frac{1}{n} \gg \frac{\log n}{n^{3/2}} \gg \frac{1}{n^{3/2}} \gg \frac{\log n}{n^2} \gg \frac{1}{n^2} \gg \ldots$$

If $\Lambda = \{\kappa, \mu, \nu\}$, the asymptotic expansion (10) of a sequence of Nilsson type becomes:

$$
\begin{aligned}
a_n \quad \sim \quad & \frac{\log n}{n} \sum_{\lambda \in \Lambda} \lambda^n S_{\lambda,1} g_{\lambda,1}\left(\frac{1}{n}\right) + \frac{\log n}{n^{3/2}} \sum_{\lambda \in \Lambda} \lambda_j^n S_{\lambda,2} g_{\lambda,2}\left(\frac{1}{n}\right) \\
& + \frac{1}{n} \sum_{\lambda \in \Lambda} \lambda_j^n S_{\lambda,3} g_{\lambda,3}\left(\frac{1}{n}\right) + \frac{1}{n^{3/2}} \sum_{\lambda \in \Lambda} \lambda_j^n S_{\lambda,4} g_{\lambda,4}\left(\frac{1}{n}\right)
\end{aligned}
$$

where $g_{\lambda,j}(x) \in \mathbb{C}[[x]]$ are formal power series in x and $S_{\lambda,j}$ are complex numbers.

4. Existence

In this section we will prove that a sequence is of Nilsson type, under some analytic continuation assumptions of its generating series. This is a well-known argument (see for example, [**C, CG, FS, GM, GIKM, M1**]) that consists of the following parts:
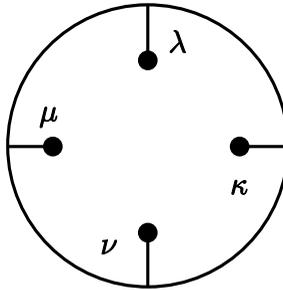
- apply Cauchy's theorem to give an integral representation of the sequence,
- deform the contour of integration to localize the computation near the singularities of the generating function,
- analyse the local computation using the local monodromy assumption of the generating function.

Let us give the details of the above existence proof. Since sequences of Nilsson type are exponentially bounded (as follows from Lemma 2.2), fix an exponentially bounded sequence (a_n) and consider its generating series

$$(11) \qquad\qquad G(z) = \sum_{n=0}^{\infty} a_n z^n$$

$G(z)$ is an analytic function for all complex numbers z that satisfy $|z| < 1/R$. Suppose now that G has analytic continuation on a disk of radius r with singularities at finitely many points $\kappa, \lambda, \mu, \nu, \ldots$. Suppose also that G has further analytic continuation on a disk of radius $r + \epsilon$ minus finitely many segments emanating from

the singularities radially as in the following figure.

Assume in addition that G has *quasi-unipotent local monodromy* at each singularity $\lambda, \mu, \nu, \kappa$ on the circle of radius r (i.e., the eigenvalues of the local monodromy are complex roots of unity).

THEOREM 4.1. *Under the above assumptions, the sequence (a_n) is of Nilsson type.*

COROLLARY 4.1. *Suppose that $G(z) = \sum_{n=0}^{\infty} a_n z^n$ is a multivalued analytic function on $\mathbb{C} \setminus \Lambda$ (where $\Lambda \subset \mathbb{C}$ is a finite set) which is regular at $z = 0$, and has quasi-unipotent local monodromy. Then, (a_n) is a sequence of Nilsson type.*

