THE SLOPE CONJECTURE FOR MONTESINOS KNOTS

STAVROS GAROUFALIDIS, CHRISTINE RUEY SHAN LEE, AND ROLAND VAN DER VEEN

Abstract. The slope conjecture relates the degree of the colored Jones polynomial of a knot to boundary slopes of incompressible surfaces. We develop a general approach that matches a state-sum formula for the colored Jones polynomial with the parameters that describe surfaces in the complement. We apply this to Montesinos knots proving the slope conjecture for Montesinos knots, with some restrictions.

Contents

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1. Introduction

1.1. **The slope conjecture and the case of Montesinos knots.** The slope conjecture relates one of the most important knot invariants, the colored Jones polynomial, to incompressible surfaces in the knot complement [Gar11b]. More precisely, the growth of the degree as a function of the color determines boundary slopes. Understanding the topological information that the polynomial detects in the knot is a central problem in quantum topology. The conjecture suggests the polynomial can be studied through surfaces, which are fundamental objects in 3-dimensional topology.

Our philosophy is that the connection follows from a deeper correspondence between terms in an expansion of the polynomial and surfaces. This would potentially lead to a purely topological definition of quantum invariants. The coefficients of the polynomial should count isotopy classes of surfaces, much like in the case of the 3D-index [**?**]. As a first test of this principle, we focus on the slope conjecture for Montesinos knots. In this case Hatcher-Oertel [HO89] provides a description of the set of incompressible surfaces of those knots. In particular they give an effective algorithm to compute the set of boundary slopes of incompressible surfaces in such knots.

We provide a state-sum formula for the colored Jones polynomial, that allows us to match the parameters of the terms of the sum that contribute to the degree of the polynomial with the parameters that describe the locally incompressible surfaces. The key innovation of our state sum is that we are able to identify those terms that actually contribute to the degree. The resulting degree function is piecewise-quadratic, allowing application of quadratic integer programming methods.

We interpret the curve systems on a Conway sphere enclosing a rational tangle in terms of these degree-maximizing skein elements in the state sum. In this paper we carry out the matching for Montesinos knots but the state-sum (10) is valid in general. In fact using this framework, one could determine the degree of the colored Jones polynomial and find candidates for corresponding incompressible surfaces in many new cases beyond Montesinos knots.

While the local theory works in general, fitting together the surfaces in each tangle to obtain a (globally) incompressible surface has yet to be done. The behavior of the colored Jones polynomial under gluing of tangles has similar patterns, which may be explored in future work.

The Montesinos knots are those which together with some well-understood algebraic knots have small Seifert fibered 2-fold branched cover [Mon73, Zie84]. For our purposes, we will not use this abstract definition, and instead construct Montesinos links by inserting rational tangles into pretzel knots. More precisely, a Montesinos link is the closure of a list of rational tangles arranged as in Figure 1 and concretely in Figure 2. See Definition 2.4.

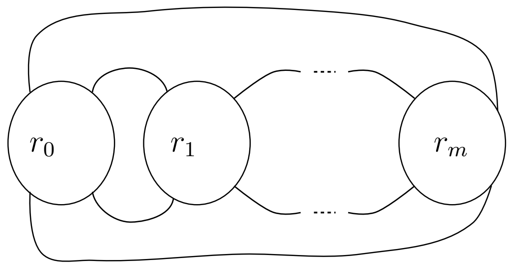


FIGURE 1. A Montesinos link.

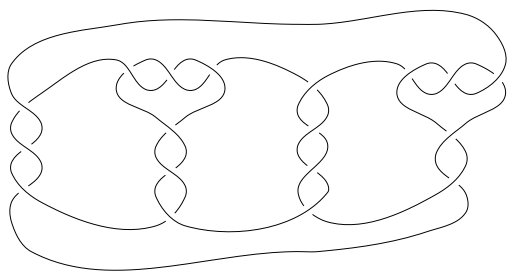


FIGURE 2. The Montesinos link $K(-\frac{1}{3}, -\frac{3}{10}, \frac{1}{4}, \frac{2}{7})$.

Rational tangles are parametrized by rational numbers, see Section 2.1, thus a Montesinos link $K(r_0, r_1, \dots, r_m)$ is encoded by a list of rational numbers $r_j \in \mathbb{Q}$. Note that $K(r_0, r_1, \dots, r_m)$ is a knot if and only if either there is only one even denominator, or, there is no even denominator and the number of odd numerators is odd. When $r_i = 1/q_i$ is the inverse of an integer, the Montesinos link $K(1/q_0, \dots, 1/q_m)$ is also known as the pretzel link $P(q_0, \dots, q_m)$.

1.2. **Our results.** Recall the colored Jones polynomial $J_{K,n}(v) \in \mathbb{Z}[v^{\pm 2}]$ of a knot K colored by the n-dimensional irreducible representation of $\mathfrak{sl}_2$ [Tur88]. Our variable v for the colored Jones polynomial is related to the skein theory variable A [Prz91] and to the Jones variable q [Jon87] by $v = A^{-1} = q^{-\frac{1}{4}}$. With our conventions, if $3_1 = P(1,1,1)$ denotes the left-hand trefoil, then $J_{3_1,2}(v) = v^{18} - v^{10} - v^6 - v^2$. For the n-colored unknot we get $J_{O,n} = \frac{v^{2n} - v^{-2n}}{v^2 - v^{-2}}$.

Let $\delta_K(n)$ denote the maximum v-degree of the colored Jones polynomial $J_{K,n}(v)$. It follows that $\delta_K(n)$ is a quadratic quasi-polynomial [Gar11a]. In other words, for every knot K there exists an $N_K \in \mathbb{N}$ such that for $n > N_K$:

$$\delta_K(n) = \mathrm{js}_K(n) n^2 + \mathrm{jx}_K(n) n + c_K(n) \tag{1}$$

where js_K, jx_K, and c_K are periodic functions.

Conjecture 1.1. (The strong slope conjecture)
For any knot K and any n, there is an n' and an essential surface $S \subset S^3 \setminus K$ with $|\partial S|$

boundary components, such that the boundary slope of S equals $\mathrm{js}_K(n) = p/q$ (reduced to lowest terms and with the assumption $q > 0$), and $\frac{2\chi(S)}{q|\partial S|} = \mathrm{jx}_K(n')$ of S.

The number $q|\partial S|$ is called the *number of sheets* of S, and $\chi(S)$ is the Euler characteristic of S. See the discussion at the beginning of Section 6 for the definition of an essential surface and boundary slope. We call a value of the function js_K a *Jones slope* and a value of the function jx_K a *normalized Euler characteristic*. The original slope conjecture is the part of Conjecture 1.1 that concerns the interpretation of js_K as boundary slopes [Gar11b], while the rest of the statement is a refinement by [KT15]. The reader may consult these two sources [Gar11b], [KT15] for additional background. By considering the mirror image $\overline{K}$ of K and the formula $J_{K,n}(v^{-1}) = J_{\overline{K},n}(v)$, the strong slope conjecture is equivalent to the statement in [KT15] that includes the behavior of the minimal degree.

The slope conjecture and the strong slope conjecture were established for many knots including alternating knots, adequate knots, torus knots, knots with at most 9 crossings, 2-fusion knots, graph knots, near-alternating knots, and most 3-tangle pretzel knots and 3-tangle Montesinos knots [Gar11b, FKP11, GvdV16, LvdV16, MT17, Lee, LLY, How]. However the general case remains intractable and most proofs simply compute the quantum side and the topology side separately, comparing only the end results.

Since the strong slope conjecture is known for adequate knots [Gar11b, FKP11, FKP13], we will ignore the Montesinos knots which are adequate. Note that when $m \geq 2$, the only non-adequate Montesinos knots $K(r_0, r_1, \dots, r_m)$ have precisely one negative or positive tangle [LT88, p.529]. Without loss of generality we need only to consider $\mathrm{js}_K(n)$ and $\mathrm{jx}_K(n)$ for a Montesinos knot with precisely one negative tangle. The positive tangle case follows from taking mirror image.

Before stating our main result on Montesinos knots we start with the case of pretzel knots as they are the basis for our argument. In fact Theorem 1.2 is the bulk of our work. For $P(q_0, \dots, q_m)$ to be a knot, at most one tangle has an even number of crossings, and if each tangle has an odd number of crossings, then the number of tangles has to be odd. In the theorem below, the condition on the parities of the q_i's and the number of tangles may be dropped if one is willing to exclude an arithmetic sub-sequence of colors n.

Theorem 1.2. *Fix an $(m+1)$-vector q of odd integers $q = (q_0, \dots, q_m)$ with $m \geq 2$ even and $q_0 < -1 < 1 < q_1, \dots, q_m$. Let $P = P(q_0, \dots, q_m)$ denote the corresponding pretzel knot. Define rational functions $s(q), s_1(q) \in \mathbb{Q}(q)$:*

$$s(q) = 1 + q_0 + \frac{1}{\sum_{i=1}^m (q_i - 1)^{-1}}, \qquad s_1(q) = \frac{\sum_{i=1}^m (q_i + q_0 - 2)(q_i - 1)^{-1}}{\sum_{i=1}^m (q_i - 1)^{-1}}. \tag{2}$$

For all $n > n_K$ we have:
(a) If $s(q) < 0$, then the strong slope conjecture holds with

$$\mathrm{js}_P(n) = -2s(q), \qquad \mathrm{jx}_P(n) = -2s_1(q) + 4s(q) - 2(m-1). \tag{3}$$

(b) If $s(q) = 0$, then the strong slope conjecture holds with

$$\mathrm{js}_P(n) = 0, \qquad \mathrm{jx}_P(n) = \begin{cases} -2(m-1) & \text{if } s_1(q) \geq 0 \\ -2s_1(q) - 2(m-1) & \text{if } s_1(q) < 0 \end{cases}. \tag{4}$$

(c) If $s(q) > 0$, then the strong slope conjecture holds with

$$\mathrm{js}_P(n) = 0, \qquad \mathrm{jx}_P(n) = -2(m-1). \tag{5}$$

Next, we consider the case of Montesinos knots. Recall that by applying Euclid's algorithm, every rational number r has a unique positive continued fraction expansion $r = [b_0, \ldots, b_{\ell'}]$, see (7), with $\ell' < \infty$, $b_0 \in \mathbb{Z}$, $|b_j| \geq 1$ for $1 \leq j \leq \ell'-1$, $|b_{\ell'}| \geq 2$, and b_j's all of the same sign as r. From this we define an even length continued fraction expansion $[a_0, \ldots, a_{\ell_r}]$ of r to be equal to $[b_0, \ldots, b_{\ell'}]$ if ℓ' is even, and we define it to be equal to $[b_0, \ldots, b_{\ell'} - 1, 1]$ (resp. $[b_0, \ldots, b_{\ell'} + 1, -1]$) if ℓ' is odd and $r > 0$ (resp. $r < 0$) . Note $[a_0, \ldots, a_{\ell_r}]$ is well-defined. We will call $[a_0, \ldots, a_{\ell_r}]$ the unique even length positive continued fraction expansion for r. Define $r[j] = a_j$ for $j = 0, \ldots, \ell_r$. Let

$$\langle r \rangle_e = \sum_{j=3,\ j=\text{even}}^{\ell_r} r[j], \quad \langle r \rangle_o = \sum_{j=3,\ j=\text{odd}}^{\ell_r} r[j], \quad \langle r \rangle = \langle r \rangle_e + \langle r \rangle_o \,.$$

For example, the fraction $63/202 = [0, 3, 4, 1, 5, 2]$ has the unique even length positive continued fraction expansion $[0, 3, 4, 1, 5, 1, 1]$. Adding up all the partial quotients of the continued fraction expansion with even indices ≥ 3, we get $\langle 63/202 \rangle_e = 5 + 1 = 6$. Similarly, adding up all the partial quotients with odd indices ≥ 3, we get $\langle 63/202 \rangle_o = 1 + 1 = 2$.

Given a Montesinos knot $K(r_0, \ldots, r_m)$, define D_K to be the diagram obtained by summing rational tangles corresponding to the unique even length positive continued fraction expansion for each r_i, and then taking the numerator closure. See Section 2.1 for how a rational tangle diagram is assigned to a continued fraction expansion of a rational number and definitions for the tangle sum and numerator closure.

By the classification [**?**], [BZ03] and the existence of reduced diagrams of Montesinos links [LT88], we will further restrict to Montesinos knots $K(r_0, \ldots, r_m)$ where $|r_i| < 1$ for all $0 \leq i \leq m$. See Section 2.2 for the discussion of why we may do so without loss of generality. Again, the condition on the parities of the q_i's and the number of tangles may be dropped if one is willing to exclude an arithmetic sub-sequence of colors n, thus proving a weaker version of the conjecture for all Montesinos knots.

Let $(r_0, \ldots, r_m) \in \mathbb{Q}^{m+1}$ denote a tuple of rational numbers, and let $(q_0, \ldots, q_m) \in \mathbb{Z}^{m+1}$ denote the associated tuple of integers where $q_i = r_i[1] + 1$ for $i > 0$ and

$$q_0 = \begin{cases} \quad r_0[1] + 1 \text{ if } \ell_{r_0} = 2 \text{ and } r_0[2] = 1. \\ \quad r_0[1] \text{ otherwise} \end{cases}$$

for the unique even length positive continued fraction expansion of r_i's.

Theorem 1.3. *Let $K = K(r_0, r_1, \ldots, r_m)$ be a Montesinos knot such that $r_0 < 0$, $r_i > 0$ for all $1 \leq i \leq m$, and $|r_i| < 1$ for all $0 \leq i \leq m$ with $m \geq 2$ even. Suppose $q_0 < -1 < 1 < q_1, \ldots, q_m$ are all odd, and q_0' is an integer that is defined to be 0 if $r_0 = 1/q_0$, and defined* *to be $r_0[2]$ otherwise. Let $P = P(q_0, \ldots, q_m)$ be the associated pretzel knot, and let $\omega(D_K)$, $\omega(D_P)$ denote the writhe of D_K, D_P with orientations. Then the strong slope conjecture holds. For all $n > N_K$ we have:*

$$\mathrm{js}_K(n) = \mathrm{js}_P(n) - q_0' - \langle r_0 \rangle - \omega(D_P) + \omega(D_K) + \sum_{i=1}^{m}(r_i[2]-1) + \sum_{i=1}^{m}\langle r_i \rangle,$$

$$\mathrm{jx}_K(n) = \mathrm{jx}_P(n) - 2\frac{q_0'}{r_0[2]} + 2\langle r_0 \rangle_o - 2\sum_{i=1}^{m}(r_i[2]-1) - 2\sum_{i=1}^{m}\langle r_i \rangle_e.$$

Example 1.4. Consider the Montesinos knot $K = K(-\frac{46}{327}, \frac{35}{151}, \frac{5}{31}, \frac{16}{35}, \frac{1}{5})$. Applying Theorem 1.2 and 1.3, we compute the Jones slope $\mathrm{js}_K(n)$ by using Euclid's algorithm to obtain the unique even length continued fraction expansion for each rational number in the definition of K. We have for the first rational number $-46/327$,

$$-\frac{46}{327} = 0 + \frac{1}{-\frac{327}{46}} = 0 + \frac{1}{-7 + (-\frac{5}{46})} = 0 + \frac{1}{-7 + \frac{1}{-\frac{46}{5}}} = 0 + \frac{1}{-7 + \frac{1}{-9 + (-\frac{1}{5})}} = [0, -7, -9, -5].$$

This is of odd length, so the unique even length continued fraction expansion for $-\frac{46}{327}$ is

$$-\frac{46}{327} = [0, -7, -9, -4, -1].$$

The rational numbers together with their unique even length continued fractions expansions are

$$-\frac{46}{327} = [0, -7, -9, -4, -1],\ \frac{35}{151} = [0, 4, 3, 5, 2],\ \frac{5}{31} = [0, 6, 5],\ \frac{16}{35} = [0, 2, 5, 2, 1],\ \frac{1}{5} = [0, 4, 1].$$

The associated pretzel knot is $P(-7, 5, 7, 3, 5)$. Theorem 1.2 applied to the pretzel knot gives that

$$s(q) = -\frac{36}{7} < 0 \text{ and } s_1(q) = -\frac{32}{7}.$$

So

$$\mathrm{js}_P(n) = (-2)(-\frac{36}{7}) = \frac{72}{7} \text{ and } \mathrm{jx}_p(n) = -2(-\frac{32}{7}) + 4(-\frac{36}{7}) - 2(4) = -\frac{122}{7}.$$

Dunfield's program [Dun01], which computes the boundary slope and other topological properties of essential surfaces for a Montesinos knot based on Hatcher and Oertel's algorithm, produces an essential surface S whose boundary slope is equal to $-2s(q) = 72/7$, and such that $2\chi(S)/(7|\partial S|) = -122/7$. Now we compute $\mathrm{js}_K(n)$ and $\mathrm{jx}_K(n)$ using Theorem 1.3. To aid in presentation, we replace each symbol in the theorem by the number computed from the example. We have

$$\mathrm{js}_K(n) = \underbrace{\mathrm{js}_P(n)}_{72/7} - \underbrace{r_0[2]}_{-9} - \underbrace{\langle r_0\rangle}_{-4+-1} - \underbrace{\omega(D_P)}_{-13} + \underbrace{\omega(D_K)}_{-43} + \underbrace{\sum_{i=1}^{m}(r_i[2]-1)}_{2+4+4} + \underbrace{\sum_{i=1}^{m}\langle r_i\rangle}_{(5+2)+(2+1)} = \frac{100}{7}.$$

$$\mathrm{jx}_K(n) = \underbrace{\mathrm{jx}_P(n)}_{-122/7} -2 + 2\underbrace{\langle r_0\rangle_o}_{-4} -2\underbrace{\sum_{i=1}^{m}(r_i[2]-1)}_{2+4+4} -2\underbrace{\sum_{i=1}^{m}\langle r_i\rangle_e}_{2+1} = -\frac{374}{7}.$$

For the Montesinos knot, Dunfield's program also produces an essential surface S which realizes the strong slope conjecture, with boundary slope $100/7$ and $2\chi(S)/7|\partial S| = -374/7$.

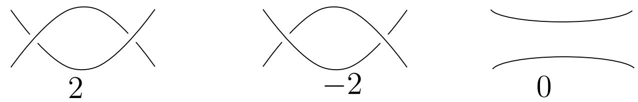
1.3. **Plan of the proof.** We divide the proof of Theorem 1.2 and Theorem 1.3 into two parts, first concerning the claims regarding the degree of the colored Jones polynomial, and the second concerning the existence of essential surfaces realizing the strong slope conjecture.

First we use a mix of skein theory and fusion, reviewed in Section 2.3, to find a formula for the degree of the dominant terms in the resulting state sum for the colored Jones polynomial in Section 3. Using quadratic integer programming techniques we determine the maximal degree of these dominant terms in Section 4, and this is applied to find the degree of the colored Jones polynomial for the pretzel knots we consider in 4.3. In Section 5 we determine the degree of the colored Jones polynomial for the Montesinos knots we consider in Theorem 1.3 by reducing to the pretzel case. Finally, we work out the relevant surfaces using the Hatcher-Oertel algorithm in Section 6, and we match the growth rate of the degree of the quantum invariant with the topology, using the analogy drawn between the parameters of the state sum and the parameters for the Hatcher-Oertel algorithm by Lemma 6.3. We explicitly describe the essential surfaces realizing the strong slope conjecture in Sections 6.5 and 6.7, and the proofs of Theorem 1.2 and Theorem 1.3 are completed in Section 6.6 and Section 6.8, respectively.

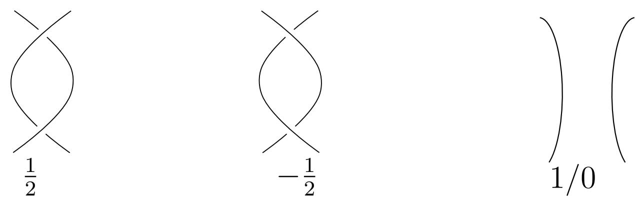
2. Preliminaries

2.1. **Rational tangles.** Let us recall how to parametrize rational tangles by rational numbers and their continued fraction expansions. Originally studied by John Conway [Con70], this material is well-known and may be found for instance in [KL04, BS]. An (m, n)-tangle is an embedding of a finite collection of arcs and circles into B^3, such that the endpoints of the arcs lie in the set of $m + n$ points on $\partial B^3 = S^2$. We consider tangles up to isotopy of the ball B^3 fixing the boundary 2-spheres. The integer m indicates the number of points on the upper hemisphere of S^2, and the integer n indicates the number of points on the lower hemisphere. We may isotope a tangle so that its endpoints are arranged on a great circle of the boundary 2-sphere S^2. A tangle diagram is then a regular projection of the tangle onto the plane of this great circle. We represent tangles by tangle diagrams, and we will refer to an (m, m)-tangle as an m-tangle. Our building blocks of rational tangles are the horizontal and the vertical 2-tangles shown below, called elementary tangles in [KL04].

- A *horizontal tangle* has n horizontal half-twists (i.e., crossings) for $n \in \mathbb{Z}$.



- A *vertical tangle* has n vertical half-twists (i.e., crossings) for $n \in \mathbb{Z}$.



The horizontal tangle with 0 half-twists will be called the 0-tangle, and the vertical tangle with 0 half-twists will be called the ∞-tangle.

Definition 2.1. A rational tangle is a 2-tangle that can be obtained by applying a finite number of consecutive twists of neighboring endpoints to the 0-tangle and the ∞-tangle.

For $2m$-tangles we define tangle addition, denoted by $\oplus$, and tangle multiplication, denoted by $*$, as follows in Figure 3. We also define the numerator closure of a $2m$-tangle as a knot or link obtained by joining the two sets of m endpoints in the upper hemisphere, and by joining the two sets of m endpoints in the lower hemisphere.

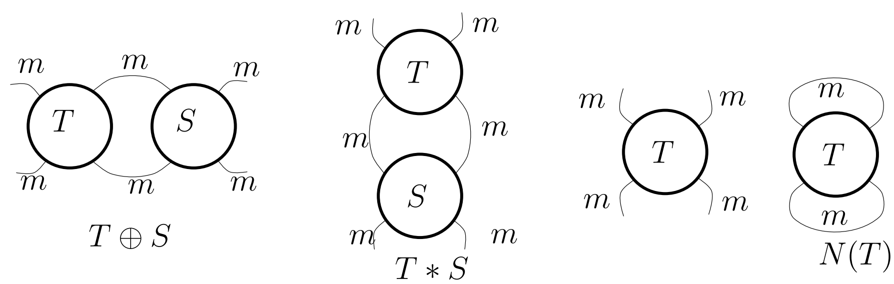


FIGURE 3. $2m$-tangle addition, multiplication, and numerator closure.

The following theorem is paraphrased from [KL04] with changes in notations for the elementary rational tangles.

