SUPER-REPRESENTATIONS OF 3-MANIFOLDS AND TORSION POLYNOMIALS

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Abstract. Torsion polynomials connect the genus of a hyperbolic knot (a topological invariant) with the discrete faithful representation (a geometric invariant). Using a new combinatorial structure of an ideal triangulation of a 3-manifold that involves edges as well as faces, we associate a polynomial to a cusped hyperbolic manifold that conjecturally agrees with the $\mathbb{C}^2$-torsion polynomial, which conjecturally detects the genus of the knot. The new combinatorics is motivated by super-geometry in dimension 3, and more precisely by super-Ptolemy assignments of ideally triangulated 3-manifolds and their $\mathrm{OSp}_{2|1}(\mathbb{C})$-representations.

Contents

Date: 12 March 2024.

Key words and phrases. super-Teichmuller space, 3-manifolds, hyperbolic geometry, Thurston, representations, torsion polynomials, Thurston norm, genus, ideal triangulations, face matrices, Neumann-Zagier matrices, super-ptolemy variables, super-shapes, ortho-symplectic super-group.

1. Introduction

1.1. **Overview.** A well-known topic in geometry and topology is the study of representations of surface groups into simple Lie groups. Recently, this topic has been extended by replacing simple Lie groups (such as $\mathrm{SL}_2(\mathbb{C})$) with super-Lie groups, and most notably by the orthosymplectic group $\mathrm{OSp}_{2|1}(\mathbb{C})$. These representations have been studied by at least three different points of view, namely as character varieties, as cluster algebras, and as super-Teichmüller space; see for instance [PZ19, IPZ18, MOZ, She] as well as [Wit].

All this is about surfaces. In our paper we extend this study in the context of 3-manifolds equipped with an ideal triangulation. Explicitly,

(a) We introduce super-Ptolemy coordinates of 3-dimensional triangulations and prove that they parametrize $\mathrm{OSp}_{2|1}(\mathbb{C})$-representations of fundamental groups of 3-manifolds; see Section 2.
(b) Using such representations, we define a 1-loop polynomial and show that it is a topological invariant; see Theorem 3.4.
(c) We show that an $\mathrm{SL}_2(\mathbb{C})$-representation lifts to an $\mathrm{OSp}_{2|1}(\mathbb{C})$-representation if and only if the 1-loop polynomial, evaluated at $t = 1$, vanishes; see Theorem 3.6.
(d) We conjecture that our 1-loop polynomial coincides with the $\mathbb{C}^2$-torsion polynomial whose degree conjecturally detects the genus of a knot [DFJ12]; see Conjecture 3.7. Both polynomials have values in the trace field of a cusped hyperbolic 3-manifold, and are explicitly computable. Doing so, we check our conjecture explicitly for the 4_1 knot.

1.2. **Torsion polynomials: a Thurstonian connection.** The complement $M = S^3 \setminus K$ of a hyperbolic knot K in 3-space has two interesting invariants, both defined by Thurston

- the genus of K, i.e., the least genus of all embedded spanning surfaces of K, generalized to the Thurston norm on $H_2(M, \partial M, \mathbb{R})$ [Thu86],
- the discrete faithful representation $\pi_1(M) \to \mathrm{PSL}_2(\mathbb{C})$ [Thu97].

These two invariants, one topological and another geometric, are beautifully linked to each other via torsion polynomials revealing, to quote Agol–Dunfield [AD20], a "remarkable Thurstonian connection between the topology and geometry of 3-manifolds". Torsion polynomials are twisted versions of the Alexander polynomial, where one twists the homology of the infinite cyclic cover of M using an $\mathrm{SL}_2(\mathbb{C})$-lift ρ_{geom} of the geometric representation of M, or a symmetric power $\mathrm{Sym}^{n-1}(\rho_{\mathrm{geom}})$ of it, the corresponding polynomial being denoted by $\tau_{M,\rho_{\mathrm{geom}},n}(t)$. These geometric invariants are Laurent polynomials in t with coefficients in the trace field of M, and a key feature is that their degrees give bounds for the genus of the knot. More precisely, one has

$$2 \cdot \mathrm{genus}(K) - 1 \geq \frac{1}{n} \deg_t \tau_{M,\rho_{\mathrm{geom}},n}(t) \tag{1}$$

for all $n \geq 2$. When $n = 3$ (or n being odd bigger than 1), examples show that the above bound is not sharp, but when $n = 2$, it was conjectured in [DFJ12], for reasons that are not entirely clear, and proven in several families that the inequality in (1) becomes an equality [AD20]. As Agol–Dunfield state, this is a remarkable Thurstonian connection between the topology and geometry of 3-manifolds.

The paper concerns two seemingly unrelated problems from 3-dimensional hyperbolic geometry and character varieties of 3-manifold groups. Below, M denotes a cusped hyperbolic 3-manifold.

Problem 1. Can one compute the $\mathbb{C}^2$-torsion polynomial $\tau_{M,\rho,2}(t)$ of an $\mathrm{SL}_2(\mathbb{C})$-representation ρ of $\pi_1(M)$ from an ideal triangulation $\mathcal{T}$ of M?

Problem 2. What is the geometric meaning of the set of $\mathrm{SL}_2(\mathbb{C})$-representations ρ of $\pi_1(M)$ whose $\mathbb{C}^2$-torsion $\tau_{M,\rho,2}(1)$ is vanishing?