REMARK 4.2. *We know of at least three ways to show that a germ $G(z)$ of an analytic function can be analytically continued to the complex plane, namely*
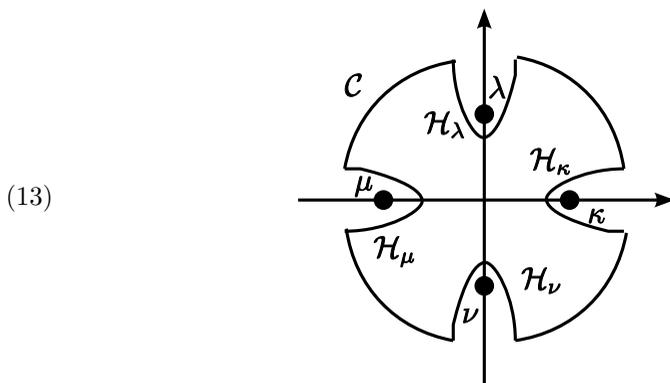
   (a) *G satisfies a linear differential equation, see for example [**Ga2**, Thm.1] reviewed in Section 5.1 below. For examples that come from Quantum Topology (specifically, spin networks) see [**GV1, GV2**].*
   (b) *G satisfies a nonlinear differential equation. See for example the instanton solutions of Painlevé I studied in detail in [**GIKM**] and the matrix models of [**GM**].*
   (c) *G is resurgent. See for example the Kontsevich-Zagier series studied in detail in [**CG**], and more generally the arithmetic resurgence conjecture of [**Ga1**] for sequences that appear in Quantum Topology.*

PROOF. (of Theorem 4.1) We begin by applying Cauchy's theorem to give an integral representation of (a_n). If γ is a circle of radius less than $1/R$ that contains the origin, then we have:

$$(12) \qquad\qquad a_n = \frac{1}{2\pi i} \int_\gamma \frac{G(z)}{z^{n+1}} dz$$

We can deform γ to a contour $\mathcal{C}$ which consists of a contour $\mathcal{H}_\lambda$ around each

singularity λ and finitely many arcs γ_i of the circle of radius $1/(r + \epsilon)$ as in the

following figure.

(13)

The contours $\mathcal{H}_\lambda$ are known as *Hankel contours* in Analysis (see [**O**]) and *Lefschetz thimbles* in Algebraic Geometry (see [**Ph, Sa**]). Cauchy's theorem implies that

$$(14) \qquad a_n = \frac{1}{2\pi i} \sum_{\lambda \in \Lambda} \int_{\mathcal{H}_\lambda} \frac{G(z)}{z^{n+1}} dz + \frac{1}{2\pi i} \int_{\gamma_{r+\epsilon}} \frac{G(z)}{z^{n+1}} dz$$

The above expression is exact, and decomposes the sequence (a_n) into a finite sum of sequences (one per singularity) and an extra term. Of course, there is nothing canonical about this decomposition, since the size of the Hankel contour and $\gamma_{r+\epsilon}$ depends on ϵ. One could make the decomposition nearly canonical by using Hankel contours that extend to infinity, but even so there are choices of directions to be made, and we will not use them here.

The integral over $\gamma_{r,\epsilon}$ can be estimated by $O((r+\epsilon)^{-n})$ since G is uniformly bounded on the arcs $\gamma_{r,\epsilon}$. Since we assume that the local monodromy of $G(z)$ around a singularity is *quasi-unipotent*, it follows (see [**M1**]) that modulo germs of holomorphic functions at zero, $G(\lambda + z)$ has a local expansion of the form

$$G(\lambda + z) = \sum_{\alpha',\beta'} z^{\alpha'} (\log z)^{\beta'} h_{\alpha',\beta'}(z)$$

where the summation is over a finite set, $\alpha' \in \mathbb{Q}$, $\beta' \in \mathbb{N}$ and $h_{\alpha',\beta'}(z)$ are germs of functions analytic at $z = 0$. For a germ $f(z)$ of a *multi-valued* analytic function at $z = 0$, let $\Delta_0 f$ denote its *variation* defined by

$$\Delta_0 f(z) = \lim_{\epsilon \to 1} f(e^{2\pi i \epsilon} z) - \lim_{\epsilon \to 0} f(e^{2\pi i \epsilon} z)$$