Theorem 2.2. [KL04, Lemma 3] *Every rational tangle can be isotoped to have a diagram in standard form, obtained by consecutive additions of horizontal tangles only on the right (or only on the left) and consecutive multiplications by vertical tangles only at the bottom (or only at the top), starting from the 0-tangle or the ∞-tangle.*

More precisely, every rational tangle diagram maybe be isotoped to have the algebraic presentation

$$(((a_\ell * \frac{1}{a_{\ell-1}}) \oplus a_{\ell-2}) * \cdots * \frac{1}{a_1}) \oplus a_0, \tag{6}$$

where $a_j \in \mathbb{Z}$.

Recall the notation of the positive continued fraction expansion [KL04, BS]:

$$[a_0, \ldots, a_\ell] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots + \cfrac{1}{a_\ell}}}} \tag{7}$$

for integers $a_j > 0$ for $1 \leq j \leq \ell$ and $a_0 \in \mathbb{Z}$. We define the rational number r associated to a rational tangle in standard form with algebraic expression (6) to be

$$r = [a_0, \ldots, a_\ell].$$

Conversely, given a positive continued fraction expansion of a rational number $r = [a_0, \ldots, a_\ell]$ we may obtain a diagram of a rational tangle given by the corresponding algebraic expression (6). See Figure 4 for an example.

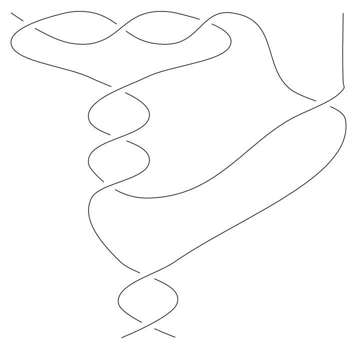


FIGURE 4. A rational tangle diagram T associated to the continued fraction expansion $[0, 2, 1, 3, 3] = 13/36$.

A rational tangle is determined by their associated rational number to a standard diagram by the following theorem.

Theorem 2.3. [Con70] *Two rational tangles are isotopic if and only if they have the same associated rational number.*

See [KL04, Theorem 3] for a proof of this statement.

Definition 2.4. A *Montesinos link* $K(r_0, r_1, \ldots r_m)$ is a link that admits a diagram obtained by summing rational tangles

$$((T_{c_0} + T_{c_1}) + T_{c_2}) \cdots + T_{c_m},$$

where c_i for each $0 \leq i \leq m$ is a choice of positive continued fraction expansion of r_j, then taking the numerator closure.

Note that a different choice of a positive continued fraction expansion for each r_i produces a different diagram of the same knot by Theorem 2.3. To simplify our arguments, we will choose a specific diagram for the Montesinos knot $K(r_0, r_1, \ldots, r_m)$ outside the family of adequate Montesinos knots by specifying the choice of a positive continued fraction expansion for each rational number r_i.

2.2. **Classification of Montesinos links.** The book [BZ03] has a complete account of the classification of Montesinos links, originally due to Bonahon [?].

Theorem 2.5. [BZ03, Theorem 12.29] *Let $K(r_0, \ldots, r_m)$ be a Montesinos link such that $m \geq 3$ and $r_0, \ldots, r_m \in \mathbb{Q} \setminus \mathbb{Z}$. Then K is determined up to isomorphism by the rational number $\sum_{i=0}^{m} r_m$ and the vector $((r_0 \mod 1), (r_1 \mod 1), \ldots, (r_m \mod 1))$, up to cyclic permutation and reversal of order.*

Note that requirement for the r_i's to be in $\mathbb{Q} \setminus \mathbb{Z}$ rules out the case where an integer rational tangle can be absorbed into another one in the tangle sum defining the Montesinos knot. We will work with *reduced* diagrams for Montesinos knots as studied by Lickorish and Thistlethwaite [LT88]. Here we follow the exposition of [FKP13, Chapter 8].

Definition 2.6. Let K be a Montesinos link. A diagram is called a *reduced Montesinos diagram* of K if it is the numerator closure of the sum of rational angles $T_0, \ldots, T_m$ corresponding to rational numbers $r_0, \ldots, r_m$ with $m \geq 2$, and both of the following hold:

(1) Either all of the r_i's have the same sign, or $0 < |r_i| < 1$ for all i.
(2) For each i, the diagram of T_i comes from a positive continued fraction expansion of r_i.

It follows as a consequence of the classification theorem that every Montesinos link $K(r_0, \ldots, r_m)$ with $m \geq 2$ has a reduced diagram. For example, if $r_i < 0$ while $r_j \geq 1$, we can subtract 1 from r_j and add 1 to r_i until condition (1) is satisfied. This does not change the link type of the Montesinos link by Theorem 2.5. Since we are focused on Montesinos links with r_i's of different signs, we may assume that $0 < |r_i| < 1$. Thus $r_i[0] = 0$ for all $0 \leq i \leq m$.

2.3. **Skein theory and the colored Jones polynomial.** We consider the skein module of link diagrams on an oriented surface F with a finite (possibly empty) collection of points specified on the boundary ∂F. This will be used to give a definition of the colored Jones polynomial from a diagram of a link. For the original reference for skein modules see [Prz91]. We will follow the approach of Lickorish [Lic97, Section 13] except for the variable substitution (our v is his A^{-1} to avoid confusion with the A for a Kauffman state). See [?] for how the skein theory gives the colored Jones polynomial, also know as the quantum $\mathfrak{sl}_2$ invariant. The word "color" refers to the weight of the irreducible representation where one evaluates the invariant.

Definition 2.7. Let v be a fixed complex number. The linear skein module $\mathcal{S}(F)$ of F is a vector space of formal linear sums over $\mathbb{C}$, of unoriented and properly-embedded tangle diagrams in F, considered up to isotopy of F fixing ∂F, and quotiented by the relations

(i) $D \sqcup \bigcirc = (-v^{-2} - v^{2})D$, and
(ii) $= v^{-1} \;)\,(\; + v$.

Here $\bigcirc$ denotes the unknot and $D \sqcup \bigcirc$ is the disjoint union of the diagram D with an unknot. Relation (ii) indicates how we can write a diagram as a sum of two diagrams with coefficients in v by locally replacing a crossing by the two splicings on the right.

We consider the linear skein module $\mathcal{S}(D^2, n, n')$ of the disc D^2 with $n+n'$-points specified on its boundary, where the boundary is viewed as a rectangle with n marked points above and n' marked points below. From the skein relations in Definition 2.7, every element in $\mathcal{S}(D^2, n, n')$ is generated by crossingless matchings between the n points on top and n' points below. For crossingless matchings $D_1 \in \mathcal{S}(D^2, n, n')$, and $D_2 \in \mathcal{S}(D^2, n', n'')$, there is a natural multiplication operation $D_1 \times D_2$ defined by identifying the bottom boundary of D_1 with the top boundary of D_2 and matching the n' common boundary points. Extending this by linearity to all elements in $\mathcal{S}(D^2, n, n)$ makes it into an algebra TL_n^n, called *Temperley-Lieb algebra*. For the original references see [TL71], [KL94]. We will simply write TL_n. There is a natural identification of $2n$-tangles with diagrams in TL_{2n}.

As an algebra, TL_n is generated by a basis $|_n, e_n^1, \ldots, e_n^{n-1}$, where $|_n$ is the identity with respect to the multiplication, and e_n^i is a crossing-less tangle diagram as specified below in Figure 5.

Suppose that v^4 is not a kth root of unity for $k \leq n$. There is an element, which we will denote by $\boxminus_n$, in TL_n called the nth *Jones-Wenzl idempotent*, which is uniquely defined by the following properties. For the original reference where the idempotent was defined and studied, see [Wen87]. Whenever n is specified we will simply refer to this element as the Jones-Wenzl idempotent.

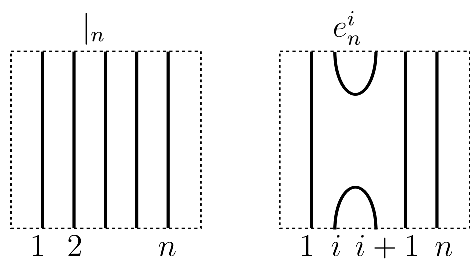


FIGURE 5. An example of the identity element 1_n and a generator e_n^i of TL_n^n for $n = 5$ and $i = 2$.

(i) $\boxminus_n \times e_n^i = e_n^i \times \boxminus_n = 0$ for $1 \leq i \leq n-1$.
(ii) $\boxminus_n - |_n$ belongs to the algebra generated by $\{e_n^1, e_n^2, \ldots, e_n^{n-1}\}$.
(iii) $\boxminus_n \times \boxminus_n = \boxminus_n$.
(iv) The image of $\boxminus_n$ in $\mathcal{S}(\mathbb{R}^2)$, obtained by embedding the disc D^2 in the plane and then joining the n boundary points on the top with those on the bottom with n disjoint planar parallel arcs outside of D^2, is equal to

$$\frac{(-1)^n(v^{-2(n+1)} - v^{2(n+1)})}{v^{-2} - v^2} \cdot \text{the empty diagram in } \mathbb{R}^2.$$

We will denote the rational function multiplying the empty diagram by $\triangle_n$.

Definition 2.8. Let D be a diagram of a link $K \subset S^3$ with k components. For each component D_i for $i \in \{1, \ldots, k\}$ take an annulus A_i via the blackboard framing. Let $\mathcal{S}(S^1 \times I)$ be the linear skein module of the annulus with no points marked on its boundary. Let

$$f_D : \underbrace{\mathcal{S}(S^1 \times I) \times \cdots \times \mathcal{S}(S^1 \times I)}_{k \text{ times}} \to \mathcal{S}(\mathbb{R}^2)$$

be the map which sends a k-tuple of elements $(s_1, \ldots, s_k)$ to $\mathcal{S}(\mathbb{R}^2)$ by immersing in the plane the someion of annuli containing the skeins such that the over- and under-crossings of

D are the over- and under-crossings of the annuli. For $n \geq 1$, the $n+1$*th unreduced colored Jones polynomial* $J_{K,n+1}(v)$ may be defined as

$$J_{K,n+1}(v) := (-v)^{\omega(D)(n^2+2n)}(-1)^n \left\langle f_D \underbrace{\left(\bigcirc_n, \bigcirc_n, \cdots, \bigcirc_n \right)}_{k \text{ times}} \right\rangle,$$

where $\langle D \rangle$ for a linear skein element in $\mathcal{S}(\mathbb{R}^2)$ is the polynomial in v multiplying the empty diagram after resolving crossings and removing disjoint circles of D using the skein relations of Definition 2.7. This is called the *Kauffman bracket* of D. To simplify notation, we will write

$$D^n = f_D \left(\bigcirc_n, \bigcirc_n, \cdots, \bigcirc_n \right).$$

A Kauffman state [Kau87], which we will denote by σ, is a choice of the A- or B-resolution at a crossing of a link diagram.

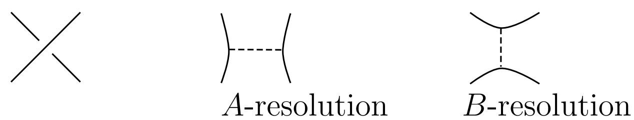


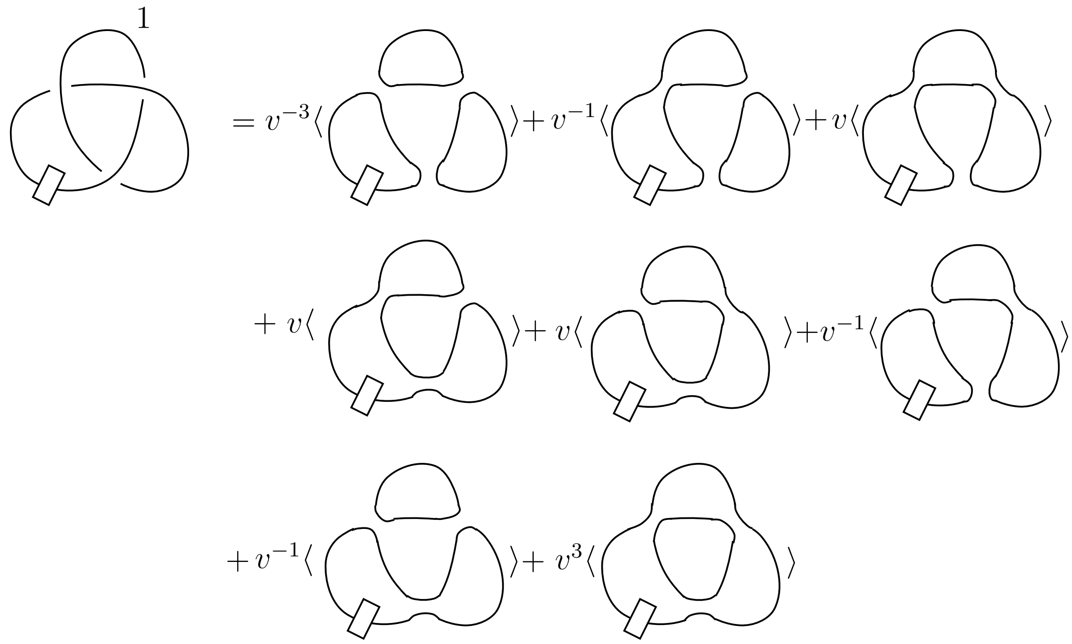
FIGURE 6. A- and B-resolutions of a crossing.

Definition 2.9. Let σ be a Kauffman state on a skein element with crossings, define

$$\operatorname{sgn}(\sigma) = (\#\text{ of } B\text{-resolutions of } \sigma) - (\#\text{ of } A\text{-resolutions of } \sigma).$$

Definition 2.10. Given a skein element $\mathcal{S}$ with crossings in $\mathcal{S}(\mathbb{R}^2)$, the *$\sigma$-state* denoted by $\mathcal{S}_\sigma$ is the set of disjoint arcs and circles, possibly connecting Jones-Wenzl idempotents, resulting from applying a Kauffman state σ to $\mathcal{S}$. The *σ-state graph* $\mathcal{S}^G_\sigma$ is the set of disjoint arcs and circles, possibly joining Jones-Wenzl idempotents, resulting from applying a Kauffman state σ to $\mathcal{S}$ along with segments recording the original locations of the crossings.

We summarize standard techniques and formulas for computing the colored Jones polynomial using Definition 2.8 that are relevant to this paper. Given a diagram D^n decorated with a single Jones-Wenzl idempotent from a link, a *state sum* for the Kauffman bracket $\langle D^n \rangle$ of D^n is an expansion of $\langle D^n \rangle$ into a sum over skein elements resulting from all the possible choices of Kauffman states on a subset of crossings in D^n. As an example, one can compute the second colored Jones polynomial of the trefoil knot by writing down the following state sum.



$$= \sum_{\sigma \text{ a Kauffman state}} v^{sgn(\sigma)} \langle (D^n)_\sigma \rangle.$$

where the choice of Kauffman states is taken over all the crossings of the diagram. We are left with disjoint arcs and circles connecting the Jones-Wenzl idempotent. These may be removed by applying skein relations and by applying properties of the idempotent.

Since we are interested in bounding degrees of the Kauffman brackets of skein elements in the state sum, we will define a few more relevant combinatorial quantities and gather some useful results.

Let $\mathcal{S}_\sigma$ be a skein element coming from applying a Kauffman state σ to a skein element $\mathcal{S}$ with crossings and decorated by Jones-Wenzl idempotents in $\mathcal{S}(\mathbb{R}^2)$, then $\overline{\mathcal{S}_\sigma}$ is the set of disjoint circles obtained from $\mathcal{S}_\sigma$ by replacing all idempotents with the identity. Let A_σ be the set of crossings of $\mathcal{S}$ on which σ chooses the A-resolution. Define $o(A_\sigma)$ to be the number of circles in $\overline{\mathcal{S}_\sigma}$.

Definition 2.11. A *sequence* s of states starting at σ_1 and ending at σ_f on a set of crossings in a skein element $\mathcal{S}$ is a finite sequence of Kauffman states $\sigma_1, \ldots, \sigma_f$, where σ_i and σ_{i+1} differ on the choice of the A- or B-resolution at only one crossing x, so that σ_{i+1} chooses the A-resolution at x and σ_i chooses the B-resolution.

Let $s = \{\sigma_1, \ldots, \sigma_f\}$ be a sequence of states starting at σ_1 and ending at σ_f In each step from σ_i to σ_{i+1} either two circles of $\overline{\mathcal{S}_{\sigma_i}}$ merge into one or a circle of $\overline{\mathcal{S}_{\sigma_i}}$ splits into two. When two circles merge into one as the result of changing the B-resolution to the A-resolution, the number of circles of the skein element decreases by 1 while the sign of the state decreases by 2. More precisely, let $\mathcal{S}_\sigma$ be the skein element resulting from applying the Kauffman state σ, we have

$$\operatorname{sgn}(\sigma_{i+1}) + \deg\langle \overline{\mathcal{S}_{\sigma_{i+1}}} \rangle = \operatorname{sgn}(\sigma_i) + \deg\langle \overline{\mathcal{S}_{\sigma_i}} \rangle - 4\,,$$

when a pair of circles merges from $\overline{\mathcal{S}_{\sigma_i}}$ to $\overline{\mathcal{S}_{\sigma_{i+1}}}$. This gives the following immediate corollary.

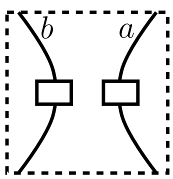
Lemma 2.12. Let $s = \{\sigma_1, \dots, \sigma_f\}$ be a sequence of states on a skein element $\mathcal{S}$ with crossings, then

$$\operatorname{sgn}(\sigma_1) + \deg\langle\overline{\mathcal{S}_{\sigma_1}}\rangle = \operatorname{sgn}(\sigma_f) + \deg\langle\overline{\mathcal{S}_{\sigma_f}}\rangle$$

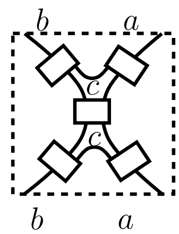
if and only if a circle is split from $\overline{\mathcal{S}_{\sigma_i}}$ to $\overline{\mathcal{S}_{\sigma_{i+1}}}$ for every $1 \le i \le f-1$. Otherwise

$$\operatorname{sgn}(\sigma_1) + \deg\langle\overline{\mathcal{S}_{\sigma_1}}\rangle > \operatorname{sgn}(\sigma_f) + \deg\langle\overline{\mathcal{S}_{\sigma_f}}\rangle.$$

We will also use standard fusion and untwisting formulas involving skein elements decorated by Jones-Wenzl idempotents for which one can consult [Lic97] and the original reference [MV94].

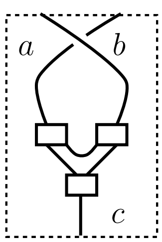


$$= \sum_{\substack{c:(a,b,c)\\ \text{admissible}}} \frac{\triangle_c}{\theta(a,b,c)}$$

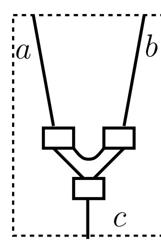


(8)

FIGURE 7. fusion formula: the skein element which locally looks like the left-hand side is equal to the sum of skein elements on the right-hand side with corresponding local replacements.



$$= (-1)^{\frac{a+b-c}{2}} v^{a+b-c+\frac{a^2+b^2-c}{2}}$$



(9)

FIGURE 8. Untwisting formula: the skein element which locally looks like the left-hand side is equal to the skein element on the right-hand side with the local replacement.

We say that a triple (a, b, c) of non-negative integers is admissible if $a + b + c$ is even and $a \le b + c$, $b \le c + a$, and $c \le a + b$. For k a non-negative integer, let $\triangle_k! := \triangle_k \triangle_{k-1} \cdots \triangle_1$, with the convention that $\triangle_0 = 1$. In the pictures above, the function $\theta(a, b, c)$ is defined by

$$\theta(a,b,c) = \frac{\triangle_{x+y+z}!\triangle_{x-1}!\triangle_{y-1}!\triangle_{z-1}!}{\triangle_{y+z-1}\triangle_{z+x-1}!\triangle_{x+y-1}!},$$

where x, y, and z are determined by $a = y + z$, $b = z + x$, and $c = x + y$.

The *degree* of a rational function $L(v)$ is the maximum power of v in the formal Laurent series expansion of $L(v)$ with finitely many positive degree terms.

3. The colored Jones polynomial of pretzel knots

From this point on we will always consider the standard diagram when referring to the pretzel knot $K = K(1/q_0, \ldots, 1/q_m)$, with $|q_i| > 1$. Throughout the section the integer $n \geq 2$ is fixed, and we will illustrate graphically using the example $K(-1/5, 1/3, 1/3, 1/3, 1/5)$.

The colored Jones polynomial for a fixed n of a knot is by Definition 2.8 the Kauffman bracket of the n-blackboard cable of a diagram of K decorated by a Jones-Wenzl idempotent, multiplied by a monomial in v raised to the power of the writhe of the diagram with orientation. We write the colored Jones polynomial as $J_{K,n+1} = (-1)^n(-v)^{\omega(K)n(n+2)}\langle K^n\rangle$.

The Jones-Wenzl idempotent is a sum of tangle diagrams with coefficients rational functions of v in the algebra TL_n. A skein element in $\mathrm{TL}^n_{n'}$ decorated by Jones-Wenzl idempotents is thus also a sum of tangle diagrams with coefficients rational functions of v by locally replacing idempotent with its sum. We extend the tangle sum operation $\oplus$ to skein elements $\mathcal{S}$ decorated by Jones-Wenzl idempotents in TL_{2n}, written

$$\mathcal{S} = \sum_{T \in \mathrm{TL}_{2n}} s(v)T,$$

as

$$\mathcal{S} \oplus \mathcal{S}' = \sum_{T, T' \in \mathrm{TL}_{2n}} s(v)s'(v) T \oplus T'.$$

Graphically, this is will be the same as joining the top right and bottom right $2n$-strands of $\mathcal{S}$ to the top left and bottom left $2n$-strands to $\mathcal{S}'$, with the presence of the idempotent indicating that this is actually a sum of such diagrams in TL_{2n}. Similarly, we extend the numerator closure to skein elements in TL_{2n}.

We will represent the diagram $K^n = N(K^n_- \oplus K^n_+)$ as the numerator closure of two $2n$-tangles decorated by Jones-Wenzl idempotents, with the label n indicating the number of parallel strands. This decomposition of K^n reflects the original splitting of $K = N(K_- \oplus K_+)$ into two 2-tangles K_- and K_+. A twist region is a vertical tangle with a nonzero number of crossings all of the same sign. Let K_- be the negative twist region consisting of $-q_0$ crossings, and K_+ the rest of the diagram K. For a fixed n double the idempotents in K^n so that four are framing the n-cable of the negative twist region consisting of $-q_0$ crossings, and four are framing the n-cable of the rest of the knot diagram. The $2n$-tangle K^n_- is the n-cable of K_- along with the four idempotents, and K^n_+ is the rest of K^n, which is the n-cable of K_+, also decorated with four idempotents. See the middle figure in Figure 9.