(see [**M1**]) when z is restricted on a line segment $[0, \epsilon)$. The variation of the building blocks z^α and $(\log z)^\beta$ are given by

$$\Delta_0(z^\alpha) = \begin{cases} (e^{2\pi i \alpha} - 1) z^\alpha & \alpha \in \mathbb{Q} \setminus \mathbb{Z} \\ \delta_0 & \alpha \in \mathbb{Z} \setminus \mathbb{N} \, , \\ 0 & \alpha \in \mathbb{N} \end{cases} \qquad \Delta_0(\log z) = 2\pi i$$

where δ_0 is the Dirac delta function (really, a distribution). For a singularity λ of $G(z)$, let $\Delta_\lambda G(z)$ denote the variation of $G(\lambda + z)$. It follows that for z in the line segment of Figure (13), we have

$$(15) \qquad \Delta_\lambda G(z) = \sum z^\alpha (\log z)^\beta \sum_{k=0}^{\infty} c'_{\alpha,\beta,\lambda,k} z^{k-1}$$

where the sum is over a finite set $\{\alpha, \beta\}$, $\alpha \in \mathbb{Q}\backslash\mathbb{N}$, $\beta \in \mathbb{N}$ and $\sum_{k=0}^{\infty} \sum_{k=0}^{\infty} c'_{\alpha,\beta,\lambda,k} z^k$ are germs of analytic functions at $z = 0$. When $\alpha \in \mathbb{Z}\backslash\mathbb{N}$, we can deform the Hankel contour into a small circle centered around λ and apply Cauchy's theorem. For the remaining cases $\alpha \in \mathbb{Q}\backslash\mathbb{Z}$, a change of variables $z \mapsto \lambda(1 + z)$ centers the Hankel contour at zero and implies that

$$(16) \qquad \int_{\mathcal{H}_\lambda} \frac{G(z)}{z^{n+1}} dz = \lambda^{-n} \int_{\mathcal{H}_0} \frac{G(\lambda(1+z))}{(1+z)^{n+1}} dz = \lambda^{-n} \int_0^\epsilon \frac{\Delta_\lambda G(\lambda z)}{(1+z)^{n+1}} dz$$

A *Beta-integral* calculation gives that

$$\int_0^\infty \frac{z^{\gamma-1}}{(1+z)^{n+1}} dz = \frac{\Gamma(\gamma)\Gamma(n+1-\gamma)}{\Gamma(n+1)}$$

and therefore

$$\int_0^\epsilon \frac{z^{\gamma-1}}{(1+z)^{n+1}} dz = \frac{\Gamma(\gamma)\Gamma(n+1-\gamma)}{\Gamma(n+1)}(1 + O((r+\epsilon)^{-n}))$$

More generally, for a natural number β let us define

$$(17) \qquad I_{\gamma,\beta}(n) = \int_0^\infty \frac{z^{\gamma-1}}{(1+z)^{n+1}}(\log z)^\beta dz$$

Then, we have

$$(18) \qquad I_{\gamma,\beta}(n) = \frac{\Gamma(\gamma)\Gamma(n+1-\gamma)}{\Gamma(n+1)} p_\beta(\gamma, n)$$

where $p_\beta(\gamma, n)$ is a polynomial in the variables $\psi^{(k)}(n+1-\gamma)$ and $\psi^{(l)}(\gamma)$ with rational coefficients, where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the logarithmic derivative of the Γ-function. For example, we have:

$$p_0(n) = 1$$
$$p_1(n) = -\psi(n+1-\gamma) + \psi(\gamma)$$
$$p_2(n) = \psi(n+1-\gamma)^2 + \psi^{(1)}(n+1-\gamma) - 2\psi(\gamma)\psi(n+1-\gamma) + \psi(\gamma)^2 + \psi^{(1)}(\gamma)$$

Compare also with [**M1**, Eqn.I.4.2] and [**M2**, Eqn.7.5]. What is important is not the exact evaluation of $I_{\gamma,\beta}(n)$ given in (18), but the fact that the sequence $I_{\gamma,\beta}(n)$ is of Nilsson type. This follows from the fact that we have an asymptotic expansion (see [**O**]):

$$(19) \qquad \frac{\Gamma(n+1-\gamma)}{\Gamma(n+1)} \sim \frac{1}{n^\gamma}\left(1 + \frac{\gamma^2-\gamma}{2n} + \frac{3\gamma^4 - 2\gamma^3 - 3\gamma^2 + 2\gamma}{24n^2} + \cdots\right)$$

Alternatively, one may show that the sequence $I_{\gamma,\beta}(n)$ is of Nilsson type by a change of variables $z = e^t - 1$ which gives

$$\int_0^\infty \frac{z^{\gamma-1}}{(1+z)^{n+1}}(\log z)^\beta dz = \int_0^\infty e^{-nt} t^{\gamma-1} A_{\gamma,\beta}(t) dt$$

where

$$A_{\gamma,\beta}(t) = \left(\frac{e^t-1}{t}\right)^{\gamma-1}\left(\log\left(\frac{e^t-1}{t}\right) - \log t\right)^\beta dt$$

is a function which can be expanded into a polynomial of $\log t$ with coefficients functions which are analytic at $t = 0$. Expand $A_{\gamma,\beta}(t)$ into power series at $t = 0$ and interchange summation and integration by applying Watson's lemma (see [**O**]) to conclude that $I_{\gamma,\beta}(n)$ is of Nilsson type.

Replace $\Delta_\lambda G(\lambda z)$ by (15) in (16) and interchange summation and integration by applying Watson's lemma (see [**O**]). It follows that

$$\frac{1}{2\pi i}\int_{\mathcal{H}_\lambda}\frac{G(z)}{z^{n+1}}dz \sim \lambda^{-n}\sum_{\alpha,\beta}\frac{1}{n^\alpha}(\log n)^\beta\sum_{k=0}^\infty c_{\alpha,\beta,\lambda,k}\frac{1}{n^k}$$

Equation (14) cocnludes that (a_n) is of Nilsson type. Strictly speaking, the above analysis works only when $\Re(\alpha) > -1$. This is a local integrability assumption of the Beta-integral. The asymptotic expansion (2) remains valid as stated even when $\Re(\alpha) \le -1$ as follows by first integrating $G(z)$ a sufficient number of times, and then applying the analysis. This is exactly what was done in [**CG**, Sec.7] at the cost of complicating the notation. $\square$

REMARK 4.3. *Since the sequence $(c'_{\alpha,\beta,\lambda,k})$ as a function of k is exponentially bounded and the asymptotic expansion (19) is Gevrey-1, it follows that the sequence $(c_{\alpha,\beta,\lambda,k})$ is Gevrey-1. Moreover, if the sequence $(c'_{\alpha,\beta,\lambda,k})$ lies in a number field K, then we can write the asymptotic expansion of (a_n) in the form (10) where $S_{\alpha,\beta,\lambda}$ are polynomials (with rational coefficients) of values of logarithmic derivatives of the Gamma function at rational numbers.*

5. G-functions

5.1. G-functions: examples of sequences of Nilsson type. In [**Ga2**, Thm.1] it was proven that balanced multisum sequences (which appear in abundance in Enumerative Combinatorics) are sequences of Nilsson type. The proof uses the theory of G-functions which verifies that the generating series of balanced multisum sequences satisfies the hypothesis of Corollary 4.1. Let us give the definition of a balanced multisum sequence, a G-function and an example.

DEFINITION 5.1. (a) A *term* $\mathfrak{t}_{n,k}$ in variables (n,k) where $k = (k_1,\ldots,k_r)$ is an expression of the form:

$$(20) \qquad \mathfrak{t}_{n,k} = C_0^n\prod_{i=1}^r C_i^{k_i}\prod_{j=1}^J A_j(n,k)!^{\epsilon_j}$$

where $C_i \in \overline{\mathbb{Q}}$ for $i = 0,\ldots,r$, $\epsilon_j = \pm 1$ for $j = 1,\ldots,J$, and A_j are integral linear forms in the variables (n,k) such that for every $n \in \mathbb{N}$, the set