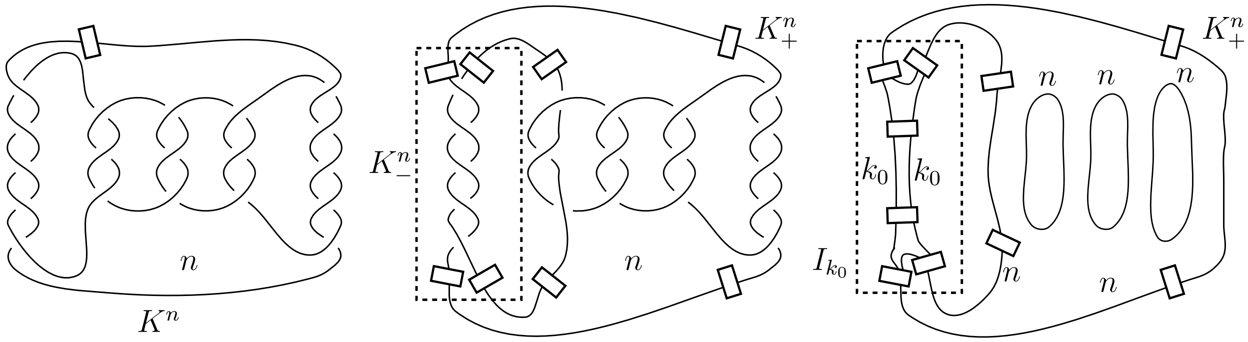


FIGURE 9. From left to right: K^n, doubling the idempotents and the splitting $K^n = N(K^n_- \oplus K^n_+)$, and $N(I_{k_0} \oplus K^n_+)_{\sigma_B}$, where σ_B is the Kauffman state that chooses the B-resolution on all the crossings in K^n_+. The dotted boxes enclose the skein elements in $\mathcal{S}(D^2, 2n, 2n)$, which are sums of $2n$-tangles.

It is convenient to compute the bracket of these $2n$-tangles first. For any tangle T write $\langle T^n \rangle$ to mean cabling each component by a JW idempotent of order n and evaluating in the Temperley-Lieb algebra TL_{2n}.

We write $\langle K^n_- \rangle = \sum_{k_0} G_{k_0}(v) I_{k_0}$ for tangles I_{k_0} with four JW idempotents of size n connected in the middle to a JW idempotent of size $2k_0$ arranged in an I-shape using the fusion and untwisting formulas. Apply the fusion formula to two strands of K^n_- going into (or coming out of) the n-cable negative twist region. Then, apply the untwisting formula to get rid of all the negative crossings. The function $G_{k_0}(v) = \frac{\triangle_{2k_0}}{\theta(n,n,2k_0)}(-1)^{n-k_0} v^{2n-2k_0+n^2-2k_0^2}$ is a rational function that is the product of two coefficient functions in v multiplying the replacement skein elements. The other tangle K^n_+ is expanded into a state sum by taking Kauffman states over all the crossings in K^n_+, leaving the four JW idempotents of size n: Let $(K^n_+)_\sigma$ denote the skein element resulting from applying a Kauffman state σ to all the crossings of K^n_+. Then, $\langle K^n_+ \rangle = \sum_\sigma v^{\mathrm{sgn}(\sigma)} (K^n_+)_\sigma$. The state sum we consider is indexed by pairs (k_0, σ) and we write

$$\langle K^n \rangle = \sum_{(k_0,\sigma)} G_{k_0}(v) v^{\mathrm{sgn}(\sigma)} \langle N(I_{k_0} \oplus (K^n_+)_\sigma) \rangle. \tag{10}$$

See the rightmost figure of Figure 9 for an example of $N(I_{k_0} \oplus (K^n_+)_\sigma)$. Using the notion of through strands, we collect like terms together in our state sum.

Definition 3.1. Consider the Temperley-Lieb algebra $\mathrm{TL}^n_{n'}$ with n inputs and n' outputs. Let T be an element of $\mathrm{TL}^n_{n'}$ with no crossings. Viewing ∂D^2 as a square, an arc in T with one endpoint on the top boundary of the disc D^2 defining $\mathrm{TL}^n_{n'}$ and another endpoint on the bottom boundary is called a *through strand* of T.

We can organize states (k_0, σ) according to the number of through strands at various levels. The *global* number of through strands of σ, denoted by $c = c(\sigma)$, is the number of through strands of $(K^n_+)_\sigma$ in TL_{2n} inside the box framed by four idempotents in K^n_+, see Figure 10 for an example.

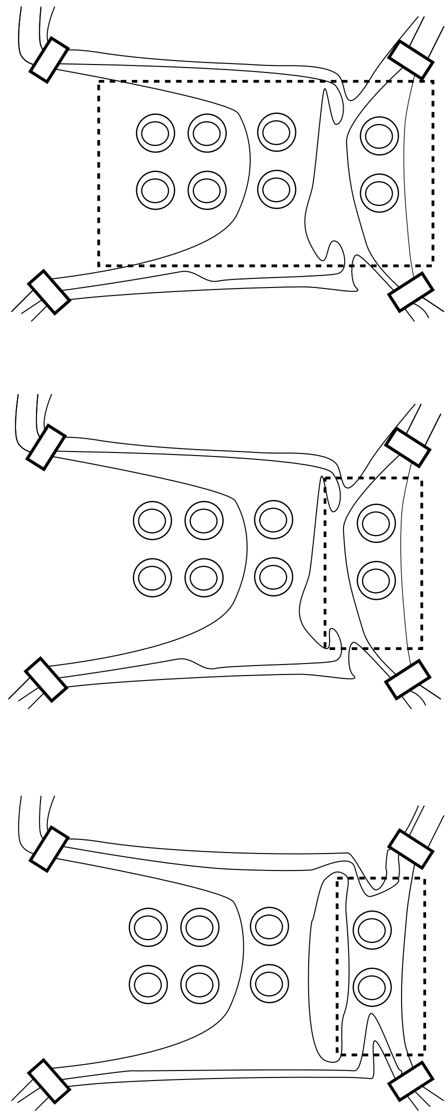


FIGURE 10. An example of $(K_+^n)_\sigma$ with $c(\sigma) = 4$. When restricting σ to the 4th twist region, we have $c_4(\sigma) = k_4(\sigma) = 2$. In the last picture we show an example of a state σ where $c_4(\sigma) = 1$ and therefore $k_4(\sigma) = 1$.

We will also define $c_i(\sigma)$ to be the number of ith *local* through strands when restricting σ to the ith twist region, that are also global through strands. The parameter corresponding to a Kauffman state σ for each twist region, k_i, will be defined as $k_i(\sigma) = \lceil \frac{c_i(\sigma)}{2} \rceil$. The intuition for these parameters is that they will be used to bound the degrees of each term in the state sum relative to each other, which is crucial to determining the degree of the nth colored Jones polynomial $J_{K,n+1}$.

With the notation $k = (k_0, \ldots, k_m)$ we set

$$\mathcal{G}_{c,k} = \sum_{k_0} \sum_{\sigma : k_i(\sigma) = k_i, c(\sigma) = c} G_{k_0}(v) v^{\mathrm{sgn}(\sigma)} \langle N(I_{k_0} \oplus (K_+^n)_\sigma) \rangle. \tag{11}$$

We prove the following theorem.

Note $0 \leq k_i \leq n$ and define the parameters c, k to be *tight* if $k_0 = k_1 + \cdots + k_m = \frac{c}{2}$.

Theorem 3.2. *Assume $|q_i| > 1$ and write $\langle K^n \rangle = \sum_{c,k} \mathcal{G}_{c,k}$ using (10) and (11). For tight c, k we have $\mathcal{G}_{c,k} = (-1)^{q_0(n-k_0)+n+k_0+\sum_{i=1}^m (n-k_i)(q_i-1)} v^{\delta(n,k)} + l.o.t.$[1] and $\delta(n,k) =$*

$$-2\left((q_0+1)k_0^2 + \sum_{i=1}^m (q_i-1)k_i^2 + \sum_{i=1}^m (-2+q_0+q_i)k_i - \frac{n(n+2)}{2}\sum_{i=0}^m q_i + (m-1)n\right) \quad (12)$$

If c, k are not tight then there exists a tight pair c', k' (coming from some Kauffan state) such that $\deg_v \mathcal{G}_{c,k} < \deg_v \mathcal{G}_{c',k'}$.

This theorem will be used in the next section to find the actual degree of $J_{K,n+1}$ using quadratic integer programming.

3.1. **Outline of the proof of Theorem 3.2.** Let $st(c,k)$ be the set of states (k_0, σ) with $c(\sigma) = c$ and $k_i(\sigma) = k_i$ for all i such that the parameters c, k are tight. A state in $st(c,k)$ is said to be *taut* if its term $G_{k_0}(v)v^{\mathrm{sgn}(\sigma)}\langle N(I_{k_0} \oplus (K_+^n)_\sigma)\rangle$ in (11) maximizes the v-degree within $st(c,k)$. For any fixed tight c, k we plan to construct all taut states. The first examples of we construct will be *minimal states*, from which we will derive all taut states. A state in $st(c,k)$ is minimal if it has the least number of A-resolutions.

We will first show that minimal states are characterized by having a certain configuration on the set of crossings where they choose the A-resolution, called *pyramidal*. This will also be used to show that c, k not tight implies $\deg_v \mathcal{G}_{c,k} < \deg_v \mathcal{G}_{c',k'}$ for some tight pair c', k'.

Then, with the construction of all taut states from minimal states, we show that $\delta(n,k)$ is the maximal degree of a taut state with parameters k, and

$$\mathcal{G}^{taut}_{c,k \text{ tight}} = (-1)^{q_0(n-k_0)+n+k_0+\sum_{i=1}^m (n-k_i)(q_i-1)} v^{\delta(n,k)} + l.o.t.,$$

where $\mathcal{G}^{taut}_{c,k \text{ tight}}$ is the double sum of $\mathcal{G}_{c,k}$ only over taut states with tight c, k. This will lead to

$$\mathcal{G}_{c,k \text{ tight}} = (-1)^{q_0(n-k_0)+n+k_0+\sum_{i=1}^m (n-k_i)(q_i-1)} v^{\delta(n,k)} + l.o.t.$$

and conclude Theorem 3.2.

Conventions for representing a Kauffman state. Throughout the rest of Section 3, we will indicate schematically a crossing-less skein element $\mathcal{S}_\sigma$, resulting from applying a Kauffman state to a skein element $\mathcal{S}$ with crossings, by the following convention. Let $\mathcal{S}_B$ be the result of applying the all-B state on the crossings of $\mathcal{S}$. For a Kauffman state σ let A_σ be the set of crossings of $\mathcal{S}$ on which σ chooses the A-resolution. The skein element $\mathcal{S}_\sigma$ is represented by $\mathcal{S}_B$ with colored edges, such that the edge in $\mathcal{S}_B$ corresponding to a crossing in A_σ is colored red, and all other edges remain black. The skein element $\mathcal{S}_\sigma$ may then be recovered by a local replacement of two arcs with a dashed segment. See Figure 11 below.

[1]The abbreviation $l.o.t.$ means lower order terms in v.

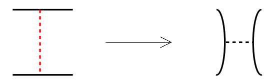


FIGURE 11. A red edge indicates the state where the B-resolution replaces the A-resolution for a Kauffman state σ.

3.2. **Simplifying the state sum and pyramidal position for crossings.** We will denote by $\mathcal{S}(k_0, \sigma)$ the skein element $N(I_{k_0} \oplus (K_+^n)_\sigma)$ as in (11).

Lemma 3.3. Fix (k_0, σ) determining a skein element $\mathcal{S}(k_0, \sigma)$ with $k_i = k_i(\sigma)$ and $c = c(\sigma)$. If $k_0 > \sum_{i=1}^m k_i$, then $\mathcal{S}(k_0, \sigma) = 0$.

Proof. Note that $\sum_{i=1}^m k_i \geq \frac{c}{2}$. Thus if $k_0 > \sum_{i=1}^m k_i$, then $k_0 > \frac{c}{2}$, and the lemma follows from [Lee, Lemma 3.2]. □

With the information of through strands $c(\sigma)$ and $\{k_i(\sigma)\}$, we describe the structure of A_σ for a Kauffman state σ. It is necessary to introduce a labeling of the crossings with respect to their positions in the all-B Kauffman state graph $\mathcal{S}(k_0, B) = N(I_{k_0} \oplus (K_+^n)_B)$.

We first further decompose $K_+^n = \mathcal{S}^t \times \mathcal{S}^w \times \mathcal{S}^b$ where $\times$ is the multiplication by stacking in TL, and let the crossings contained in those skeins be denoted by C^t, C^w, and C^b, respectively. See Figure 12 for an example.

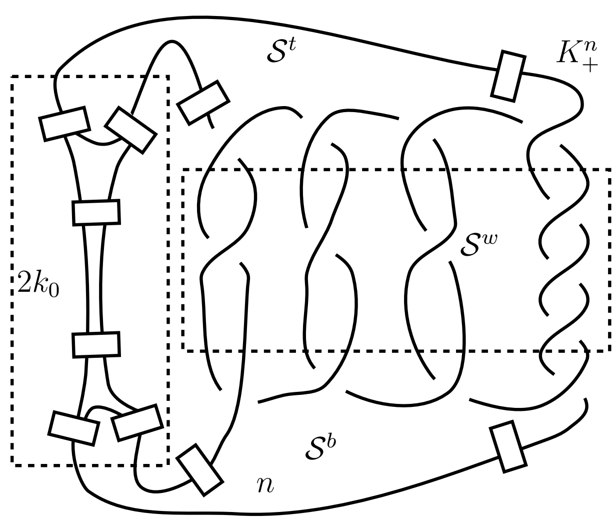


FIGURE 12. Skein element $\mathcal{S} = I_{k_0} \cdot (\mathcal{S}^t \times \mathcal{S}^w \times \mathcal{S}^b)$ of the pretzel knot $K(-1/5, 1/3, 1/3, 1/3, 1/5)$. We have $\mathcal{S}^t \in \mathrm{TL}_{2mn}^{2n}$, $\mathcal{S}^w \in \mathrm{TL}_{2mn}$, and $\mathcal{S}^b \in \mathrm{TL}_{2n}^{2mn}$.

See Figure 13 for a guide to the labeling. The skein element T_B consists of n arcs on top in the region defining $\mathcal{S}^t$, n arcs on the bottom in the region defining $\mathcal{S}^b$, and $q_i - 1$ sets of n circles for the ith twist region in the region defining $\mathcal{S}^w$. The n upper arcs are labeled by $S_1^u, \ldots, S_n^u$, and the n lower arcs are labeled by $S_1^\ell, \ldots, S_n^\ell$, respectively. C_j^u is the set of crossings whose corresponding segments in T_B lie between the arcs S_j^u and S_{j+1}^u. Similarly we define C_j^ℓ by reflection.

For the crossings in the region defining $\mathcal{S}^w$, we divide each state circle into upper and lower half arcs as also shown in Figure 12, and use an additional label s for $1 \leq s \leq q_i$. Thus, the notation $C^{\ell,s}_{i,j}$, where $1 \leq s \leq q_i$ for each twist region with q_i crossings and $1 \leq j \leq n$ indicating a circle in the n-cable, means the crossings between the state circles $S^{\ell,s}_{i,j}$ and $S^{\ell,s}_{i,j+1}$, see Figure 13.

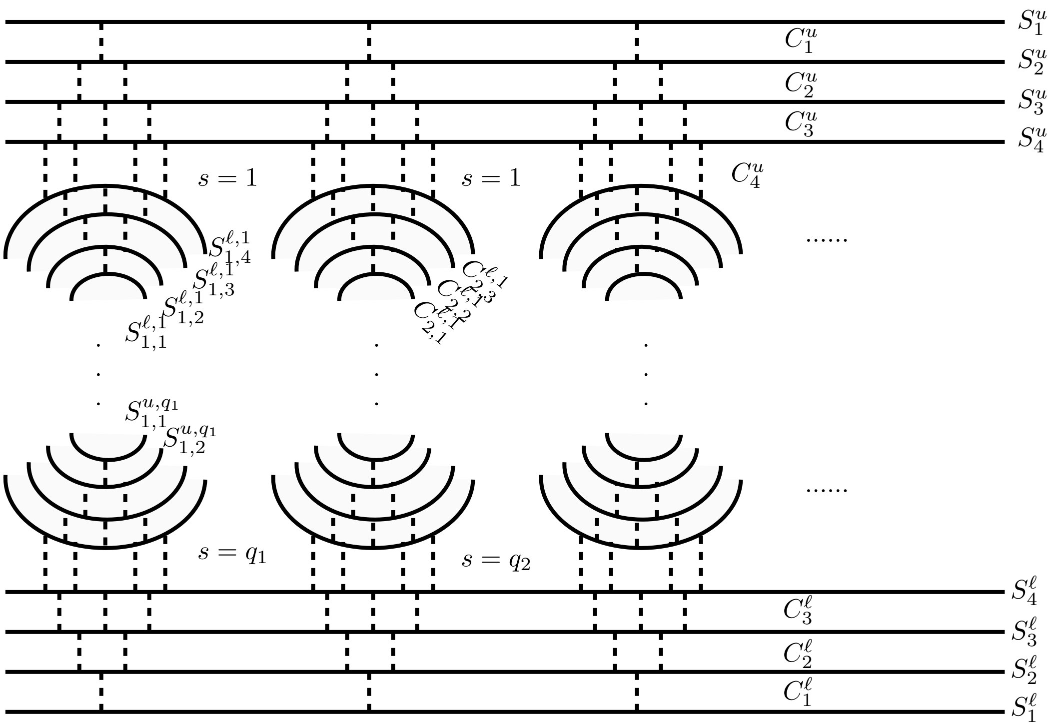


FIGURE 13. Labeling of crossings, arcs, and circles from applying the all-B state on T^n. In this example $n = 4$.

It is helpful to see a local picture at each n-cabled crossing in T^n.

The goal of this subsection is to prove the following theorem.

Theorem 3.4. *Suppose a skein element $\mathcal{S}(k_0, \sigma)$ has parameters $k_i = k_i(\sigma)$ and $c = c(\sigma)$. Then there is a subset $A'_\sigma \subseteq A_\sigma$ of crossings on which the Kauffman state σ chooses the A-resolution, such that we have $A'_\sigma = A^t_\sigma \cup A^w_\sigma \cup A^b_\sigma$ denoting the crossings in the regions determining $\mathcal{S}^t$, $\mathcal{S}^w$, and $\mathcal{S}^b$, respectively, and*

(i) $|A^w_\sigma| = \sum_{i=1}^m (q_i - 2)k_i^2$. *The set $A^w_\sigma = \cup_{i=1}^m \cup_{s=1}^{q_i} \cup_{j=1}^{k_i} (u^s_{i,j} \cup \ell^s_{i,j})$ is a union of crossings with $u^s_{i,j} \subset C^{u,s}_{i,j}$ and $\ell^s_{i,j} \subset C^{\ell,s}_{i,j}$, such that*
  - *For each $n - k_i + 1 \leq j \leq n$, $u^s_{i,j}$, $\ell^s_{i,j}$ each has $j - n + k_i$ crossings.*
  - *For each $n - k_i + 2 \leq j \leq n$ and a pair of crossings x, x' in $u^s_{i,j}$ (resp. $\ell^s_{i,j}$) whose corresponding segments e, e' in T_B are adjacent (i.e., there is no other edge in $u^s_{i,j}$ between e and e'), there is a crossing x'' in $u^s_{i,j-1}$ (resp. $\ell^s_{i,j-1}$), where the end of the corresponding segment e'' on $S^s_{i,j}$ lies between the ends of e and e'.*

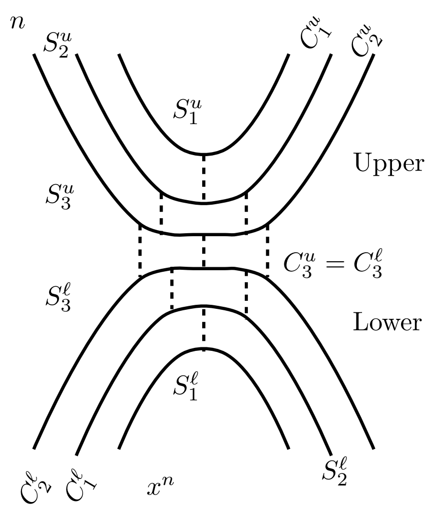


FIGURE 14. Local labeling of n^2 crossings on the all-B state of an n-cabled crossing. In this example $n = 3$.

(ii) $|A^t_\sigma| = |A^b_\sigma| = \frac{c^2/4 - c/2 + \sum_{i=1}^m (k_i^2 + k_i)}{2}$. *The set* $A^t_\sigma = \cup_{j=n-c/2+1}^n u_j$ *is a union of crossings* $u_j \subset C_j^u$, *and the set* $A^b_\sigma = \cup_{j=n-c/2+1}^n \ell_j$ *is a union of crossings* $\ell_j \subset C_j^\ell$ *satisfying:*
  – *For* $n - \frac{c}{2} + 1 \leq j \leq n$, u_j *(resp.* ℓ_j*) has* $j - n + \frac{c}{2}$ *crossings.*
  – *For each* $n - \frac{c}{2} + 2 \leq j \leq n$ *and a pair of crossings* x, x' *in* u_j *(resp.* ℓ_j*) whose corresponding segments* e, e' *in* T_B *are adjacent (i.e., there is no other crossing in* u_j *whose corresponding segment is between* e *and* e'*), there is a crossing* x'' *in* u_{j-1} *(resp.* ℓ_{j-1}*), where the end of the corresponding segment* e'' *on* S_j^u *(resp.* S_j^ℓ*) lies between the ends of* e *and* e'.

It follows that $|A'_\sigma| = |A^t_\sigma| + |A^w_\sigma| + |A^b_\sigma| = \frac{c^2}{4} - \frac{c}{2} + \sum_{i=1}^m (k_i^2 + k_i) + \sum_{i=1}^m (q_i - 2)k_i^2$. The set of crossings A'_σ is said to be in *pyramidal position*.

Proof. Statement (i) is a direct application to every set of n-cabled crossings in each twist region of $\mathcal{S}^w$ of the following result from [Lee].

Lemma 3.5. [Lee, Lem. 3.7] Let $\mathcal{S}$ be a skein element in TL_{2n} consisting of a single n-cabled positive crossing x^n with labels as shown in Figure 14.