$$(21) \qquad \mathrm{supp}(\mathfrak{t}_{n,\bullet}) := \{k \in \mathbb{Z}^r \mid A_j(n,k) \ge 0, \ j = 1,\ldots,J\}$$

is finite. We will call a term *balanced* if in addition it satisfies the *balance condition*:

$$(22) \qquad \sum_{j=1}^J \epsilon_j A_j = 0.$$

(b) A *(balanced) multisum sequence* (a_n) is a sequence of complex numbers of the form

$$(23) \qquad a_n = \sum_{k\in\mathrm{supp}(\mathfrak{t}_{n,\bullet})}\mathfrak{t}_{n,k}$$

where $\mathfrak{t}$ is a (balanced) term and the sum is over a finite set that depends on $\mathfrak{t}$.

For example, the following sequence is a balanced multisum

$$(24) \quad a_n = \sum_{k,l} \binom{n}{k+l}^2 \binom{n+k}{k}^3 \binom{n+l}{l} = \sum_{k,l} \frac{(n+k)!^3(n+l)!}{k!^3 l! n!^2 (k+l)!^2 (n-k-l)!^2}$$

where the summation is over the set of pairs of integers (k,l) that satisfy $0 \leq k,l$ and $k+l \leq n$.

Let us now recall what us a G-function. The latter were introduced by Siegel in [**Si**] with motivation being arithmetic problems in elliptic integrals, and transcendence problems in number theory. For further information about G-functions and their properties, see [**An1, An2**].

DEFINITION 5.2. We say that series $G(z) = \sum_{n=0}^{\infty} a_n z^n$ is a G-*function* if

(a) the coefficients a_n are algebraic numbers,
(b) there exists a constant $C > 0$ so that for every $n \in \mathbb{N}$ the absolute value of every conjugate of a_n is less than or equal to C^n,
(c) the common denominator of $a_0, \ldots, a_n$ is less than or equal to C^n,
(d) $G(z)$ is holonomic, i.e., it satisfies a linear differential equation with coefficients polynomials in z.

G-functions satisfy the hypothesis of Corollary 4.1; see [**An1, An2**]. Indeed, they satisfy a linear differential equation which analytically continues them in the complex plane. Moreover, the arithmetic hypothesis ensures that the local monodromy is quasi-unipotent. We can now state the main result of [**Ga2**].

THEOREM 5.3. [**Ga2**] (a) If (a_n) is a balanced multisum sequence, its generating function $G(z) = \sum_{n=0}^{\infty} a_n z^n$ is a G-function.
(b) In that case, it follows that (a_n) is a sequence of Nilsson type.

The reader may have noticed that we defined the notion of a sequence of Nilsson type only when $\limsup |a_n|^{1/n} > 0$. In case the generating series is a G-function, the remaining case is taken care by the following lemma.

LEMMA 5.4. *If* $G(z) = \sum_{n=0}^{\infty} a_n z^n$ *is a G-function and* $\limsup |a_n|^{1/n} = 0$, *then* $a_n = 0$ *for all but finitely many n.*

PROOF. The assumption implies that $G(z)$ is an entire G-function. Since those are regular-singular at infinity, it follows that $G(z)$ is a polynomial; see also [**An1, An2**]. The result follows.   □

5.2. Classical spin networks: examples of G-functions. In [**GV1, GV2**] it was proven that the evaluation of a quantum spin network at a fixed root of unity is a balanced multisum sequence, and consequently it is a sequence of Nilsson type.

6. Effective computations

6.1. Exact computations. Proposition 2.1 is a uniqueness statement about the asymptotics of a sequence of Nilsson type, and Theorem 4.1 is an existence statement which is not effective. There are two types of effective computations, exact and numerical. The exact computations use as an input a linear recursion relation of the sequence. The following proposition is elementary and is discussed in detail for example in [**FS, WZ2**].