If $\mathcal{S}_\sigma$ for a Kauffman state σ on x^n has $2k$ through strands, then σ chooses the A-resolution on a set of k^2 crossings C_σ of x^n, where $C_\sigma = \cup_{j=n-k}^n (u_j \cup \ell_j)$ is a union of crossings $u_j \subseteq C_j^u$ and $\ell_j \subseteq C_j^\ell$, such that

- For each $n - k + 1 \leq j \leq n$, u_j, ℓ_j each has $j - n + k$ crossings.
- For each $n - k + 2 \leq j \leq n$, and a pair of crossings x, x' in u_j (resp. ℓ_j) whose corresponding segments c, c' in the all-B state of x^n are adjacent (i.e., there is no other edge in C_σ between c and c'), there is a crossing x'' in u_{j-1} (resp. ℓ_{j-1}), where the end of the corresponding segment c'' on S_j^u (resp. S_j^ℓ) lies between the ends of c and c'.

The same proof applies to the crossings in the strip $\mathcal{S}^t$, see Figure 15. Reflection with respect to the horizontal axis will show (ii) for $\mathcal{S}^b$.

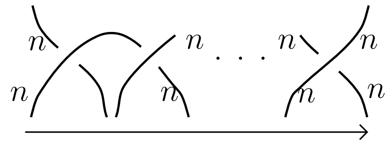


FIGURE 15. The arrow indicates the direction from left to right of the crossings in $\mathcal{S}^t$.

□

We will now apply what we know about the crossings on which a state σ chooses the A-resolution from Theorem 3.4 to construct degree-maximizing states for given global through strands $c(\sigma)$ and parameters $\{k_i(\sigma)\}$. See Figure 16 for an example of a pyramidal position of crossings.

FIGURE 16. A minimal state τ is shown with $n = 3$ and $c(\tau) = 6$ global through strands. From the representation on the left one can see the pyramidal position of the crossings A'_τ as described by Theorem 3.4. The skein element $\mathcal{S}(k_0, \tau)$ with $k = (k_0, 0, 0, 2, 1)$ resulting from applying τ is shown on the right.

3.3. **Minimal states are taut and their degrees are $\delta(n, k)$.** The contribution of the state (k_0, σ) to the state sum is $G_{k_0}(v)v^{\mathrm{sgn}(\sigma)}\langle N(I_{k_0} \oplus (K_+^n)_\sigma)\rangle$ as in (11). We denote its v-degree by $d(k_0, \sigma)$.

Recall the skein element $\mathcal{S}(k_0, \sigma) = N(I_{k_0} \oplus (K_+^n)_\sigma)$. Also recall A_σ denotes the set of crossings on which σ chooses the A-resolution, and $|A_\sigma|$ is the number of crossings in A_σ. A *minimal state* with tight parameters c, k $(k_0 = k_1 + \cdots + k_m = \frac{c}{2})$ has the least $|A_\sigma|$. Let $\mathrm{o}(A_\sigma)$ denote the number of circles of $\overline{\mathcal{S}(k_0, \sigma)}$, which is the skein element obtained by replacing all the JW idempotents in $\mathcal{S}(k_0, \sigma)$ by the identity, respectively.

Lemma 3.6. A minimal state (k_0, τ) with $c(\tau)$ through strands and tight c, k has A_τ in pyramidal position as specified in Theorem 3.4 and distance $|A_\tau|$ from the all-B state given by

$$|A_\tau| = 2\left((\sum_{i=1}^m k_i)\frac{(\sum_{i=1}^m k_i - 1)}{2} + \sum_{i=1}^m \frac{k_i(k_i+1)}{2} \right) + \sum_{i=1}^m (q_i - 2)k_i^2 \,.$$

Moreover,

$$G_{k_0}(v)v^{\mathrm{sgn}(\sigma)}\langle I_{k_0} \cdot T_\sigma^n\rangle = (-1)^{q_0(n-k_0)+n+k_0+\sum_{i=1}^m (n-k_i)(q_i-1)} v^{\delta(k,n)} + l.o.t. \tag{13}$$

Proof. Observe that minimal states τ have corresponding crossings A_τ in pyramidal position. Moreover, if A_τ is pyramidal, then $|A_\tau|$ determines the number of circles $\mathrm{o}(A_\tau)$. The skein element $\mathcal{S}(k_0, \tau)$ is adequate as long as $k_0 \leq \sum_i^m k_i$, thus by [Arm13, Lem. 4], we have

$$\deg v^{\mathrm{sgn}(\tau)}\langle \mathcal{S}(k_0, \tau)\rangle = \deg v^{\mathrm{sgn}(\tau)}\langle \overline{\mathcal{S}(k_0, \tau)}\rangle \,,$$

and we simply need to determine the number of circles in $\overline{\mathcal{S}(k_0,\tau)}$ and $\operatorname{sgn}(\tau)$ in order to compute the degree of the Kauffman bracket. This is completely specified by the pyramidal configuration of A_τ by just applying the Kauffman state. With the assumption that $k_0 = \sum_{i=1}^m k_i = \frac{c}{2}$ since c, k is tight, the degree is then

$$
\begin{aligned}
d(k_0,\tau) = {} & \underbrace{\sum_{i=1}^m q_i n^2 - 2(2\left(\frac{\left(\sum_{i=1}^m k_i\right)\left(\left(\sum_{i=1}^m k_i\right)-1\right)}{2} + \sum_{i=1}^m \frac{k_i(k_i+1)}{2}\right) + \sum_{i=1}^m (q_i-2)k_i^2)}_{\operatorname{sgn}(\tau)} \\
& + \underbrace{2\left(2n - (\left(\sum_{i=1}^m k_i\right) - k_0) + \sum_{i=1}^m (n-k_i)(q_i-1)\right)}_{2o(A_\tau)} \\
& + \underbrace{q_0(2n - 2k_0 + \frac{2n^2-4k_0^2}{2}) + 2k_0 - 2n}_{\text{fusion and untwisting}}.
\end{aligned}
$$

The sign of the leading term is given by

$$
(-1)^{\overbrace{q_0(n-k_0)+n+k_0}^{\text{fusion and untwisting}}+o(A_\tau)} = (-1)^{q_0(n-k_0)+n+k_0+\sum_{i=1}^m (n-k_i)(q_i-1)}.
$$

□

Lemma 3.7. Minimal states are taut. In other words, given c, k tight, we have

$$
\max_{\sigma: c(\sigma)=c, k_i(\sigma)=k_i} d(k_0,\sigma) = d(k_0,\tau),
$$

where τ is a minimal state with $c(\tau) = c$ and $k_i(\tau) = k_i$.

Proof. Note that for *any* state σ with corresponding skein element $\mathcal{S}(k_0,\sigma)$

$$
A_\tau \subseteq A_\sigma
$$

for a minimal state τ with the same parameter set (c, k) by Theorem 3.4 and $d(k_0,\tau) = d(k_0,\tau')$ for two minimal states τ, τ' with the same parameters $c(\tau) = c(\tau')$ and $k_i(\tau) = k_i(\tau')$ by Lemma 3.6. This implies $d(k_0,\sigma) \leq d(k_0,\tau)$. □

3.3.1. *Constructing minimal states.*

Lemma 3.8. A minimal state exists for any tight c, k, where c is an even integer between 0 and $2n$ and $k_0 = \sum_{i=1}^m k_i = \frac{c}{2}$.

Proof. It is not hard to see that at an n-cabled crossing x^n in a twist region with q_i crossings, for any $0 \leq k_i \leq n$ there is always a minimal state giving $2k_i$ through strands. For an n-cabled crossing x^n in $\mathcal{S}^t$ or $\mathcal{S}^b$, it is also not hard to see that we may take the pyramidal position P for the minimal state for the bottom half (or upper half, for $\mathcal{S}^b$) of the crossings in x^n in C_n^u and $C_{i,j}^{\ell,1}$ for each twist region.

What remains to be shown is that a minimal state always exists, given the set of parameters $\{k_i\}$ and c total through strands for crossings in the top and bottom strips delimited by

$\{S^u_j\}_{j=1}^n$ and $\{S^\ell_j\}_{j=1}^n$. To see this, we take the leftmost configuration with $\{k_i\}$ through strands for the bottom half of the crossings in x^n for each twist region, which we already know to exist. Given two crossings x and x' in C^u_n whose corresponding segment in $\mathcal{S}(k_0, B)$ has ends on S^u_n we can always find another crossing x'' in C^u_{n-1} , the end of whose corresponding segment on S^u_n lies between those of x and x', because the previously chosen crossings in C^u_n are leftmost. Pick the leftmost possible and repeat to choose crossings in C^u_j for $n-k+1 \leq j \leq n-2$. We pick crossings in the bottom strip by reflection. $\square$

Lemma 3.9. Let σ be a state with $c = c(\sigma)$ and $k_i = k_i(\sigma)$ which is not tight, that is, $\sum_{i=1}^m k_i > \frac{c}{2}$ or $k_0 < \frac{c}{2}$, then $d(k_0, \sigma) < d(k_0, \tau)$, where τ is a minimal state with $c(\tau) = c$ through strands.

Proof. For the case $\sum_{i=1}^m k_i > \frac{c}{2}$, we can apply Theorem 3.4 to conclude that there is a minimal state τ (there may be multiple such states) such that

$$A_\tau \subset A_\sigma,$$

with $k_i(\tau) \leq k_i(\sigma)$ for each i. There must be some i for which $k_i(\tau) < k_i(\sigma)$. Applying the B-resolution to the additional crossings to obtain a sequence of states from τ to σ, we see that it must contain two consecutive terms that merge a pair of circles.

If $k_0 < \frac{c}{2}$ with similar arguments we can see that $o(A_\sigma) < o(A_\tau)$. $\square$

3.4. **Enumerating all taut states.** By Lemma 3.7, we have shown that every taut state contains a minimal state. Next we show that every taut state is obtained from a unique such minimal state τ by changing the resolution from B-to A-on a set of crossings F_τ. We show that any taut state σ with $c(\sigma) = c(\tau)$ and $k_i(\sigma) = k_i(\tau)$ containing τ as the *leftmost* minimal state, to be defined below, satisfies $A_\sigma = A_\tau \cup p$, where p is any subset of F_τ.

All the circles here in the definitions and theorems are understood with possible extra labels u, ℓ, s, i, j indicating where they are in the regions defining $\mathcal{S}^t, \mathcal{S}^w$, and $\mathcal{S}^b$. To simplify notation we do not show these extra labels.

Definition 3.10. For each $x \in A_\tau$ between S_{j-1} and S_j, let R_x be the set of crossings to the right of x between S_{j-1} and S_j, but to the left of any $x' \in A_\tau$ between S_{j-2} and S_{j-1}, and any $x'' \in A_\tau$ between S_j and S_{j+1}. We define the following possibly empty subset F_τ of crossings of K^n.

$$F_\tau := \cup_{x \in A_\tau} R_x \,.$$

See Figure 17 and 18 for examples.

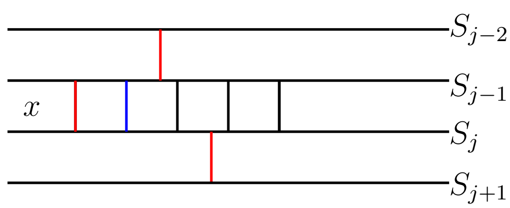


FIGURE 17. Only the blue edge is in R_x because of the presence of the top and bottom red edges.

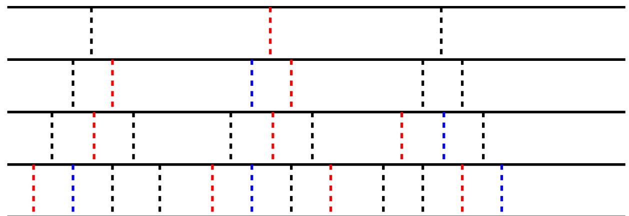


FIGURE 18. An example of F_τ with edges shown in blue with the minimal state τ shown as red edges.

Definition 3.11. Given a set of crossings C of K^n, a crossing $x \in C$, and $1 \leq j \leq n$, define the *distance* $|x|_C$ *of a crossing* $x \in C$ *from the left* to be

$$|x|_C := \text{For } x \in C_j, \text{ the \# of edges in } \mathcal{S}(k_0, B) \text{ to the left of } x \text{ between } S_j \text{ and } S_{j+1},$$

the *distance* of the set C from the left is

$$\sum_{x \in C} |x|_C \,.$$

Given any state σ with tight parameters c, k, we extract the *leftmost* minimal state τ_σ where $A_{\tau_\sigma} \subseteq A_\sigma$, i.e., there is no other minimal state τ' such that $A_{\tau'} \subset A_\sigma$, and the distance of $A_{\tau'}$ from the left is less than the distance of A_{τ_σ} from the left.

FIGURE 19. On the left, a taut state having the same degree as a minimal state but is not equal to it. We have $c = 6$, $k_1 = 0$, $k_2 = 0$, $k_3 = 2$ and $k_4 = 1$ as the minimal state in Figure 16, and the thickened red edges indicate the difference from a minimals state with the same parameters. Choosing the B-resolution at each of the thickened red edges splits off a circle.

Lemma 3.12. A Kauffman state σ with tight parameters $c(\sigma), \{k_i(\sigma)\}$ is taut if and only if A_σ may be written as

$$A_\sigma = A_{\tau_\sigma} \cup p$$

where τ_σ is the leftmost minimal state from σ such that $A_{\tau_\sigma} \subseteq A_\sigma$, and p is a subset of F_{τ_σ}. See Figure 19 for an example of a taut state that is not a minimal state, and how it is obtained from the leftmost minimal state that it contains.

Proof. By construction, if a state σ is such that

$$A_\sigma = A_{\tau_\sigma} \cup p$$

where p is a subset of F_{τ_σ}, then σ is a taut state.

Conversely, suppose by way of contradiction that σ is taut, which means that it has the same parameters (c, k) as its leftmost minimal state τ_σ with the same degree, but that there is a crossing $x \in A_\sigma$ and $x \notin F_{\tau_\sigma}$. Then there are two cases

(1) x is to the left or to the right of all the edges in A_{τ_σ}.
(2) $x \in C_j$ is between $x', x'' \in C_j$ in A_{τ_σ} for some j.

In both cases we consider the state σ' where

$$A_{\sigma'} = A_\tau \cup \{x\}\,,$$

and we assume that taking the A-resolution on x splits off a circle from the skein element $\overline{\mathcal{S}(k_0, \sigma)}$ otherwise by Lemma 2.12,

$$\deg v^{\mathrm{sgn}(\sigma)}\langle\overline{\mathcal{S}(k_0,\sigma)}\rangle < \deg v^{\mathrm{sgn}(\tau_\sigma)}\langle\overline{\mathcal{S}(k_0,\tau_\sigma)}\rangle\,,$$

a contradiction to σ being taut.

In case (1), the state σ' has parameters (c, k') such that $\sum_{i=1}^m k'_i < \sum_i^m k_i$. If each step of a sequence from σ' to σ splits a circle in order to maintain the degree, then the parameters for σ, and hence the number of global through strands of $\mathcal{S}(k_0, \sigma)$ will differ from $\mathcal{S}(k_0, \tau_\sigma)$, a contradiction.

In case (2), we have that $x \notin F_{\tau_\sigma}$ must be an edge of the following form between a pair of edges x', x'' as indicated in the generic local picture shown in Figure 20, since τ_σ is assumed to be leftmost.

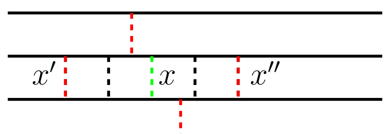


FIGURE 20. The crossing x corresponds to the green edge.

Choosing the A-resolution at x merges a pair of circles which means that $d(k_0, \sigma) < d(k_0, \tau_\sigma)$, a contradiction. □

3.5. **Adding up all taut states in $st(c, k)$.** Note that in general there may be many taut states σ with fixed parameters k.

Theorem 3.13. *Let $c, k = \{k_i\}_{i=1}^m$ be tight. The sum*

$$\sum_{\sigma \text{ taut}: c(\sigma)=c, k_i(\sigma)=k_i} v^{\mathrm{sgn}(\sigma)}\langle\mathcal{S}(k_0,\sigma)\rangle = (-1)^{q_0(n-k_0)+n+k_0+\sum_{i=1}^m (n-k_i)(q_i-1)} v^{d(k_0,\tau)} + l.o.t., \quad (14)$$

where τ is a minimal state in the sum.

We are finally ready to prove Theorem 3.13.

Proof. Every minimal state with parameters c, k may be obtained from the leftmost minimal state of the entire set of minimal states $\mathcal{M}$ by transposing to the right. Now we organize the sum (14) by putting it into equivalence classes of states indexed by the leftmost minimal state τ_σ. We may write

$$\sum_{\sigma \text{ taut}: c(\sigma)=c, k_i(\sigma)=k_i} v^{\mathrm{sgn}(\sigma)}\langle\mathcal{S}(k_0,\sigma)\rangle = \sum_{\tau \text{ minimal}} \sum_{\sigma\,:\,\tau_\sigma=\tau} v^{\mathrm{sgn}(\sigma)}\langle\mathcal{S}(k_0,\sigma)\rangle.$$

By Lemma 3.12, this implies

$$\sum_{\sigma \text{ taut}: c(\sigma)=c, k_i(\sigma)=k_i} v^{\mathrm{sgn}(\sigma)} \langle \mathcal{S}(k_0,\sigma)\rangle = \sum_{\tau \text{ minimal}} \sum_{j=0}^{|F_\tau|} \binom{n}{j} v^{\mathrm{sgn}(\tau)-2j}(-v^2-v^{-2})^{o(A_\tau)+j}.$$

If $F_\tau \neq \emptyset$, then by a direct computation,

$$\begin{aligned}\deg\left(\sum_{j=0}^{|F_\tau|}\binom{n}{j} v^{\mathrm{sgn}(\tau)-2j}(-v^2-v^{-2})^{o(A_\tau)+j}\right) &= \mathrm{sgn}(\tau)+2o(A_\tau)-4|F_\tau| \\ &< \deg\left(v^{\mathrm{sgn}(\tau)}\langle \mathcal{S}(k,\tau)\rangle\right) = \delta(n,k)\end{aligned}$$

by Lemma 3.6.

Every taut state can be grouped into a nontrivial canceling sum except for the rightmost minimal state. Thus it remains and determines the degree of the sum. □

3.6. **Proof of Theorem 3.2.** Recall that $J_{K,n+1} = \sum_{c,k} \mathcal{G}_{c,k}$ and

$$\mathcal{G}_{c,k} = \sum_{k_0} G_{k_0}(v) \sum_{\sigma: k_i(\sigma)=k_i, c(\sigma)=c} v^{\mathrm{sgn}(\sigma)}\langle N(I_{k_0} \oplus (K_+^n)_\sigma)\rangle$$

By the fusion and untwisting formulas we have

$$G_{k_0}(v) = (-1)^{q_0(k_0+n)} \frac{\triangle_{2k_0}}{\theta(n,n,2k_0)} v^{q_0(2n-2k_0+n^2-k_0^2)}.$$

We apply the previous lemmas to compute for each c, k the v-degree of the sum

$$\sum_{\sigma: k_i(\sigma)=k_i, c(\sigma)=c} v^{\mathrm{sgn}(\sigma)}\langle N(I_{k_0} \oplus (K_+^n)_\sigma)\rangle.$$

When c, k is tight the top degree part of the sum is $\mathcal{G}_{c,k}^{taut}$. By Theorem 3.13, we have that the coefficient and the degree of the leading term are given by a minimal state τ with parameters c, k. The degree is computed to be $\delta(n,k)$ in Lemma 3.6, which also determines the leading coefficient.

When σ is a state such that c, k is not tight, and $k_0 \geq c(\sigma)$ or $k_0 \geq \sum_{i=1}^m k_i(\sigma)$, Lemma 3.3 says that $\mathcal{S}(k_0,\sigma)$ is zero. Otherwise, Lemma 3.9 says that there exists a taut state corresponding to a tight $\tilde{c}, \tilde{k}$ that has strictly higher degree. □

4. Quadratic integer programming

In this section we collect some facts regarding real and lattice optimization of quadratic functions.

4.1. **Quadratic real optimization.** We begin with considering the well-known case of real optimization.

Lemma 4.1. Suppose that A is a positive definite $m \times m$ matrix and $b \in \mathbb{R}^m$. Then, the minimum

$$\min_{x \in \mathbb{R}^m} \frac{1}{2} x^t A x + b \cdot x \tag{15}$$

is uniquely achieved at $x = -A^{-1}b$ and equals $-\frac{1}{2} b^t A b$.

Proof. The function is proper with the only critical point at $x = -A^{-1}b$ which is a local minimum since the Hessian of A is positive definite. □

For a vector $v \in \mathbb{R}^m$, we let v_i for $i = 1, \dots, m$ denote its ith coordinate, so that $v = (v_1, \dots, v_m)$. When v_i's are nonzero for all i, we set $v^{-1} = (v_1^{-1}, \dots, v_m^{-1})$.

The next lemma concerns optimization of convex separable functions $f(x)$, that is, functions of the form

$$f(x) = \sum_{i=1}^m f_i(x_i), \qquad f_i(x_i) = a_i x_i^2 + b_i x_i \tag{16}$$

where $a_i > 0$ and b_i are real for all i. The terminology follows Onn [Onn10, Sec.3.2].

Lemma 4.2. (a) Fix a separable convex function $f(x)$ as in (16) and a real number $t \in \mathbb{R}$. Then the minimum

$$\min\{f(x) \mid \sum_i x_i = t, \; x \in \mathbb{R}^m\} \tag{17}$$

is uniquely achieved at $x^*(t)$ where

$$x_i^*(t) = \frac{a_i^{-1} t + \frac{1}{2} \sum_j (b_j - b_i) a_i^{-1} a_j^{-1}}{\sum_j a_j^{-1}}, \tag{18}$$

and

$$f(x^*(t)) = \frac{1}{1 \cdot a^{-1}} t^2 + \frac{b \cdot a^{-1}}{1 \cdot a^{-1}} t + s_0(a, b) \tag{19}$$

where $1 \in \mathbb{Z}^m$ denotes the vector with all coordinates equal to 1, and $s_0(a, b)$ is a rational function in coordinates $a_1, \dots, a_m$ and $b_1, \dots, b_m$.

(b) If $t \gg 0$, then the minimum

$$\min\{f(x) \mid \sum_i x_i = t, \; x \in \mathbb{R}^m, \; 0 \le x_i, \;\; i = 1, \dots, m\} \tag{20}$$

is uniquely achieved at (18) and given by (19).

Note that the coordinates of the minimizer $x^*(t)$ are linear functions of t for $t \gg 0$; we will call such minimizers linear. It is obvious that the minimal value is then quadratic in t for $t \gg 0$.

Proof. Let $f(x) = \sum_j a_j x_j^2 + b_j x_j$ and $g(x) = \sum_j x_j$ and use Lagrange multipliers.

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = t \,. \end{cases}$$

So, $2a_jx_j + b_j = \lambda$ for all j, hence $x_j + b_j/(2a_j) = \lambda/(2a_j)$ for all j. Summing up, we get $t + \sum_j b_j/(2a_j) = \lambda \sum_j 1/(2a_j)$. Solving for λ, we get $\lambda = \frac{2t+\sum_j b_j a_j^{-1}}{\sum_j a_j^{-1}}$ and using

$$x_i = \frac{\lambda - b_i}{2a_i} = \frac{2t + \sum_j (b_j - b_i)a_j^{-1}}{2a_i \sum_j a_j^{-1}} = \frac{a_i^{-1}t + \frac{1}{2}\sum_j (b_j - b_i)a_i^{-1}a_j^{-1}}{\sum_j a_j^{-1}},$$

Equation (18) follows. Observe that $x^*(t)$ is an affine linear function of t. It follows that $f(x^*(t))$ is a quadratic function of t. An elementary calculation gives (19) for an explicit rational function $s_0(a, b)$.

If in addition $t \gg 0$ observe that $x^*(t) = \frac{t}{1\cdot a^{-1}} a^{-1} + O(1)$, therefore $x^*(t)$ is in the simplex $x_i \geq 0$ for all i and $\sum_i x_i = t$. The result follows. □

4.2. **Quadratic lattice optimization.** In this section we discuss the lattice optimization problem

$$\min\{f(x) \mid Ax = t,\ x \in \mathbb{Z}^m,\ 0 \leq x \leq t\} \tag{21}$$

for a nonnegative integer t, where $A = (1, 1, \dots, 1)$ is a $1 \times m$ matrix and $f(x)$ is a convex separable function (16) with $a, b \in \mathbb{Z}^m$ with $a > 0$. We will follow the terminology and notation from Onn's book [Onn10]. In particular the set $x \in \mathbb{Z}^m$ satisfying the above conditions $Ax = t$ and $0 \leq x_i \leq t$ is called the feasible set. Lemma 3.8 of Onn [Onn10] gives a necessary and sufficient condition for a lattice vector x to be optimal. In the next lemma, suppose that a feasible $x \in \mathbb{Z}^m$ is non-degenerate, that is, $x_i < t$ and $x_j > 0$ for all i, j. Note that this is not a serious restriction since otherwise the problem reduces to a lattice optimization problem of the same shape in one dimension less.

Lemma 4.3. [Onn10] Fix a feasible $x \in \mathbb{Z}^m$ which is non-degenerate. Then it is optimal (i.e., a lattice optimizer for the problem (21)) if and only if it satisfies the certificate

$$2(a_ix_i - a_jx_j) \leq (a_i + a_j) - (b_i - b_j). \tag{22}$$

Proof. Lemma 3.8 of Onn [Onn10] implies that x is optimal if and only if $f(x) \leq f(x + g)$ for all $g \in G(A)$ where $G(A)$ is the Graver basis of A. In our case, the Graver basis is given by the roots of the A_{m-1} lattice, i.e., by

$$G((1, 1, \dots, 1)) = \{e_j - e_i \mid 1 \leq i, j \leq m,\ i \neq j\}.$$

Let $g = e_j - e_i \in G(A)$ and $f(x)$ as in (16). Then $f(x) \leq f(x + g)$ is equivalent to (22). □

Below, we will call a vector quasi-linear if its coordinates are linear quasi-polynomials.

Proposition 4.4. (a) Every non-degenerate lattice optimizer $x^*(t)$ of (21) is quasi-linear of the form

$$x_i^*(t) = \frac{a_i^{-1}}{\sum_j a_j^{-1}} t + c_i(t) \tag{23}$$

for some ϖ-periodic functions c_i, where

$$\varpi = \sum_i \prod_{j \neq i} a_j. \tag{24}$$

(b) When $t \gg 0$ is an integer, the minimum value of (21) is a quadratic quasi-polynomial

$$\frac{1}{1 \cdot a^{-1}} t^2 + \frac{b \cdot a^{-1}}{1 \cdot a^{-1}} t + s_0(a,b)(t) \tag{25}$$

where $s_0(a,b)$ is a ϖ-periodic function.

Note that in general there are many minimizers of (21). Comparing with (18) it follows that any lattice minimizer of (21) is within $O(1)$ from the real minimizer.

Proof. Let $A_i = \prod_{j \neq j} a_j = a_1 \dots \hat{a}_j \dots a_m$, then $\varpi = A_1 + \cdots + A_m$. Suppose x^* satisfies the optimality criterion (22) and $Ax^* = t$ where $A = (1, 1, \dots, 1)$. Let $x^{**} = x^* + (A_1, \dots, A_m)$. Since $a_i A_i - a_j A_j = 0$ for $i \neq j$, it follows that

$$2(a_i x_i^* - a_j x_j^*) = 2(a_i x_i^{**} - a_j x_j^{**}) \,.$$

Hence x^* satisfies the optimality criterion (22) if and only if x^{**} does. Moreover, $Ax^{**} = Ax^* + \varpi = t + \varpi$. Since $a_i^{-1}/(\sum_j a_j^{-1}) = A_i/\varpi$, it follows that every minimizer $x^*(t)$ satisfies the property that $x_i^*(t) - \frac{a_i^{-1}}{\sum_j a_j^{-1}} t$ is a ϖ-periodic function of t. Part (a) follows. For part (b), write $x^*(t) = \frac{t}{1 \cdot a^{-1}} a^{-1} + c(t)$ and use the fact that $Ac(t) = 0$ to deduce that $f(x^*(t))$ is a quadratic quasi-polynomial of t with constant quadratic and linear term given by (2) $\square$

4.3. **Application: the degree of the colored Jones polynomial.** Recall that our aim is to compute the maximum of the degree function $\delta(k) = \delta(k,n)$ of the states in the state sum of the colored Jones polynomial with tight parameters $k_0 = \sum_{i=1}^m k_i$, see Theorem 3.2. Here $k = (k_0, k_1, \dots, k_m)$ and $q = (q_0, q_1, \dots, q_m)$ are $(m+1)$-vectors and we make use of the assumption that q_i is odd for all $0 \leq i \leq m$. We will compute the maximum in two steps,

Step 1: We will apply Proposition 4.2 to the function $\delta(k)$ (divided by -2, and ignoring the terms that depend on n and q but not on k):

$$-\frac{1}{2}\delta(k) = \sum_{i=1}^m (q_i - 1) k_i^2 + (q_0 + 1)\Big(\sum_{i=1}^m k_i\Big)^2 + \sum_{i=1}^m k_i(-2 + q_0 + q_i) \,. \tag{26}$$

under the usual assumptions that $q_0 < 0$, $q_i > 0$ for $i = 1, \dots, m$. We assume that $k = (k_1, \dots, k_m) \in \mathbb{Z}^m$. Restricting $\delta(k)$ to the simplex $k_i \geq 0$ and $t = k_1 + \cdots + k_m$ and using Proposition 4.4, it follows that

$$\min_{\substack{k_i \geq 0 \\ \sum_i k_i = t}} \delta(k) = Q_0(t), \qquad \text{where} \qquad Q_0(t) = s(q)t^2 + s_1(q)t + s_0(q)(t) \,, \tag{27}$$

and $s(q)$, $s_1(q)$ are given by (2) and $s_0(q)$ is a ϖ-periodic function of t where ϖ is the denominator of $s(q)$.

Step 2: Since

$$\min_{\substack{k_i \geq 0 \\ \sum_i k_i \leq n}} \delta(k) = \min_{0 \leq t \leq n} Q_0(t) \,,$$

it remains to compute the minimum

$$\min_{0 \leq t \leq n} Q_0(t)$$

of a quadratic function of t (the fact that this is a quasi-polynomial whose constant term is a periodic function of t does not affect the argument, since we can work in a fixed congruence). It follows that $Q_0(t)$ is positive definite, degenerate, or negative definite if and only if $s(q) > 0$, $s(q) = 0$, or $s(q) < 0$, respectively.

Case 1: $s(q) < 0$. Then $Q_0(t)$ is negative definite and the minimum is achieved at the boundary $t = n$ (since this has lower value than that of $t = 0$). It follows that

$$\min_{\substack{k_i \geq 0 \\ \sum_i k_i \leq n}} \delta(k) = s(q)n^2 + s_1(q)n + s_0(q)(n)\,.$$

Case 2a: $s(q) = 0$, $s_1(q) \neq 0$. Then $Q_0(t)$ is a linear function of t and the minimum is achieved at $t = 0$ or $t = n$ depending on whether $s_1(q) \geq 0$ or $s_1(q) < 0$, so we have

$$\min_{\substack{k_i \geq 0 \\ \sum_i k_i \leq n}} \delta(k) = \begin{cases} s_0(q)(n) & \text{if } s_1(q) \geq 0 \\ s_1(q)n + s_0(q)(n) & \text{if } s_1(q) < 0\,. \end{cases}$$

Case 2b: $s(q) = 0 = s_1(q)$. Now $t = 0$ and $t = n$ both contribute equally so cancellation may occur. It does not because the sign of the leading term is constant due to the parity of the q_i's.

Case 3: $s(q) > 0$. Then $Q_0(t)$ is positive definite and Proposition 4.4 implies that the lattice minimizers are near $-s_1(q)/(2s(q))$ or at 0, when $s_1(q) < 0$ or $s_1(q) \geq 0$ and the minimum value is given by:

$$\min_{\substack{k_i \geq 0 \\ \sum_i k_i \leq n}} \delta(k) = \begin{cases} -\frac{s_1(q)^2}{2s(q)} & \text{if } s_1(q) < 0 \\ s_0(q)(n) & \text{if } s_1(q) \geq 0\,. \end{cases}$$

Again cancellation of multiple lattice minimizers is ruled out because the signs of the leading terms are always the same due to the assumption on the parity of the q_i's.

Remark 4.5. For future reference it may be of interest to note that there are very few pretzel knots with $s(q) \geq 0$ and $s_1(q) = 0$. These are cases 2b and 3 above where cancellations might occur if we had no control on the sign of the leading coefficients. The case $P(-3, 5, 5)$ is mentioned in [LvdV] for its colored Jones polynomial with growing leading coefficient.

Lemma 4.6. (Exceptional Pretzel knots)
The only pretzel knots with $q_0 \leq -2 < 3 \leq q_1, \dots q_m$ for which $s(q) \geq 0$ and $s_1(q) = 0$ are

(1) $P(-3, 5, 5)$, $P(-3, 4, 7)$, $P(-2, 3, 5, 5)$, with $s(q) = 0$.
(2) $P(-2, 3, 7)$, with $s(q) = \frac{1}{2}$.

Proof. Changing variables to $f_i = q_i - 1$ turns the two equations $s(q) \geq 0$ and $s_1(q) = 0$ into: $f_0(f_1^{-1} + \cdots + f_m^{-1}) + m = 0$ and $2 + f_0 + \frac{1}{f_1^{-1} + \cdots + f_m^{-1}} = c$ for some $c \geq 0$. Solving for f_0 yields $f_0 = (c-2)\frac{m}{m-1}$. Since $f_0 \leq -3$ we must have $0 \leq c \leq 2 - 3\frac{m-1}{m}$. This means there can only be such c when $m = 2$ or 3. Suppose $m = 2$ then $c = 0$ or $c = \frac{1}{2}$. In the first case we find $f_2 = \frac{2f_1}{f_1 - 2}$ so the positive integer solutions are $(f_1, f_2) \in \{(3, 6), (4, 4), (6, 3)\}$. In the case $c = \frac{1}{2}$ we find $f_2 = \frac{3f_1}{2f_1 - 3}$ so $(f_1, f_2) \in \{(2, 6), (3, 3), (6, 2)\}$. Finally the case $m = 3, c = 0, f_0 = -3$ yields $(f_1, f_2, f_3) \in \{(2, 4, 4), (2, 3, 6), (3, 3, 3)\}$ and permutations. $\square$

5. The colored Jones polynomial of Montesinos knots

In this section we will extend Theorem 3.2 to the class of Montesinos knots. For a Montesinos knot $K = K(r_0, r_1, \ldots, r_m)$, we always consider the standard diagram coming from the unique continued fraction expansion of even length in each tangle as in the case of pretzel knots. To build the diagram from simpler diagrams we introduce the tangle replacement move (in short, TR-move), and study its effect on the state-sum formula for the colored Jones polynomial.

5.1. The TR-move.

The TR-move is a local modification of a link diagram D. Suppose D contains a twist region T. Viewing T as a rational tangle $T = \frac{1}{t}$ for some integer t we may consider a new diagram D_1 obtained by replacing T by the rational tangle $T_1 = r * \frac{1}{t}$ for some non-zero integer r with the same sign as t. Alternatively, viewing T as an integer tangle t we replace it with $T_2 = \frac{1}{r} \oplus t$, also with r having the same sign. Collectively these two operations are referred to as the TR-*move*. Recall from Section 2.1, Equation (6) that we can reconstruct a diagram of any rational tangle by a combination of TR-moves, see also Figure 21 and 22. We extend this to n-cabled tangle diagrams by labeling each arc in the diagram by n.

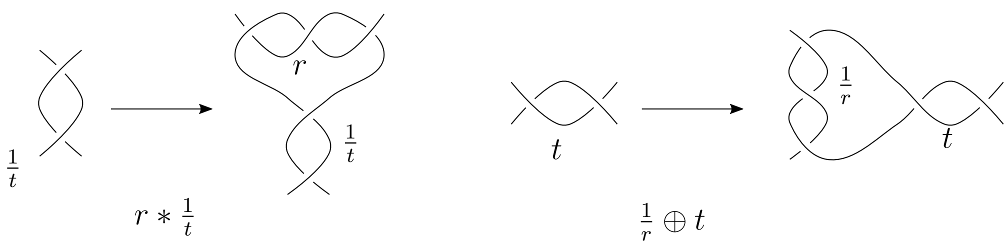


FIGURE 21. Two types of TR-move.

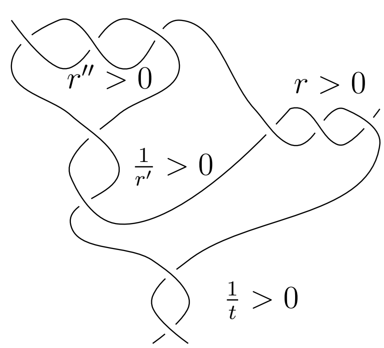


FIGURE 22. Any rational tangle is produced by a combination of TR-moves. In the picture shown, we have performed three TR-moves: first on $1/t$, then r, then $1/r'$.

We will use the TR-moves to reduce a Montesinos knot to a pretzel knot.

5.2. **Special Montesinos knot case.** We start by considering the case of Montesinos knots $K(r_0,\ldots,r_m)$ where $\ell_{r_i}=2$ for all $i\geq 0$. This includes the pretzel knots by choosing the unique even length continued fraction expansion with $r_i[2]=1$. We call these knots *special Montesinos knots*. We will prove the main theorem for such special Montesinos knots where the $r_i[1]$'s are even for all $0\leq i\leq m$ and $r_0[2]=1$.

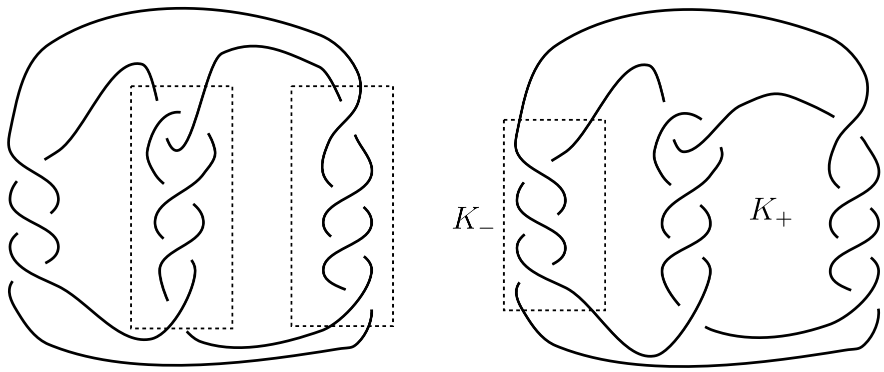


FIGURE 23. The special case $K(-\frac{1}{2+\frac{1}{1}},\frac{1}{3+\frac{1}{2}},\frac{1}{3+\frac{1}{1}})$.

As in the case of pretzel knots we use a customized state sum to compute the colored Jones polynomial, splitting $K=N(K_-\oplus K_+)$. In this case K_- is the single twist region $1/(r_0[1]+1)$ and K_+ is the 2-tangle that is the rest of the diagram. As before we apply the fusion (8) and untwisting formulas (9) to K_- and the usual Kauffman state sum to K_+ after cabling with the Jones-Wenzl idempotent of size n for the nth colored Jones polynomial. See Figure 23.

The methods used previously on the pretzel knots also apply to this case with minor modifications. In particular the notion of global through strands $c(\sigma)$ for a Kauffman state σ on K_+^n still makes sense and $k_i(\sigma)$ is still well defined by restricting σ to the ith-tangle. In this case $c_i(\sigma)$ means the number of through strands of the ith tangle of K_+^n that are also global through strands, and as before $k_i=\lceil\frac{c_i}{2}\rceil$. Let

$$\mathcal{G}_{c,k}=\sum_{k_0}\sum_{\sigma:k_i(\sigma)=k_i,c(\sigma)=c}G_{k_0}(v)v^{\mathrm{sgn}(\sigma)}\langle N(I_{k_0}\oplus(K_+^n)_\sigma)\rangle.$$

We prove the following theorem.

Theorem 5.1. *Consider* $K=K(\frac{1}{r_0[1]+\frac{1}{1}},\frac{1}{r_1[1]+\frac{1}{r_1[2]}},\ldots,\frac{1}{r_m[1]+\frac{1}{r_m[2]}})$. *Assume* $|r_i[1]|>1$, $r_i[2]>0$, *and let* $q_i=r_i[1]+1$ *for* $0\leq i\leq m$, $q_i'=r_i[2]$ *for* $1\leq i\leq m$. *Referring to the above state sum* $\langle K^n\rangle=\sum_{c,k}\mathcal{G}_{c,k}$ *we have the following. For a state* σ, *define the parameters* $c=c(\sigma),k=k(\sigma)$ *to be tight if* $k_0=k_1+\cdots+k_m=\frac{c}{2}$. *For tight* c,k *we have* $\mathcal{G}_{c,k}=(-1)^{q_0(n-k_0)+n+k_0+\sum_{i=1}^m(n-k_i)(q_i-1)}v^{\delta(n,k)}+l.o.t.$[2] *and* $\frac{-\delta(n,k)}{2}=$

$$(q_0+1)k_0^2+\sum_{i=1}^m(q_i-1)k_i^2+\sum_{i=1}^m(-2+q_0+q_i)k_i-\frac{n(n+2)}{2}\sum_{i=0}^m q_i+(m-1)n-\frac{n^2}{2}\sum_{i=1}^m(q_i'-1). \quad (28)$$

[2]The abbreviation *l.o.t.* means lower order terms in v.

If c, k are not tight then there exists a tight pair c', k' (coming from some Kauffan state) such that $\deg_v \mathcal{G}_{c,k} < \deg_v \mathcal{G}_{c',k'}$.

Proof. The proof is analogous to that of Theorem 3.2 for pretzel knots. As in the pretzel case we identify the minimal states and show that they maximize the degree and do not cancel out. Since these arguments are exactly the same we focus on describing the minimal states, one for each tight parameters of through strands c, k. [consider adding a figure here.] The minimal states are produced by choosing a minimal state for the pretzel knot $P = K(\frac{1}{q_0}, \ldots, \frac{1}{q_m})$ and extending it to a Kauffman state of $\langle K_+^n \rangle$ by choosing a pyramidal configuration on the remaining twist regions. The new pyramidal configuration has exactly k_i^2 extra A-states for each $i > 0$, so the degree of the minimal pretzel state is increased by $\sum_{i=1}^m q_i' n^2 - 2k_i^2$ in the new state sum. The number of additional circles in the pyramidal configuration is $\sum_{i=1}^m n - k_i$. Adjusting the degree accordingly concludes the proof. $\square$

5.3. **The general case.** Given $K = K(r_0, r_1, \ldots, r_m) = N(K_- \oplus K_+)$, where K_- consists of the negative twist region $1/r_0[1]$ if $r_0[2] \neq 1$, or $1/(r_0[1]+1)$ if $r_0[2] = 1$ and $\ell_{r_0} = 2$, we further split K_+ into $K_+ = D \cup V$ where D is the set of rational tangles that is the union $r_i[2] * \frac{1}{r_i[1]}$ of the first two (with respect to the continued fraction expansion) twist regions of each rational tangle r_i in K_+ and V is the remaining tangle. See Figure 24 for an illustration.

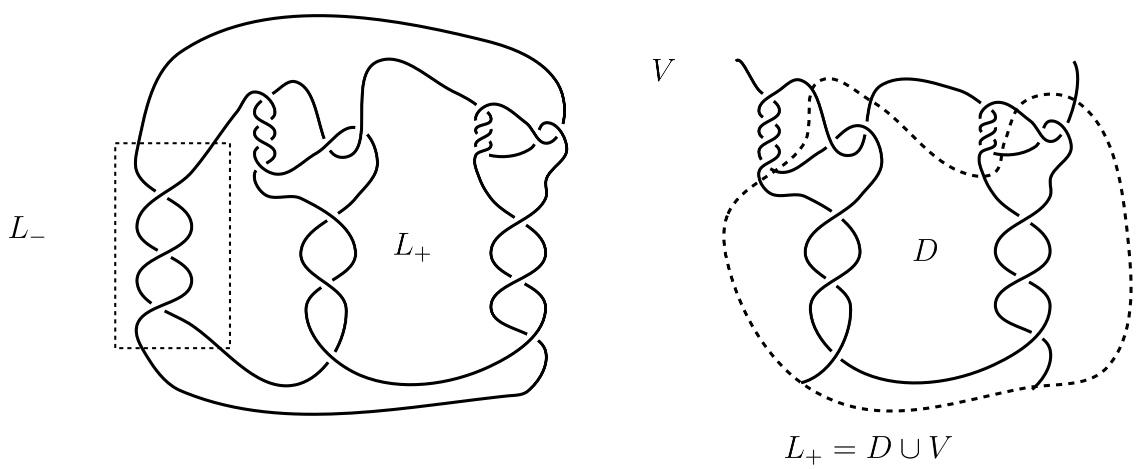


FIGURE 24. Left: a Montesinos knot $K = K(-1/3, 31/9 = \frac{1}{3+\frac{1}{2+\frac{1}{3+\frac{1}{1}}}}, 31/9) = N(K_- \oplus K_+)$. Right: further decomposition of K_+ into $D \cup V$.