PROPOSITION 6.1. *Given a linear recursion relation for a sequence (a_n) of Nilsson type, one can compute exactly λ, α, β and the power series $g_{\alpha,b,\lambda}(x)$ that appear in Equation* (10).

In particular, a linear recursion relation computes exactly the asymptotics of a sequence of Nilsson type, up to a finite number of *unknown Stokes constants*.

To apply Proposition 6.1 one needs to find a linear recursion for a sequence (a_n). This comes from the fundamental theorem of Wilf-Zeilberger which states that a balanced multisum sequence is holonomic, i.e., satisfies a linear recursion with coefficients polynomials in n; see [**Z, WZ1, PWZ**]. The proof of the above theorem has been computer implemented and works well for single sums and reasonably well for double sums; see [**PWZ, PR1, PR2**]. As an example, consider the following sequence from [**GV1**, Sec.10]

$$a_n \;=\; \frac{n!^6}{(3n+1)!^2} \sum_{k=3n}^{4n} \frac{(-1)^k (k+1)!}{(k-3n)!^4 (4n-k)!^3}$$

Using the language of [**GV1**], (a_n) is the evaluation of the tetrahedron spin network (also known as $6j$-symbol) when all edges are equal to n. The command

```
<< zb.m
```

loads the package of [**PR2**] into Mathematica. The command

```
teucl[n_, k_] := n!^6 / (3 n + 1) !^2 (-1) ^k (k + 1) ! 1 / ((4 n - k) !^3 (k - 3 n) !^4)
```

defines the summand of the sequence (a_n), and the command

```
Zb[teucl[n, k], {k, 3 n, 4 n}, n, 2]
```

computes the following second order linear recursion relation for the sequence (a_n)

$-9 \, (1+n) \, (2+3 \, n)^2 \, (4+3 \, n)^2 \, (451+460 \, n+115 \, n^2) \, a[n] \; + $
$\quad (3+2 \, n) \, (319\,212+1\,427\,658 \, n+2\,578\,232 \, n^2+2\,423\,109 \, n^3+1\,255\,139 \, n^4+340\,515 \, n^5+37\,835 \, n^6) \, a[1+n] \; - $
$\quad 9 \, (2+n) \, (5+3 \, n)^2 \, (7+3 \, n)^2 \, (106+230 \, n+115 \, n^2) \, a[2+n] == 0$

This linear recursion has two formal power series solutions of the form

$$
\begin{aligned}
a_{\pm,n} \;=\;\; & \frac{1}{n^{3/2}} \Lambda_{\pm}^n \left(1 + \frac{-432 \pm 31 i\sqrt{2}}{576n} + \frac{109847 \mp 22320 i\sqrt{2}}{331776 n^2} + \frac{-18649008 \pm 4914305 i\sqrt{2}}{573308928 n^3} \right. \\[2mm]
& + \frac{14721750481 \pm 45578388960 i\sqrt{2}}{660451885056 n^4} + \frac{-83614134803760 \pm 7532932167923 i\sqrt{2}}{380420285792256 n^5} \\[2mm]
& \left. + \frac{-31784729861796581 \mp 212040612888146640 i\sqrt{2}}{657366253849018368 n^6} + O\left(\frac{1}{n^7}\right) \right)
\end{aligned}
$$

where

$$\Lambda_{\pm} = \frac{329 \mp 460 i\sqrt{2}}{729} = e^{\mp i6 \arccos(1/3)}$$

are two complex numbers of absolute value 1. The coefficients of the formal power series $a_{\pm,n}$ are in the number field $K = Q(\sqrt{-2})$.

6.2. Numerical computations. When a sequence (a_n) is given by a multi-dimensional balanced sum, the computed implemented WZ method may not terminate. In that case, one may develop numerical methods for finding λ, α, β as in Equation (10). An example of this method is the asymptotics of the evaluation of the Cube Spin Network that appears in the Appendix of [**GV1**]. Effective methods for numerical computations of asymptotics have been developed by several authors, and have also been studied by Zagier.

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School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160, USA, `http://www.math.gatech.edu/~stavros`