Let
$$q_0 = \begin{cases} r_0[1] + 1 \text{ if } \ell_{r_0} = 2 \text{ and } r_0[2] = 1. \\ r_0[1] \text{ otherwise} \end{cases}.$$
$L = K(\frac{1}{q_0}, \frac{1}{r_1[1]+\frac{1}{r_1[2]}}, \ldots, \frac{1}{r_m[1]+\frac{1}{r_m[2]}})$ is a special Montesinos knot. We approach the general case as insertion of the rational tangle V into this special Montesinos knot. The essential feature of V^n is that its all-B state acts like the identity on $\langle L^n \rangle$ plus some closed loops, see Figure 25.

Lemma 5.2. Take the standard diagram of a Montesinos knot $K = K(r_0, r_1, \ldots, r_m) = N(K_- \oplus (D \cup V))$, where $L = K(\frac{1}{q_0}, \frac{1}{r_1[1]+\frac{1}{r_1[2]}}, \ldots, \frac{1}{r_m[1]+\frac{1}{r_m[2]}})$ is a special Montesinos knot. If $q_0 < -1$ is odd, and $q_i - 1 = r_i[1] > 1$ is even for every $i > 0$, then we have

$$\deg_v \langle K^n \rangle = \deg_v \langle L^n \rangle + c(V)n^2 + 2n\, o(V_B),$$

where $o(V_B)$ is the number of disjoint circles resulting from applying the all-B state to V.

Proof. Applying quadratic integer programming to the formula of Theorem 5.1 for the degree-maximizing states of $\langle L^n \rangle$, discarding any terms that depend only on q_i and n, we see that there are minimal states of the state sum of any special Montesinos knot that attain the maximal degree. Fix one such minimal state τ. Denote the skein element resulting from applying such a state to L^n by $\mathcal{S}(k_0, \tau)$, and the degree by $\delta(n, k) = d(k, \tau)$.

Now we consider the effect of inserting V^n into L^n to obtain K^n. Taking the all-B state on V^n preserves the states of L^n. Because V is a disjoint union of alternating tangles, we have

$$\deg G_{k_0}(v) v^{\mathrm{sgn}(\sigma)+\mathrm{sgn}(B_V)} \langle \mathcal{S}(k_0, \sigma) \cup (V^n)_B \rangle > \deg G_{k_0}(v) v^{\mathrm{sgn}(\sigma)+\mathrm{sgn}(\sigma')} \langle \mathcal{S}(k_0, \sigma) \cup (V^n)_{\sigma'} \rangle,$$

where σ' is any other state on V^n and V_B indicates the all-B state on V^n. Thus for a minimal state τ maximizing the degree in the state sum $\langle L^n \rangle$, the term $G_{k_0}(v) v^{\mathrm{sgn}(\tau)} \langle \mathcal{S}(k_0, \tau) \cup (V^n)_B \rangle$ also maximizes the degree in the new Montesinos state sum. The leading terms all have the same sign because of the assumption on the parity of the q_i's and Theorem 5.1. Thus there is no cancellation of these maximal term, and we can determine $\deg_v \langle K^n \rangle$ relative to $\deg_v \langle L^n \rangle$ by counting the number of disjoint circles $o(V_B)$, giving the formula in the lemma. ☐

It is useful to reformulate the above lemma in a more relative sense, pinpointing how the degree changes as a result of applying a TR-move. For our purposes it is more convenient to work with the composite moves $\mathsf{TR}_2^-(T) = (\frac{1}{r_1} \oplus r_2) * T$, and $\mathsf{TR}^+(T) = (r_1 * \frac{1}{r_2}) \oplus T$.

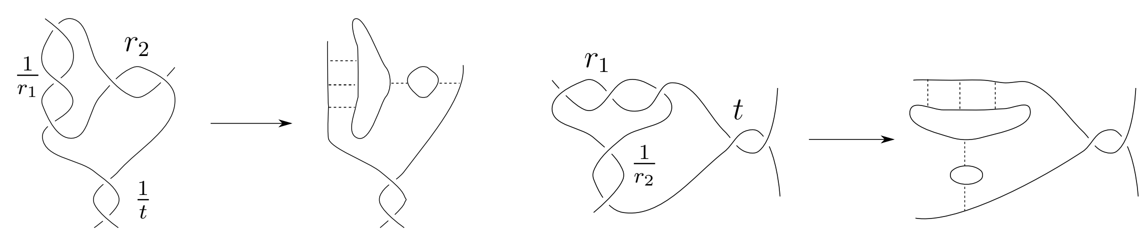


FIGURE 25. Examples of applying the all-B state and the resulting disjoint circles for moves sending tangles $\frac{1}{t}$ to $(\frac{1}{r_1} \oplus r_2) * \frac{1}{t}$ and sending t to $(r_1 * \frac{1}{r_2}) \oplus t$. [indicate where V is in this picture.]

Lemma 5.3. Suppose two standard diagrams K, L of Montesinos links satisfying the conditions of Lemma 5.2 where K is obtained from L by applying one of the moves $\mathsf{TR}_1^-, \mathsf{TR}_2^-, \mathsf{TR}^+$, locally replacing tangle $(T)^n$ by $(T')^n$, then the degree of the colored Jones polynomial changes as follows. See Figure 25 for examples of the moves TR_2^-, TR^+.

TR_1^- move: Suppose $r, t < 0$, and $T = \frac{1}{t}$ is a vertical twist region, and $T' = r * \frac{1}{t}$, then

$$\deg \langle K^n \rangle = \deg \langle L^n \rangle - rn^2 + 2(-r-1)n.$$

TR_2^- move: Suppose $r_1, r_2, t < 0$, $T = \frac{1}{t}$ is a vertical twist region, and $T' = (\frac{1}{r_1} + r_2) * \frac{1}{t}$, then

$$\deg\langle K^n\rangle = \deg\langle L^n\rangle - (r_1+r_2)n^2 - 2r_2 n.$$

TR^+ move: Suppose $r_1, r_2, t > 0$, $T = t$ is a horizontal twist region, and $T' = (r_1 * \frac{1}{r_2}) + t$, then

$$\deg\langle K^n\rangle = \deg\langle L^n\rangle + (r_1+r_2)n^2 + 2r_2 n.$$

Proof. Applying Lemma 5.2 we may simply count the number of crossings and state circles added to the degree in applying the all-B state to the newly added tangle V in each of these cases. ☐

We use Lemma 5.3 to prove the part of Theorem 1.3 concerning the degree of the colored Jones polynomial for the Montesinos knots that we consider.

Theorem 5.4. *Let $K = K(r_0, r_1, \ldots, r_m)$ be a Montesinos knot such that $r_0 < 0$, $r_i > 0$ for all $1 \le i \le m$, and $|r_i| < 1$ for all $0 \le i \le m$ with $m \ge 2$ even. Suppose $q_0 < -1 < 1 < q_1, \ldots, q_m$ are all odd, and q_0' is an integer that is defined to be 0 if $r_0 = 1/q_0$, and defined to be $r_0[2]$ otherwise. Let $P = P(q_0, \ldots, q_m)$ be the associated pretzel knot, and let $\omega(D_K)$, $\omega(D_P)$ denote the writhe of D_K, D_P with orientations. Then for all $n > N_K$ we have:*

$$\mathrm{js}_K(n) = \mathrm{js}_P(n) - q_0' - \langle r_0\rangle - \omega(D_P) + \omega(D_K) + \sum_{i=1}^m (r_i[2]-1) + \sum_{i=1}^m \langle r_i\rangle,$$

$$\mathrm{jx}_K(n) = \mathrm{jx}_P(n) - 2\frac{q_0'}{r_0[2]} + 2\langle r_0\rangle_o - 2\sum_{i=1}^m (r_i[2]-1) - 2\sum_{i=1}^m \langle r_i\rangle_e.$$

Proof. Suppose $K = K(r_0, r_1, \ldots, r_m) = N(K_- \oplus K_+)$ is a Montesinos knot, then K is obtained from a special Montesinos knot $L = K(\frac{1}{q_0}, \frac{1}{r_1[1]+\frac{1}{r_1[2]}}, \ldots, \frac{1}{r_m[1]+\frac{1}{r_m[2]}}) = N(L_- \oplus L_+)$ by a combination of TR^+ moves on the tangles in L_+ following the unique even length positive continued fraction expansions of r_i for $1 \le i \le m$. Each rational tangle diagram has an algebraic expression of the form

$$(((r_i[\ell_{r_i}] * \frac{1}{r_i[\ell_{r_i}-1]}) \oplus r_i[\ell_{r_i}-2]) * \cdots * \frac{1}{r_i[1]}).$$

The diagram is obtained by applying successive TR^+ moves to $r_i[j] * 1/r_i[j-1]$, sending $r_i[j]$ to $((r_i[j+2] * 1/r_i[j+1]) \oplus r_i[j])$ for each even $0 \le j \le \ell_{r_i}$ starting with $j = 2$.

The rational tangle in K_- is constructed from the special Montesinos knot L by applying the TR_2^- moves to $\frac{1}{r_0[j]}$, sending $\frac{1}{r_0[j]}$ to $(\frac{1}{r_0[j+2]} \oplus r_0[j+1]) * \frac{1}{r_0[j]}$, for odd $0 \le j \le \ell_{r_0}$ starting with $j = 1$, with a final TR_1^--move sending $\frac{1}{r_0[\ell_{r_0}-1]}$ to $(\frac{1}{r_0[\ell_{r_0}]} * \frac{1}{r_0[\ell_{r_0}-1]})$. These moves extend to the n-cables of the tangle diagrams.

We have two cases for the degree of $\langle K^n\rangle$ relative to $\langle L^n\rangle$:

(1) r_0 is the inverse of an integer $= q_0$. In this first case, we count the change in the degree of $\langle K^n\rangle$ given by Theorem 5.4 from applying TR^+ to L_+^n of the special Montesinos knot via Lemma 5.3. Each application of TR^+ adds $(r_i[j+2]+r_i[j+1])n^2+2r_i[j+1]n$

from even $j \geq 2$ for each $1 \leq i \leq m$. We have

$$\deg_v \langle K^n \rangle$$
$$= \deg_v \langle L^n \rangle + n^2 \sum_{i=1}^{m} \langle r_i \rangle + 2n \sum_{i=1}^{m} \langle r_i \rangle_o$$

Substituting (28) for $\deg_v \langle K^n \rangle$, we get

$$\deg_v \langle K^n \rangle$$
$$= -2((q_0+1)k_0^2 + \sum_{i=1}^{m}(q_i-1)k_i^2 + \sum_{i=1}^{m}(-2+q_0+q_i)k_i - \frac{n(n+2)}{2}\sum_{i=0}^{m} q_i + (m-1)n - \frac{n^2}{2}\sum_{i=1}^{m}(q_i'-1))$$
$$+ n^2 \sum_{i=1}^{m} \langle r_i \rangle + 2n \sum_{i=1}^{m} \langle r_i \rangle_o.$$

Apply quadratic integer programming, ignoring the part of the degree function that only depends on n, q_i, and q_i''s, we see that as long as the q_i's for $0 \leq i \leq m$ satisfy the hypothesis of the theorem,

$$\deg_v \langle K^n \rangle$$
$$= (\mathrm{js}_P(n) - \omega(D_p))n^2 - 2n\omega(D_p) - 2s_1(q)(n)n + s_0(q)(n) + n^2 \sum_{i=1}^{m}(q_i'-1) + n^2 \sum_{i=1}^{m} \langle r_i \rangle + 2n \sum_{i=1}^{m} \langle r_i \rangle_o.$$

Gathering the coefficients multiplying n^2 and accounting for the writhe of D_K, we get

$$\mathrm{js}_K = \mathrm{js}_P - \omega(D_p) + \omega(D_K) + \sum_{i=1}^{m}(q_i'-1) + \sum_{i=1}^{m} \langle r_i \rangle$$

Note that $q_i' = r_i[2]$, and q_0', $\langle r_0 \rangle$ are both 0 for this case, and so

$$\mathrm{js}_K = \mathrm{js}_P - q_0' - \langle r_0 \rangle + \omega(D_p) + \omega(D_K) + \sum_{i=1}^{m}(r_i[2]-1) + \sum_{i=1}^{m} \langle r_i \rangle.$$

Now we compute jx_K by considering $\deg_v \langle L^{n-1} \rangle$ and collecting coefficients of n. This gives me

$$\mathrm{jx}_K = \mathrm{jx}_P + 2\sum_{i=1}^{m}(q_i'-1) + 2\sum_{i=1}^{m} \langle r_i \rangle_o - 2\sum_{i=1}^{m} \langle r_i \rangle$$
$$= \mathrm{jx}_P + 2\sum_{i=1}^{m}(q_i'-1) + 2\sum_{i=1}^{m} \langle r_i \rangle_e.$$

(2) r_0 is not the inverse of an integer. In this case, we account for the degree change for the TR^+ moves applied to K_+^n in the same way as in case (1). It remains to account for the change to the degree based on applying TR_2^- moves with a final TR_1^-

move to the negative tangle of the special Montesinos knot K. Each application of the TR_2^--move adds $-(r_0[j+2]+r_0[j+1])n^2 - 2(r_0[j+1])n$ to the degree, and the final application of the TR_1^--move adds $-r_0[\ell_{r_0}]n^2 + 2(-r_0[\ell_{r_0}]-1)n$. We sum the contribution over j odd and $1 \leq j < \ell_{r_0}$.

$$\sum_{j \text{ odd }, 1\leq j<\ell_{r_0}} -(r_0[j+2]+r_0[j+1])n^2 - 2(r_0[j+1])n$$
$$= -(r_0[2]+\langle r_0\rangle - r_0[\ell_{r_0}])n^2 - 2(\langle r_0\rangle_e - r_0[\ell_{r_0}] + r_0[2])n$$

When we plug in $n-1$ for n, we get

$$\begin{aligned} &= -(r_0[2]+\langle r_0\rangle - r_0[\ell_{r_0}])n^2 + 2(r_0[2]+\langle r_0\rangle - r_0[\ell_{r_0}])n - 2(\langle r_0\rangle_e - r_0[\ell_{r_0}] + r_0[2])n \\ &+ \text{terms that do not grow with } n. \\ &= -(r_0[2]+\langle r_0\rangle - r_0[\ell_{r_0}])n^2 + (2\langle r_0\rangle_o)n + \text{ constant terms that do not grow with } n. \end{aligned} \tag{29}$$

For the purpose of computing js_K and jx_K, we may ignore the constant terms that don't grow with n. We compute similarly the quadratic growth rate and the linear growth rate of the final TR_1^--move.

$$\begin{aligned} &- r_0[\ell_{r_0}](n-1)^2 + 2(-r_0[\ell_{r_0}]-1)(n-1) \\ &= -r_0[\ell_{r_0}]n^2 + 2(-r_0[\ell_{r_0}])n + 2(-r_0[\ell_{r_0}]-1)n + \text{constant terms that do not grow with } n. \\ &= -r_0[\ell_{r_0}]n^2 + 2(-2r_0[\ell_{r_0}]-1)n + \text{constant terms that do not grow with } n. \end{aligned} \tag{30}$$

When we add the coefficients multiplying n^2 and the coefficients multiplying n from (29), (30) from the moves on K_+^n, we get in this case

$$\mathrm{js}_K = \left(\mathrm{js}_P - \omega(D_p) + \omega(D_K) + \sum_{i=1}^m (q_i'-1) + \sum_{i=1}^m \langle r_i\rangle\right) - (r_0[2]+\langle r_0\rangle) \tag{31}$$

and

$$\mathrm{jx}_K = (\mathrm{jx}_P + 2\sum_{i=1}^m (q_i'-1) + 2\sum_{i=1}^m \langle r_i\rangle_e) + 2\langle r_0\rangle_o - 2\frac{q_0'}{r_0[2]}. \tag{32}$$

□

[should probably remind overall that we are doing $J_{n+1} = \langle K^n\rangle$.]

6. Essential surfaces of Montesinos knots

Let Σ be a compact, connected, and properly embedded surface in a compact, orientable 3-manifold M that is not boundary parallel. We say that Σ is *essential* if the map on fundamental groups $\iota^* : \pi_1(\Sigma) \to \pi_1(M)$ induced by inclusion is injective. The surface Σ is *incompressible* in the 3-manifold M if for each disc $D \subset M$ with $D \cap \Sigma = \partial D$, there is a disc $D' \subset \Sigma$ with $\partial D' = \partial D$. The surface Σ is called *∂-incompressible* if for each disk $D \subset M$

with $D \cap S = \alpha$, $D \cap \partial M = \beta$ (α and β are arcs), and $\alpha \cup \beta = \partial D$ and $\alpha \cap \beta = S^0$, there is a disk $D' \subset S$ with $\partial D' = \alpha' \cup \beta'$ such that $\alpha' = \alpha$ and $\beta' \subset \partial \Sigma$.

Given an essential surface Σ with nonempty boundary in a compact orientable manifold M with torus boundary, consider the first homology class of $[\partial S]$ in $H_1(\partial M)$. Write $[\partial S] = p\mu + q\lambda$ where μ and λ are a meridian and longitude basis of the torus. The *boundary slope* of S is the fraction p/q, reduced to lowest terms. Hatcher showed that the set of boundary slopes of a compact orientable manifold is finite [Hat82].

An orientable surface is essential if and only if it is incompressible. On the other hand, a non-orientable surface is essential if and only if its orientable double cover in the ambient manifold is incompressible. In an irreducible orientable 3-manifold whose boundary consists of tori (such as a link complement), an orientable incompressible surface is either ∂-incompressible or a ∂-parallel annulus [?]. Therefore, the problem of finding boundary slopes for Montesinos knots may be reduced to the problem of finding orientable incompressible and ∂-incompressible surfaces, and we will only consider such surfaces for the rest of the paper.

In this section, we summarize the Hatcher-Oertel algorithm for finding all boundary slopes of Montesinos knots [HO89], based on the classification of orientable incompressible and ∂-incompressible surface of rational (2-bridge) knots in [HT85]. For every Jones slope that we find in Section 4.3, we will use the algorithm to produce an orientable, incompressible and ∂-incompressible surface, whose boundary slope, number of boundary components, and Euler characteristic realize the strong slope conjecture. This completes the proof of Theorem 1.2 and Theorem 1.3.

We will follow the conventions of [HO89] and [HT85]. For further exposition of the algorithm, the reader may consult [?]. It will be useful to introduce the negative continued fraction expansion [BS, Ch.13]

$$[[a_0, \ldots, a_\ell]] = [a_0, -a_1, \ldots, (-1)^\ell a_\ell] = a_0 - \cfrac{1}{a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \cdots - \cfrac{1}{a_\ell}}}}. \tag{33}$$

with $a_i \in \mathbb{Z}$ and $a_i \neq 0$ for $i > 0$.

6.1. **Incompressible and ∂-incompressible surfaces for a rational knot.** A notion originally due to Haken [Hak61], a *branched surface* B in a 3-manifold M is a subspace locally modelled on to the space as shown in the following figure. This means every point has a neighborhood diffeomorphic to the neighborhood of a point in the model space. A properly embedded surface Σ in M is *carried by* B if Σ can be isotoped so that it runs nearly parallel to B.

Using branched surfaces, Hatcher and Thurston [HT85] classify all orientable, incompressible and ∂-incompressible surfaces with nonempty boundary for a rational knot $K(r)$ where $r \in \mathbb{Q} \cup \{1/0\}$ in terms of negative continued fraction expansions of r. For each negative continued fraction expansion $[[b_0, b_1, \ldots, b_k]]$ of r as in (43) they construct a branched surface $\Sigma(b_1, \ldots, b_k)$ and associated surfaces $S_n(n_1, \ldots, n_k)$ carried by $\Sigma(b_1, \ldots, b_k)$, where

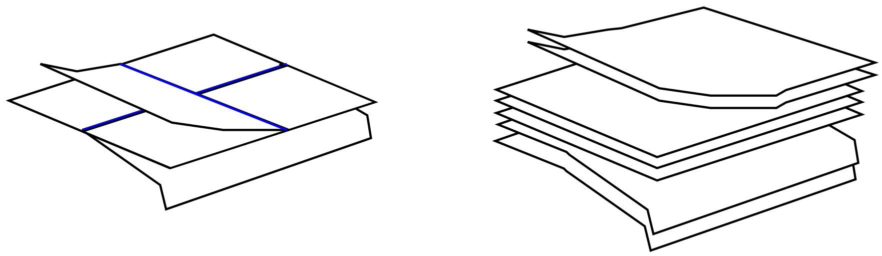


FIGURE 26. Left: local picture of a branched surface, with the blue lines indicating the singularities. Right: a surface carried by the branch surface.

$n \geq 1$ and $0 \leq n_i \leq n$. They show that every non-closed incompressible, ∂-incompressible surface in $S^2 \setminus K(r)$ is isotopic to $S_n(n_1, \ldots, n_k)$ for some n and n_i's . Furthermore, a surface $S_n(n_1, \ldots, n_k)$ carried by $\Sigma(b_1, \ldots, b_k)$ is incompressible and ∂-incompressible if and only if $|b_i| \geq 2$ for each i [HT85, Theorem 1.(b) and (c)].

In general, for a fixed branched surface B, all the boundary circles of surfaces carried by B in the same torus boundary component of M are homologous [Hat82, Lemma 1]. Thus to compute a boundary slope it suffices to specify a branched surface. Floyd and Oertel have shown that there is a finite, constructible set of branched surfaces for every Haken 3-manifold which carries all the two-sided, incompressible and ∂-incompressible surfaces [FO84]. For the general theory of branched surfaces applied to the question of finding boundary slopes in a 3-manifold, interested readers may consult these references. We will continue to specialize to the case of rational knots.

Of particular importance to us is their representation of a surface $S_M(M_1, \ldots M_k)$ carried by a branched surface $\Sigma(b_1, \ldots, b_k)$ in terms of an *edge-path* on a one-complex $\mathcal{D}$. Here, $\mathcal{D}$ is the Farey ideal triangulation of $\mathbb{H}^2$ on which $\mathrm{PSL}_2(\mathbb{Z})$ is the group of orientation-preserving symmetries, see Figure 27. Recall that the vertices (in the natural compactification) of $\mathcal{D}$ are $\mathbb{Q} \cup \infty$ and we set $\infty = \frac{1}{0}$ in projective coordinates. A typical vertex of $\mathcal{D}$ will be denoted by $\langle \frac{p}{q} \rangle$ for coprime integers p, q with q nonnegative. There is an edge between two vertices $\langle \frac{p}{q} \rangle$ and $\langle \frac{r}{s} \rangle$, denoted by $\langle \frac{p}{q} \rangle$—$\langle \frac{r}{s} \rangle$, whenever $|ps - rq| = 1$. An *edge-path* is simply a path on the 1-skeleton of $\mathcal{D}$ which may have endpoints on an edge rather than on a vertex.

Given a negative continued fraction expansion $[[b_0, \ldots, b_k]]$ of r, the vertices of the corresponding edge-path are the sequence of partial sums

$$[[b_0, b_1, \ldots, b_k]], [[b_0, b_1, \ldots, b_{k-1}]], \ldots, [[b_0, b_1]], [[b_0]], \infty.$$

Such an edge-path determines a candidate $S_M(M_1, \ldots, M_k)$ for an incompressible and ∂-incompressible surface in the exterior of $K(r)$ as follows. We isotope the 2-bridge knot presentation of $K(r)$ so that it lies in $S^2 \times [0, 1]$, with the two bridges intersecting $S^2 \times \{1\}$ in two arcs of slope ∞, and the arcs of slope r lying in $S^2 \times \{0\}$. See [HT85, p. 1 Fig. 1(b)]. The slope here is determined by the lift of those arcs to $\mathbb{R}^2$, where $S^2 \times \{i\} \setminus K$ is identified with the orbit space of Γ, the isometry group of $\mathbb{R}^2$ generated by 180°-degree rotation about the integer lattice points.

Given an edge-path, with vertices $\{v\}$, choose heights $\{i_v\} \in [0, 1]$ respecting the ordering of the vertices in the path. At $S^2 \times \{0\}$, we have $2M$ arcs of slope r, and at $S^2 \times \{1\}$ we have $2n$ arcs of slope ∞. For a fixed M, each vertex $\langle v \rangle$ of an edge-path determines a curve system on $S^2 \times \{i_v\}$, consisting of $2M$ arcs of slope v with ends on the four punctures representing the intersection with the knot. The surface $S_M(M_1, \ldots, M_k)$ is constructed by having its intersections with $S^2 \times i_v$ coincide with the curve system at $\langle v \rangle$. Between one vertex $\langle v \rangle$ to another $\langle v' \rangle$, M saddles are added to change all M arcs of slope v to M arcs of slope v', with M_i indicating one of the two possible choices of such saddles. At the end of the edge-path, $2M$ disks are added to the slope ∞ curve system corresponding to closing the knot by the two bridges. For more details, see [HT85].

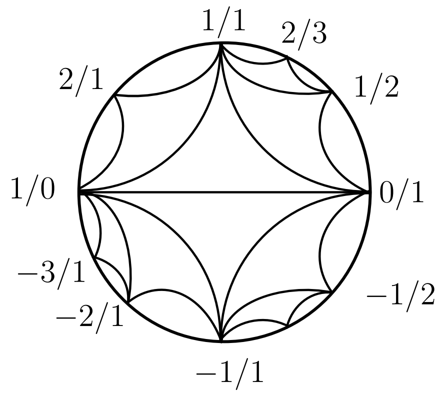


FIGURE 27. The 1-complex $\mathcal{D}$.

6.2. **Edge-paths and candidate surfaces for Montesinos knots.** Hatcher and Oertel [HO89] give an algorithm that provides a complete classification of boundary slopes of Montesinos knots by decomposing $K(r_0, r_1, \ldots, r_m)$ via a system of Conway spheres $\{S_i^2\}_{i=1}^m$, each of which contains a rational tangle T_{r_i}. Their algorithm determines the conditions under which the incompressible and the ∂-incompressible surfaces in the exterior of each rational tangle, as classified by [HT85] and put in the form as discussed in the previous section, may be glued together across the system of Conway spheres to form an incompressible surface in $S^3 \setminus K(r_0, r_1, \ldots, r_m)$.

To describe the algorithm, it is now necessary to give coordinates to curve systems on a Conway sphere. The curve system $S \cap S_i^2$ for a connected surface $S \subset S^3 \setminus K(r_0, r_1, \ldots, r_m)$ may be described by homological coordinates A_i, B_i, and C_i as shown in Figure 28 [**?**].

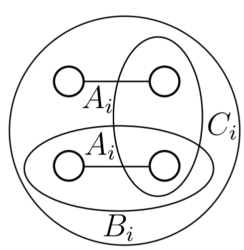


FIGURE 28. The Conway sphere containing the tangle corresponding to r_i and the curve system on it.

Since an incompressible and ∂-incompressible surface S must also be incompressible and ∂-incompressible when restricting to a tangle inside a Conway sphere, the classification of

[HT85] applies, and Hatcher and Oertel also represent such surfaces by specifying an edge-path for each restriction of the surface. However, the edge-paths lie instead in an augmented 1-complex $\hat{\mathcal{D}}$ in the plane obtained by splitting open $\mathcal{D}$ along the slope ∞ edge and adjoining constant edge-paths $\langle \frac{p}{q}\rangle$——$\langle \frac{p}{q}\rangle$. See [HO89, Fig. 1.3]. The additional edges in $\hat{\mathcal{D}}$ deal with the new possibilities of curve systems that arise when gluing the surfaces following the tangle sum.

Again, an edge-path in $\hat{\mathcal{D}}$ is a path in the 1-skeleton of $\hat{\mathcal{D}}$ which may or may not end on a vertex. It describes a surface in the complement of a rational tangle in $K(r_0, r_1, \dots, r_m)$ consisting of saddles joining curve systems corresponding to vertices, as in the last paragraph of Section 6.1. The main adjustment is that the endpoint of an edge-path describes a curve system on the Conway sphere enclosing the tangle. In order for the curve system to represent the intersection of an incompressible and ∂-incompressible surface, the endpoints must be on an edge $\langle \frac{p}{q}\rangle$——$\langle \frac{r}{s}\rangle$ and has the form

$$\frac{K}{M}\langle\frac{p}{q}\rangle + \frac{M-K}{M}\langle\frac{r}{s}\rangle,$$

for nonnegative integers K, M. If $\frac{p}{q} \neq \frac{r}{s}$, this describes a curve system on a Conway sphere consisting of K arcs of slope p/q, of (A, B, C) coordinates $K(1, q-1, p)$ and $M-K$ arcs of slope r/s, of (A, B, C) coordinates $(M-K)(1, s-1, r)$. If $\frac{p}{q} = \frac{r}{s}$, this describes a curve system on a Conway sphere consisting of $(M-K)$ arcs of slope p/q, of (A, B, C) coordinates $(M-K)(1, q-1, p) = (M-K, (M-K)(q-1), (M-K)p)$, and $M-K$ circles of slope p/q, of (A, B, C) coordinates $K(0, q, p) = (0, Kq, Kp)$. The curve system coordinates (A, B, C) corresponding to this point is obtained by adding the (A, B, C)-coordinates of $\langle \frac{p}{q}\rangle$ and $\langle \frac{r}{s}\rangle$.

The algorithm is as follows.

(1) For each fraction r_i, pick an edge-path γ_i in the 1-complex $\hat{\mathcal{D}}$ corresponding to a continued fraction expansion
$$r_i = [[b_0, b_1, \dots, b_k]], b_i \in \mathbb{Z},$$
or a constant edge-path.

(2) For each edge $\langle \frac{p}{q}\rangle$——$\langle \frac{r}{s}\rangle$ in γ_i, determine the integer parameters $\{K_i\} \geq 0$, $\{M_i\} \geq 0$ satisfying the following constraints.
   (a) $A_i = A_j$ and $B_i = B_j$ for all the A-coordinates A_i and the B-coordinates B_i of the point
   $$\frac{K_i}{M_i}\langle\frac{p}{q}\rangle + \frac{M_i-K_i}{M_i}\langle\frac{r}{s}\rangle.$$
   (b) $\sum_{i=0}^m C_i = 0$ where C_i is the C-coordinate of the point
   $$\frac{K_i}{M_i}\langle\frac{p}{q}\rangle + \frac{M_i-K_i}{M_i}\langle\frac{r}{s}\rangle.$$

The edge-paths chosen in (1) with endpoints specified by the solutions to (a) and (b) determine a candidate edge-path system $\{\gamma_i\}_{i=0}^m$, corresponding to a connected surface S in $S^3 \setminus K(r_0, r_1, \dots, r_m)$. We call this the *candidate surface* associated to a candidate edge-path system.

(3) Apply incompressibility criteria [HO89, Prop. 2.1, Cor. 2.4, Prop. 2.5-2.9] to determine if a candidate surface is an incompressible and ∂-incompressible surface and actually gives a boundary slope.

Remark 6.1. We would like to remark that Dunfield [Dun01] has written a computer program implementing the Hatcher-Oertel algorithm, which will output the set of boundary slopes given a Montesinos knot and give other information like the set of edge-paths representing an incompressible, ∂-incompressible surface, Euler characterstic, etc. The program has provided most of the data we use in our examples in this paper. Interested readers may download the program at his website https://faculty.math.illinois.edu/~nmd/montesinos/index.html.

We will write $S(\{\gamma_i\}_{i=0}^m)$ to indicate a candidate surface associated to a candidate edge-path system $\{\gamma_i\}_{i=0}^m$. Note that for a candidate edge-path system, M_i is identical for $i = 0, \dots, m$ by condition (2a) in the algorithm, so we will simply write M for M_i for a candidate surface S.

We will mainly be applying [HO89, Corollary 2.4], which we restate here. Note that for an edge $\langle \frac{p}{q} \rangle$—$\frac{k}{m}\langle \frac{p}{q} \rangle + \frac{m-k}{m}\langle \frac{r}{s} \rangle$ with $0 < q < s$, the ∇-value (called the "r-value" in [HO89]) is 0 if $\frac{p}{q} = \frac{r}{s}$ or if the edge is vertical, and the ∇-value is $q - s$ when $\frac{p}{q} \neq \frac{r}{s}$.

Theorem 6.2. [HO89, Corollary 2.4] *A candidate surface $S(\{\gamma_i\}_{i=0}^m)$ is incompressible unless the cycle of ∇-values for the final edges of the γ_i's is of one of the following types: $\{0, \nabla_1, \dots, \nabla_m\}$, $\{1, 1, \dots, 1, \nabla_m\}$, or $\{1, \dots, 1, 2, \nabla_m\}$.*

6.3. **The boundary slope of a candidate surface.** The *twist number* $\mathrm{tw}(S)$ for a candidate surface $S = \{\gamma_i\}_{i=0}^m$ is defined as

$$\mathrm{tw}(S) := \frac{2}{M}\sum_{i=0}^{m}(s_i^- - s_i^+) = 2\sum_{i=0}^{m}(e_i^- - e_i^+),$$

where s_i^- is the number of slope-decreasing saddles of γ_i, s_i^+ is the number of slope-increasing saddles of γ_i, and M is the number of sheets of S. Let an edge be given by $\langle \frac{p}{q} \rangle$—$\langle \frac{r}{s} \rangle$, we say that the edge decreases slope if $\frac{r}{s} < \frac{p}{q}$, and that the edge increases slope if $\frac{r}{s} > \frac{p}{q}$. In terms of edge-paths, $\mathrm{tw}(S)$ can be written in terms of the number e_i^- of edges of γ_i that decreases slope and e_i^+, the number of edges of γ_i that increases slope as shown. For an interpretation of the twist number in terms of the lifts of these arcs in $\mathbb{R}^2/\Gamma$, see [HO89, p. 460]. If γ_i has a final edge

$$\left\langle \frac{p}{q} \right\rangle — \frac{K_i}{M}\left\langle \frac{p}{q} \right\rangle + \frac{M-K_i}{M}\left\langle \frac{r}{s} \right\rangle.$$

Then the final edge of γ_i is called a fractional edge and counted as a fraction $\frac{M-K_i}{M}$. Finally, the boundary slope $\mathrm{bs}(S)$ of a candidate surface S is given by

$$\mathrm{bs}(S) = \mathrm{tw}(S) - \mathrm{tw}(S_0) \tag{34}$$

where S_0 is a Seifert surface that is a candidate surface from the Hatcher-Oertel algorithm.

6.4. **The Euler characteristic of a candidate surface.** We compute the Euler characteristic of a candidate surface S associated to an edge-path system $\{\gamma_i\}_{i=0}^m$, where none of the γ_i's are constant or end in $1/0$ as follows. M is again the number of sheets of the surface S. We begin with $2M$ disks which intersect $S_i^2 \times 0$ in slope $r_i = \frac{p_i}{q_i}$ arcs in each B_i.

- From left to right in an edge-path γ_i, each non-fractional edge $\langle \frac{p}{q} \rangle$—$\langle \frac{r}{s} \rangle$ is constructed by gluing M of saddles that change $2M$ arcs of slope $\frac{p}{q}$ (representing the intersections with $S_i^2 \times i_{\frac{p}{q}}$) to slope $\frac{r}{s}$ (representing the intersections with $S_i^2 \times i_{\frac{r}{s}}$), therefore decreasing the Euler characteristic by M.
- A fractional final edge of γ_i of the form $\langle \frac{p}{q} \rangle$—$\frac{K}{M}\langle \frac{p}{q} \rangle + \frac{M-K}{M}\langle \frac{r}{s} \rangle$ changes $2(M-K)$ out of $2M$ arcs of slope $\frac{p}{q}$ to $2(M-K)$ arcs of slope $\frac{r}{s}$ via $M-K$ saddles, thereby decreasing the Euler characteristic by $M-K$.

This takes care of the individual contribution of an edge-path $\{\gamma_i\}$. Now the identification of the surfaces on each of the 4-punctured sphere will also affect the Euler characteristic of the resulting surface. In terms of the common (A, B, C)-coordinates of each edge-path, there are two cases:

- The identification of hemispheres between neighboring balls B_i and B_{i+1} identifies $2M$ arcs and B_i half circles. Thus it subtracts $2M + B_i$ from the Euler characteristic for each identification.
- The final step of identifying hemispheres from B_0 and B_m on a single sphere adds B_i to the Euler characteristic.

6.5. **Matching the growth rate to topology for pretzel knots.** We consider two candidate surfaces from the Hatcher-Oertel algorithm whose boundary slope, Euler characteristic, and number of sheets will be shown to match the growth rate of the degree of the colored Jones polynomial from the previous section as predicted by the strong slope conjecture.

6.5.1. *The surface $S(M, x^*)$.* For $1 \le i \le m$ write

$$x_i^* = \frac{a_i^{-1}}{\sum_j a_j^{-1}} \text{ and } x_{i,0}^* = \frac{1}{2}\frac{\sum_j (b_j - b_i) a_i^{-1} a_j^{-1}}{\sum_j a_j^{-1}}, \tag{35}$$

where $a_i = q_i - 1$ and $b_i = q_0 + q_i - 2$. The x_i^*'s come from the coefficients of t in (19), and $x_{i,1}^*$'s come from the constant term. Let M be the least common multiple of the denominators of $\{x_i^*\}_{i=1}^m$, reduced to lowest terms. We will use $x^*(M)$ to denote the vector $(x_0^* M + x_{0,0}^*, x_1^* M + x_{1,0}^*, \dots, x_m^* M + x_{m,0}^*)$. For example, suppose we have the pretzel knot $P(-11, 7, 9)$, then

$$x_1^* = \frac{\frac{1}{7-1}}{\frac{1}{7-1} + \frac{1}{9-1}} = \frac{4}{7}, \; x_2^* = \frac{\frac{1}{8}}{\frac{1}{7-1} + \frac{1}{9-1}} = \frac{3}{7},$$

and M is 7.

Lemma 6.3. Suppose $q = (q_0, q_1, \dots, q_m)$ is such that $s(q) \le 0$. There is a candidate surface $S(M, x^*)$ from the Hatcher-Oertel algorithm with M sheets and C-coordinates

$$\{-M, Mx_1^*, Mx_2^*, \dots, Mx_m^*\}.$$

Proof. Directly from the proof of Lemma 4.1, the elements of the set $\{x^*_i\}_{i=0}^m$ satisfy the following equations.

$$x^*_i(q_i-1) = x^*_j(q_j-1), \text{ for } i \neq j, \text{ and}$$
$$\sum_{i=1}^m x^*_i = 1. \tag{36}$$

Consider the edge-path systems determined by the following choice of continued fraction expansions for $\{1/q_i\}_{i=0}^m$.

$$1/q_0 = [[-1, \underbrace{-2,-2,\ldots,-2}_{q_0-1}]]$$
$$1/q_i = [[0,-q_i]], \text{ for } i \neq 0.$$

Let $K_i = Mx^*_i$ for $1 \leq i \leq m$, and $0 \leq K_0 \leq M$, $q_0 \leq -q \leq -2$ such that

$$K_0 + M(q-2) = K_1(q_1-1), \tag{37}$$

We specify a candidate surface $S(M,x^*)$ in terms of edge-paths $\{\gamma_i\}_{i=0}^m$:

The edge-path γ_0 for q_0 is

$$\langle\frac{1}{q_0}\rangle — \langle\frac{1}{q_0+1}\rangle — \cdots — \frac{K_0}{M}\langle\frac{1}{-q}\rangle + \frac{M-K_0}{M}\langle\frac{1}{-q+1}\rangle.$$

For $i \neq 0$, we have the edge-path γ_i:

$$\langle\frac{1}{q_i}\rangle — \frac{K_i}{M}\langle\frac{1}{q_i}\rangle + \frac{M-K_i}{M}\langle\frac{0}{1}\rangle.$$

Provided that K_0, q satisfying (45) exist, this edge-path system satisfies the equations coming from (a) and (b) of Step (2) of the algorithm. Thus, there is a candidate surface with $\{-M, Mx^*_1, Mx^*_2, \ldots, Mx^*_m\}$ as the C-coordinates in the tangle corresponding to r_i.

It remains to show that the assumption $s(q) \leq 0$ implies the existence of K_0, q satisfying (45). The positive integer M divides $\sum_{i=1}^m (q_i-1)$ by definition. Recall $s(q) \leq 0$ means

$$1 + q_0 + \frac{1}{\sum_{i=1}^m (q_i-1)^{-1}} \leq 0$$
$$1 + q_0 + \frac{\Pi_{i=1}^m (q_i-1)}{\sum_{i=1}^m (q_i-1)} \leq 0.$$

Multiply both sides by M, we get

$$M(1+q_0) + \overline{\Pi_{i=1}^m (q_i-1)} \leq 0,$$

where $\overline{\Pi_{i=1}^m (q_i-1)}$ denotes the numerator in the reduced fraction $\frac{\Pi_{i=1}^m (q_i-1)}{\sum_{i=1}^m (q_i-1)}$. This implies that a pair of integers K_0, q such that $0 \leq K_0 \leq M$, $q_0 \leq q \leq -2$ exist such that (45) is satisfied, since by definition

$$\overline{\Pi_{i=1}^m (q_i-1)} = K(q_1-1).$$

So if $M > K(q_1 - 1)$, we can choose $q = 2$ and $K_0 = K_1(q_1 - 1)$. Otherwise, we choose some $q_0 \leq -q \leq -2$ such that

$$0 \leq K_1(q_1 - 1) - M(q - 2) \leq M.$$

Let K_0 be the difference $K_1(q_1 - 1) - M(q - 2)$. $\square$

The twist number of $S(M, x^*)$. With the given edge-path system in the proof of Lemma 6.3 and applying the formula for computing the boundary slope in Section 6.3, we compute the twist number of $S(M, x^*)$. For the edge-path γ_0 of q_0, since $q_0 < 0$, each edge of the edge-path is slope-decreasing. Similarly, each edge in γ_i for q_i is slope-decreasing (since $q_i > 0$, the edge $\langle 1/q_i \rangle$ — $\langle 0/1 \rangle$ is decreasing). Each non-fractional path contributes $+1$, and then the single fractional edge at the end contributes $(M - K_0)/M$. Thus

$$\frac{\mathrm{tw}(S(M, x^*))}{2} = \underbrace{(-q - q_0)}_{\text{contribution of the non-fractional edges of } \gamma_0} + \underbrace{M - \frac{K_0}{M}}_{\text{contribution of the single fractional edge at the end of } \gamma_0} .$$

$$+ \underbrace{\sum_{i=1}^{m} \frac{M - K_i}{M}}_{\text{contribution of the single fractional edge for each of the } \gamma_i\text{'s for } i \geq 1.}$$

By construction, $\sum_{i=1}^{m} \frac{K_i}{M} = 1$ and $q + \frac{M - K_0}{M} = \frac{K_1}{M}(q_1 - 1) + 2$, so

$$\mathrm{tw}(S(M, x^*)) = 2(-q_0 - x_i^*(q_1 - 1) + m - 2). \tag{38}$$

The Euler characteristic of $S(M, x^*)$. With the given edge-path system and applying the formula for computing the Euler characteristic in Section 6.4, we compute the Euler characteristic over the number of sheets for $S(M, x^*)$. We start with $2M$ disks for each tangle. Each non-fractional edge of an edgepath in $\{\gamma_i\}_{i=0}^{m}$ subtracts M to the Euler characteristic, while the final fractional edges adds

$$\sum_{i=0}^{m} M - K_i.$$

At the final step of gluing surfaces across Conway spheres, we subtract $2M + B_i$ for each identification out of m identifications, then add a single B_i back. We have, since $B_i = (q_i - 1)K_i = B_j$,

$$\sum_{i=1}^{m} B_i = m(q_i - 1)K_i.$$

Adding all these contributions, the Euler characteristic over the number of sheets of $S(M, x^*)$ is given by

$$\frac{2\chi(S(M, x^*))}{\#S(M, x^*)} = 2\left(\frac{2M(m+1)}{M} - \frac{(-q - q_0)M + (\sum_{i=0}^{m} M - K_i)}{M} - 2m - \frac{m(q_i - 1)K_i}{M} + \frac{(q_i - 1)K_i}{M}\right) \tag{39}$$

Using the substitutions that we previously used for computing the twist number, we get

$$\begin{aligned} &= 4 - \mathrm{tw}(S(M,x^*)) - 2(m-1)(x_i^*-1)(q_1-1) \\ &= 4 - 2(-q_0 - x_i^*(q_1-1) + m - 2) - 2(m-1)x_i^*(q_1-1) \\ &= 8 - 2m + 2q_0 - 2(m-2)x_i^*(q_1-1). \end{aligned} \tag{40}$$

The cycle of ∇-values of $S(M,x^*)$. For $i=0$, the last edge of the edge-path γ_0 is

$$\langle\frac{1}{-q}\rangle \text{---} \frac{K_0}{M}\langle\frac{1}{-q}\rangle + \frac{M-K_0}{M}\langle\frac{1}{-q+1}\rangle,$$

so the ∇-value for this edge-path is $|-q-(-q+1)| = 1$. For $1 \le i \le m$, the final edge of the edge-path γ_i is of the form

$$\langle\frac{1}{q_i}\rangle \text{---} \frac{K_i}{M}\langle\frac{1}{q_i}\rangle + \frac{M-K_i}{M}\langle\frac{0}{1}\rangle.$$

So the value of each $1 \le i \le m$ is $q_i - 1$ following the discussion preceding Theorem 6.2.

The cycle of ∇-values for the edge-path system is $(1, q_1-1, q_2-1, \ldots q_m-1)$.

6.5.2. *The reference surface R.* Note that the sequence of parameters $(0)_{i=0}^m$ also trivially satisfy the equations from Step 2(a) and 2(b) of the Hatcher-Oertel algorithm with the choice of continued fraction expansion $1/q_i = [[0,-q_i]]$ for $0 \le i \le m$, and therefore defines a connected candidate surface in the complement of $K(1/q_0,\ldots,1/q_m)$. We will call this surface the *reference* surface R. By the Hatcher-Oertel algorithm, the reference surface is incompressible except the ones for $K(-\frac{1}{2},\frac{1}{3},\frac{1}{3}), K(-\frac{1}{2},\frac{1}{3},\frac{1}{4})$, and $K(-\frac{1}{2},\frac{1}{3},\frac{1}{5})$.

In the framework of the Hatcher-Oertel algorithm, the edge-path corresponding to the reference surface has the following form for each q_i:

$$\langle\frac{1}{q_i}\rangle \text{---} \langle 0\rangle\,.$$

The twist number of R. With the exception of γ_0, which has a single slope-increasing edge, each γ_i is slope-decreasing of length 1, thus the twist number of the reference surface R is

$$\mathrm{tw}(R) = 2(m-1)\,. \tag{41}$$

The Euler characteristic of R. The surface R has 1 sheet and the Euler characteristic, and therefore $\chi(R)/\#R$, is

$$\frac{\chi(R)}{\#R} = 1 - m. \tag{42}$$

The cycle of ∇-values of R. The cycle of ∇-values of R is $(-q_0-1, q_1-1, \ldots, q_m-1)$.

6.5.3. *Matching the Jones slope.* The results of Section 4.3 applied to the class of pretzel knots we consider gives the degree of the nth colored Jones polynomial. We show that the quadratic growth rate with respect to n matches the boundary slope of an incompressible surface. The claim is that the Jones slope is either realized by the surface $S(M,x^*)$ or the reference surface R in Section 6.5 depending on $s(q)$ and $s_1(q)$. Note that both $S(M,x^*)$

if $(s(q) \leq 0)$ and R are incompressible by an immediate application of Theorem 6.2, since $m \geq 2$ and $|q_i| > 2$ for all i.

Instead of simply taking $s(q)$, $s_1(q)$ from Section 4.3 and comparing them to the boundary slopes and relevant topological quantities of $S(M, x^*)$ and R, which may be computed from their descriptions in terms of edge-paths, we will take a different approach. We will show that these surfaces may be directly represented by skein elements in the state sum used in Section 3 for certain choices of the color n.

For the pretzel knots that we consider, these skein elements "compete" for the maximum of the degree in the state sum for these choices n. The winner determines the degree of the colored Jones polynomial for the specific color. Quadratic integer programming shows this pattern persists for other colors, where the quadratic growth rate $s(q)$ and $s_1(q)$ remain unchanged. Thus, we may also speak of "surfaces" competing with each other. This provides some topological insight into the strong slope conjecture.

Suppose $s(q) \leq 0$, so that by Lemma 6.3, there is a candidate incompressible surface $S(M, x^*)$. Fixing the color $n = M$, we associate to $S(M, x^*)$ the skein element $\mathcal{S}(M, \tau^*)$, where τ^* is a minimal state with through strands $k(\tau^*) = x^*(M)$ such that $\delta(M, k(\tau^*)) = \delta(M, k)$. Note that by Lemma 3.8, we can construct this skein element. See below Definition 3.1 to recall the definition of $k(\tau^*)$. To R we associate the skein element $\mathcal{S}(0, \tau_0)$, where τ_0 is the Kauffman state that chooses the B-resolution on all the crossings in K^n_+. The Kauffman state τ_0 is also minimal.

Let $bs(R)$ denote the boundary slope of R and $bs(S(M, x^*))$ denote the boundary slope of $S(M, x^*)$. Let τ be a minimal state whose corresponding skein element realizes the degree of the Mth colored Jones polynomial. Using the result of section 4.3 there exist numbers $s = s(q)$ and $s_1 = s_1(q)$ and a periodic function $s_0(n) = s_0(q)(n)$ such that

$$d(k_0, \tau) = \delta(n, k) = sM^2 + s_1 M + s_0(M)\,.$$

Define

$$\mathrm{js}(\mathcal{S}(k_0, \tau)) = \omega(K) + s$$

where $\omega(K)$ is the writhe of K.

Lemma 6.4. Suppose $s(q) \leq 0$. Let R be the reference surface associated to $\mathcal{S}(0, \tau_0)$, and $S(M, x^*)$ the surface associated to the unique degree-maximizing skein element $\mathcal{S}(M, \tau^*)$ from the minimal state τ^* with boundary slope $bs(S(M, x^*))$ and $bs(R)$, respectively. If

$$\mathrm{js}(\mathcal{S}(M, \tau^*)) - \mathrm{js}(\mathcal{S}(0, \tau_0)) = \mathrm{tw}(S(M, x^*)) - \mathrm{tw}(R),$$

then $\mathrm{js}(\mathcal{S}(M, \tau^*))$ is the boundary slope of the surface $S(M, x^*)$.

Proof. Note that

$$\mathrm{js}(\mathcal{S}(0, \tau_0)) = bs(R),$$

by [FKP11, Lemma 4], and

$$bs(R) = \mathrm{tw}(R) - \mathrm{tw}(S_0)$$

where S_0 is a Seifert surface from the Hatcher-Oertel algorithm by (34).

Then by assumption,

$$\begin{aligned}\mathrm{js}(\mathcal{S}(M,\tau^*)) - \mathrm{js}(\mathcal{S}(0,\tau_0)) &= \mathrm{tw}(S(M,x^*)) - \mathrm{tw}(R)\\ \mathrm{js}(\mathcal{S}(M,\tau^*)) &= \mathrm{tw}(S(M,x^*)) - \mathrm{tw}(R) + \mathrm{tw}(R) - \mathrm{tw}(S_0)\\ \mathrm{js}(\mathcal{S}(M,\tau^*)) &= \mathrm{tw}(S(M,x^*)) - \mathrm{tw}(S_0) = bs(S(M,x^*)).\end{aligned}$$

□

Theorem 6.5. *With the same assumptions as Lemma 6.4, We have:*

$$\mathrm{js}(\mathcal{S}(M,\tau^*)) - \mathrm{js}(\mathcal{S}(0,\tau_0)) = \mathrm{tw}(S(M,x^*)) - \mathrm{tw}(R).$$

Proof. We have

$$\begin{aligned}\mathrm{js}(\mathcal{S}(M,\tau^*)) - \mathrm{js}(\mathcal{S}(0,\tau_0)) &= \omega(K) + s(M,x^*) - (\omega(K) + s(M,0))\\ &= s(M,x^*) - s(M,0).\end{aligned}$$

The reference surface R comes from the Kauffman state that chooses the A-resolution on all the crossings in the n-cabled negative twist region with $-q_0$ crossings and the B-resolution everywhere else. Therefore,

$$s(M,0) = \sum_{i=0}^{m} q_i.$$

The number $\mathrm{js}(\mathcal{S}(M,\tau^*))$ is obtained by plugging in $k(\tau^*) = x^*(M)$ into $d(M,\tau^*)$ and extracting the coefficient multiplying M^2.

$$s(M,x^*) = -2\left((q_0+1) + \sum_{i=1}^{m}(q_i-1)(x_i^*)^2 - \frac{1}{2}\sum_{i=0}^{m} q_i\right)$$

Using $(q_i-1)x_i^* = (q_j-1)x_j^*$ and $\sum_{i=1}^{m} x_i^* = 1$, so this may be written as

$$\begin{aligned}&= -2\left((q_0+1) + (q_i-1)x_i^*\sum_{i=1}^{m} x_i^* - \frac{1}{2}\sum_{i=0}^{m} q_i\right)\\ &= -2\left((q_0+1) + (q_i-1)x_i^* - \frac{1}{2}\sum_{i=0}^{m} q_i\right).\end{aligned}$$

By Equations (38) and (41) for the twist numbers of R and $S(M,x^*)$, respectively,

$$s(M,x^*) - s(M,0) = \mathrm{tw}(S(M,x^*)) - \mathrm{tw}(R).$$

□

6.5.4. *Matching the Euler characteristic.* Again we write

$$d(k_0,\tau) = sn^2 + s_1 n + s_0(n)$$

and define

$$\mathrm{jx}(\mathcal{S}(k_0,\tau)) = s_1 - 2s.$$

It is also immediate from the description of the reference surface R in terms of a Kauffman state and [FKP11, Lemma 4] that

$$\mathrm{jx}(\mathcal{S}(0,\tau_0)) = \chi(R) = \frac{\chi(R)}{\#R}.$$

For the proof, see [Lee].

Lemma 6.6. We have

$$\mathrm{jx}(\mathcal{S}(M,\tau^*)) = 2\frac{\chi(S(M,x^*))}{\#S(M,x^*)},$$

where $\chi(S(M,x^*))$ is the Euler characteristic and $\#S(M,x^*)$ is the number of sheets M of the surface $S(M,x^*)$.

Proof. We have by (39),

$$\frac{2\chi(S(M,x^*))}{\#S(M,x^*)} = 8 - 2m + 2q_0 - 2(m-2)x_i^*(q_1-1).$$

The quantity $\mathrm{jx}(\mathcal{S}(M,\tau^*)) = s_1(M,x^*) - 2s(M,x^*)$ is given by plugging in $x^*(M)$ into $d(M,\tau^*)$ and extracting the coefficient multiplying M^2 and M as in the proof of Theorem 6.5.

$$\begin{aligned}
&s_1(M,x^*) - 2s(M,x^*)\\
&= -4\sum_{i=1}^{m}(q_i-1)(x_i^* x_{i,0}^*) - 2\sum_{i=1}^{m}(-2+q_0+q_i)x_i^* + 2\sum_{i=0}^{m} q_i - 2(m-1)\\
&- 2(-2((q_0+1)+(q_i-1)x_i^* - \frac{1}{2}\sum_{i=0}^{m} q_i)).
\end{aligned}$$

From the definition of $x_{i,0}^*$, we have that $\sum_{i=1}^m x_{i,0}^* = 0$, and we also have $\sum_{i=1}^m x_i^* = 1$, $(q_i-1)x_i^* = (q_j-1)x_j^*$. We simplify the equation to

$$\begin{aligned}
&s_1(M,x^*) - 2s(M,x^*)\\
&= -4(q_i-1)x_i^* \underbrace{\sum_{i=1}^{m} x_{i,0}^*}_{0} - 2\sum_{i=1}^{m}(q_0-1+q_i-1)x_i^* + 2\sum_{i=0}^{m} q_i - 2(m-1)\\
&+ 4q_0 + 4 + 4(q_i-1)x_i^* - 2\sum_{i=0}^{m} q_i.
\end{aligned}$$

Gathering terms and rearranging, we get

$$\begin{aligned}
&s_1(M,x^*) - 2s(M,x^*) \\
&= -2(q_0-1)\underbrace{\sum_{i=1}^{m} x_i^*}_{1} - 2m(q_i-1)x_i^* - 2(m-1) + 4q_0 + 4 + 4(q_i-1)x_i^* \\
&= 8 - 2m + 2q_0 - 2(m-2)(q_i-1)x_i^* = \frac{2\chi(S(M,x^*))}{\#S(M,x^*)}.
\end{aligned}$$

□

6.6. **Proof of Theorem 1.2.** Now we prove Theorem 1.2. Fix odd integers $q_0,\dots,q_m$ with $q_0 < -1 < 1 < q_1,\dots,q_m$. Let $P = P(q_0,\dots,q_m)$ denote the pretzel knot $K(\frac{1}{q_0},\frac{1}{q_1},\dots,\frac{1}{q_m})$. By Theorem 6.2, both of the surfaces $S(M,x^*)$ (if $s(q)\le 0$) and R are incompressible by examining their edge-paths and computing their ∇-values. In Section 6.5, Theorem 6.5, Lemma 6.6, and previous work of [FKP11] say that $\mathrm{js}(\mathcal{S}(M,\tau^*)) = bs(S(M,x^*))$, $\mathrm{js}(\mathcal{S}(0,\tau_0)) = bs(R)$, $\mathrm{jx}(\mathcal{S}(M,\tau^*)) = 2\frac{\chi(S(M,x^*))}{\#S(M,x^*)}$, and $\mathrm{jx}(\mathcal{S}(0,\tau_0)) = 2\frac{\chi(R)}{\#R}$.

From Section 4.3, we have the following cases for the degree of the colored Jones polynomial $J_{P,n}(v)$. The choice of the surface detected by the Jones slope swings between the surface $S(M,x^*)$ and the reference surface R.

Case 1: $s(q)<0$. We have that the maximum of $\delta(n,k)$ is given by

$$\delta_P(n) = -2s(q)n^2 - 2s_1(q)n - 2(m-1)n + (n^2+2n)\sum_{i=0}^{m} q_i - 2s_0(q)(n),$$

where recall that $s(q)$ and $s_1(q)$ are explicitly defined by (2) and $s_0(q)(n)$ is a periodic function. By Lemma 4.2, we see that $s(q)$ and $s_1(q)$ for any n are actually the same as when n is equal to the multiple of M, where there is a unique minimal state τ^* with parameters M, x^* realizing $\delta_P(M)$. Thus the fact that $\mathrm{js}(\mathcal{S}(M,\tau^*)) = bs(S(M,x^*))$ and $\mathrm{jx}(\mathcal{S}(M,\tau^*)) = 2\frac{\chi(S(M,x^*))}{\#S(M,x^*)}$ verifies the strong slope conjecture in this case.

Case 2a: $s(q)=0$, $s_1(q)\neq 0$. If $s_1(q)>0$, the maximum $-2s_0(q)(n)$ of $\delta(n,k)$ has no quadratic or linear term, and the reference surface R verifies the conjecture. If $s_1(q)<0$. Then the maximum

$$-2s_1(q)n - 2(m-1)n + (n^2+2n)\sum_{i=0}^{m} q_i - 2s_0(q)(n)$$

of $\delta(n,k)$ is found at maximizers τ^* with parameters n,k^*, again all satisfying $n = k_0^* = k_1^* + \cdots + k_m^*$. Thus the surface $S(M,x^*)$ verifies the conjecture.

Case 2b: $s(q) = s_1(q) = 0$. There is no quadratic or linear term of the maximum of $\delta(n,k)$, thus the reference surface R verifies the conjecture.

Case 3: $s(q)>0$. In this case the maximum of $\delta(n,k)$ also does not have quadratic/linear terms, and the reference surface R verifies the conjecture.

Remark 6.7. With the analogy between the C-curve system coordinates K_i and the real maximizers x^* as established by Lemma 6.3, it is interesting to note that for $n\neq M$, the degrees of the terms in the state sum of the colored Jones polynomial seem to correspond to disconnected surfaces with the same C-curve system coordinates. The boundary slope and normalized Euler characteristic of the disconnected surfaces approximate the connected one associated to the real maximizers when $n = M$.

6.7. **Matching the growth rate to topology for Montesinos knots.** Let $K(r_0, \dots, r_m)$ be a Montesinos knot satisfying the assumptions of Theorem 1.3, and let $P(q_0, \dots, q_m)$ be the associated pretzel knot. Similar to the case of pretzel knots, we define a surface $S(M, x^*)$ that corresponds to a real maximizer $\mathcal{S}(M, \tau^*)$ of $\delta(n, k)$ as in Theorem 5.1, when we apply the method of Lagrange multipliers as in Lemma 4.2 to (28). We give the explicit description of the surface in terms of edge-path systems from the Hatcher-Oertel algorithm below. We will see that these surfaces are built from extending the surfaces of the associated pretzel knots.

6.7.1. *The surface $S(M, x^*)$.* The edge-path system of $S(M, x^*)$ is described as follows.

For $i = 0$, say $r_0 = [0, a_1, a_2, \dots, a_{\ell_{r_0}}]$ for $a_i < 0$, we take the following continued fraction expansion

$$[[-1, \underbrace{-2, \dots, -2}_{-a_1-1 \text{ times}}, a_2 - 1 - 1, \underbrace{-2, \dots, -2}_{-a_3-1 \text{ times}}, a_{2j} - 1 - 1, \underbrace{-2, \dots, -2}_{-a_{2j+1}-1 \text{ times}}, \dots]], \tag{43}$$

with corresponding edge-path

$$\left\langle [[0, a_1, a_2, \dots, a_{\ell_{r_0}}]] \right\rangle \text{———} \cdots \text{———} \langle [[-1, -2, -2]] \rangle \text{———} \langle [[-1, -2]] \rangle \text{———} \langle -1 \rangle .$$

For $1 \le i \le m$, say $r_i = [0, a_1, a_2, \dots, a_{\ell_{r_i}}]$ for $a_i > 0$, we take the following continued fraction expansion

$$[[0, -a_1 - 1, \underbrace{-2, \dots, -2}_{a_2-1 \text{ times}}, -a_3 - 1 - 1, \underbrace{-2, \dots, -2}_{a_4-1 \text{ times}}, -a_{2j+1} - 1 - 1, \underbrace{-2, \dots, -2}_{a_{2j+2}-1 \text{ times}}, \dots]], \tag{44}$$

with corresponding edge-path

$$\left\langle [[0, a_1, a_2, \dots, a_{\ell_{r_i}}]] \right\rangle \text{———} \cdots \text{———} \langle [[0, -a_1 - 1, -2]] \rangle \text{———} \langle [[0, -a_1 - 1]] \rangle \text{———} \langle 0 \rangle .$$

We let $n = M$ be the least common multiple of the denominators of $\{x^*_{i,1}\}$ as given below, reduced to lowest terms. Write

$$x^*_i(M) = x^*_{i,1} M + x^*_{i,0}, \text{ so } x^*_{i,1} = \frac{a_i^{-1}}{\sum_j a_j^{-1}} \text{ and } x^*_{i,0} = \frac{1}{2} \frac{\sum_j (b_j - b_i) a_i^{-1} a_j^{-1}}{\sum_j a_j^{-1}}.$$

where $a_i = q_i$ and $b_i = q_0 + q_i - 1$. $S(M, x^*)$ is the candidate surface from the Hatcher-Oertel algorithm with M sheets and C-coordinates $\{-M, Mx^*_1, Mx^*_2, \dots, Mx^*_m\}$. [elaborate?]

We similarly have

Lemma 6.8. Suppose $q = (q_0, q_1, \dots, q_m)$ is such that $s(q) \le 0$. There is a candidate surface $S(M, x^*)$ from the Hatcher-Oertel algorithm with M sheets and C-coordinates

$$\{-M, Mx^*_1, Mx^*_2, \dots, Mx^*_m\}.$$

Proof. Let $K_i = Mx_i^*$ for $1 \le i \le m$, and $0 \le K_0 \le M$, $q_0 \le -q \le -2$ such that
$$K_0 + M(q-2) = K_1(q_1 - 1), \tag{45}$$
We specify a candidate surface $S(M, x^*)$ in terms of edge-paths $\{\gamma_i\}_{i=0}^m$, by tacking onto the existing edge-path system for the associated pretzel knot $P(q_0, q_1, \dots, q_m)$:

The edge-path γ_0 for r_0 is
$$\big\langle [[0, a_1, a_2, \dots, a_{\ell_{r_0}}]] \big\rangle \text{---} \cdots \text{---} \langle \frac{1}{q_0} \rangle \text{---} \langle \frac{1}{q_0+1} \rangle \text{---} \cdots \text{---} \frac{K_0}{M} \langle \frac{1}{-q} \rangle + \frac{M-K_0}{M} \langle \frac{1}{-q+1} \rangle.$$
For $i \neq 0$, we have the edge-path γ_i:
$$\big\langle [[0, a_1, a_2, \dots, a_{\ell_{r_i}}]] \big\rangle \text{---} \cdots \text{---} \langle \frac{1}{q_i} \rangle \text{---} \frac{K_i}{M} \langle \frac{1}{q_i} \rangle + \frac{M-K_i}{M} \langle \frac{0}{1} \rangle.$$
Provided that K_0, q satisfying (45) exist, this edge-path system satisfies the equations coming from (a) and (b) of Step (2) of the algorithm. We have already verified that K_0, q exist when $s(q) \le 0$ in Lemma 6.3. Thus, there is a candidate surface with $\{-M, Mx_1^*, Mx_2^*, \dots, Mx_m^*\}$ as the C-coordinates in the tangle corresponding to r_i. $\square$

We also define a reference surface R for $K(r_0, r_1, \dots, r_m)$ associated to the skein $\mathcal{S}(0, \tau_0)$, where τ_0 is the all-B state on K_+^n.

6.7.2. *The reference surface R.* For the reference surface R, we have for each r_i, the edge-path system corresponding to the following continued fraction expansion
For $r_0 = [0, a_1, a_2, \dots, a_{\ell_{r_0}}]$ for $a_i < 0$, we take the following continued fraction expansion.
$$[[0, -a_1, a_2 - 1, \underbrace{-2, \dots, -2}_{-a_3 - 1 \text{ times}}, a_4 - 1 - 1, \underbrace{-2, \dots, -2}_{-a_5 - 1 \text{ times}}, a_{2j} - 1 - 1, \underbrace{-2, \dots, -2}_{-a_{2j+1} - 1 \text{ times}}, \dots]], \tag{46}$$
with corresponding edge-path
$$\big\langle [[0, a_1, a_2, \dots, a_{\ell_{r_0}}]] \big\rangle \text{---} \cdots \text{---} \langle [[0, -a_1]] \rangle \text{---} \langle 0 \rangle .$$
For $1 \le i \le m$, say $r_i = [0, a_1, a_2, \dots, a_{\ell_{r_i}}]$ for $a_i > 0$, we take the following continued fraction expansion.
$$[[0, -a_1 - 1, \underbrace{-2, \dots, -2}_{a_2 - 1 \text{ times}}, -a_3 - 1 - 1, \underbrace{-2, \dots, -2}_{a_4 - 1 \text{ times}}, -a_{2j+1} - 1 - 1, \underbrace{-2, \dots, -2}_{a_{2j+2} - 1 \text{ times}}, \dots]], \tag{47}$$
with corresponding edge-path
$$\big\langle [[0, a_1, a_2, \dots, a_{\ell_{r_i}}]] \big\rangle \text{---} \cdots \text{---} \langle [[0, -a_1 - 1, -2]] \rangle \text{---} \langle [[0, -a_1 - 1]] \rangle \text{---} \langle 0 \rangle .$$
Again, both R and $S(M, x^*)$ are incompressible by a direct application of Proposition 6.2.

6.8. **Proof of Theorem 1.3.** [Putting everything together we prove Theorem 1.3.

Proof. Theorem 5.4 gives js_K and jx_K in terms of the Jones slope and the normalized Euler characteristic of the associated pretzel knot P. The resulting formulas are matched with the boundary slope and normalized Euler characteristic of incompressible surfaces by Lemma **??**. $\square$

]

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