We will answer both problems by introducing super-Ptolemy assignments for $\mathcal{T}$ (see Section 2) and defining a 1-loop polynomial, which is a topological invariant (see Theorem 3.4). The conjectural equality of the 1-loop polynomial with the $\mathbb{C}^2$-torsion polynomial (see Conjecture 3.7) gives a solution to the first problem. The vanishing of the 1-loop polynomial at $t = 1$ gives a necessary and sufficient condition for an $\mathrm{SL}_2(\mathbb{C})$-representation of $\pi_1(M)$ to lift to an $\mathrm{OSp}_{2|1}(\mathbb{C})$-representation (see Theorem 3.6) thus answering the second problem.

Both the 1-loop and the $\mathbb{C}^2$-torsion polynomials have coefficients in the trace field of the representation and can be exactly computed, and doing so we can confirm our conjecture for the 4_1 knot. Moreover, the conjecture is proven in subsequent work (using a different set of ideas) for all fibered cusped hyperbolic 3-manifolds [GYa]. Aside from the fibered case, Nathan Dunfield has confirmed the conjecture numerically for all manifolds in the `OrientableCuspedCensus` and beyond by using the methods of `SnapPy` [CDGW].

Our super-Ptolemy assignments lead to new combinatorics of ideal triangulations beyond the well-known Neumann–Zagier matrices, the latter encoding which tetrahedra wind around each edge. This newly found combinatorics involves linear equations associated to faces and tetrahedra, dictated by the representations of $\pi_1(M)$ into $\mathrm{OSp}_{2|1}(\mathbb{C})$; see Section 1.3 below for a detailed discussion.

1.3. **Super-geometry in dimension 3.** A byproduct of our paper is a solution to the following problem that advances recent work of many people on super-Riemann surfaces and their coordinates into the third dimension. For detailed description of super-geometry in dimension 2, super-Teichmüller space and super-Riemann surfaces, we refer the reader to [Wit, PZ19, IPZ18, APT17, She, MOZ, OS19].

Problem 3. What are super-Ptolemy assignments of 3-dimensional triangulations?

A solution to the above problem is detailed in Section 2. Here, we highlight the important features of super-Ptolemy assignments in dimension 3, postponing their precise definitions, notations and properties for later.

Recall that a Ptolemy assignment is a map $c : \mathcal{T}^1 \to \mathbb{C}^*$ from the set of oriented edges of an ideal triangulation $\mathcal{T}$ that satisfies $c(-e) = -c(e)$ for all edges e and the equation

$$c_{01}c_{23} - c_{02}c_{13} + c_{03}c_{12} = 0 \qquad (2)$$

for each tetrahedron, where $c_{ij} = c(e_{ij})$ and e_{ij} is the (i, j)-edge of a tetrahedron [GTZ15] (see also [GGZ15b]). There are canonical bijections between the sets of generically decorated $(\mathrm{SL}_2(\mathbb{C}), N_2)$-representations, solutions to the Ptolemy equations, and natural $(\mathrm{SL}_2(\mathbb{C}), N_2)$-cocycles of the truncated triangulation[1] studied in detail in [GTZ15].

[1]where $N_2 = \{\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)\}$ is the unipotent subgroup of $\mathrm{SL}_2(\mathbb{C})$

Whereas a Ptolemy assignment c assigns a nonzero complex number to every edge of an ideal triangulation, a super-Ptolemy assignment is a pair of maps (c, θ) that assign an invertible even element of a Grassmann algebra at each edge and an odd element at each face of an ideal triangulation. Instead of Equation (2), a super-Ptolemy assignment satisfies one equation

$$c_{01}c_{23} - c_{02}c_{13} + c_{03}c_{12} + c_{01}c_{03}c_{12}c_{13}c_{23}\theta_0\theta_2 = 0 \tag{3}$$

for each tetrahedron as well as one equation for each face

$$\begin{aligned} c_{12}\theta_0 - c_{02}\theta_1 + c_{01}\theta_2 &= 0 \qquad & c_{13}\theta_0 - c_{03}\theta_1 + c_{01}\theta_3 &= 0 \\ c_{23}\theta_0 - c_{03}\theta_2 + c_{02}\theta_3 &= 0 \qquad & c_{23}\theta_1 - c_{13}\theta_2 + c_{12}\theta_3 &= 0 \end{aligned} \tag{4}$$

of each tetrahedron of $\mathcal{T}$. Here $c_{ij} = c(e_{ij})$ where e_{ij} is the edge (i, j) and $\theta_k = \theta(f_k)$ where f_k is the face opposite to the vertex k as in Figure 3.

Super-Ptolemy assignments lead to a fundamental correspondence described by a pair of bijections

$$\left\{ \begin{matrix} \text{Generically decorated} \\ (\mathrm{OSp}_{2|1}(\mathbb{C}), N)\text{-reps on } M \end{matrix} \right\} \xleftrightarrow{1-1} P_{2|1}(\mathcal{T}) \xleftrightarrow{1-1} \left\{ \begin{matrix} \text{Natural } (\mathrm{OSp}_{2|1}(\mathbb{C}), N)\text{-} \\ \text{cocycles on } \mathring{\mathcal{T}} \end{matrix} \right\} \tag{5}$$

which are given explicitly in Figures 1 and 2.

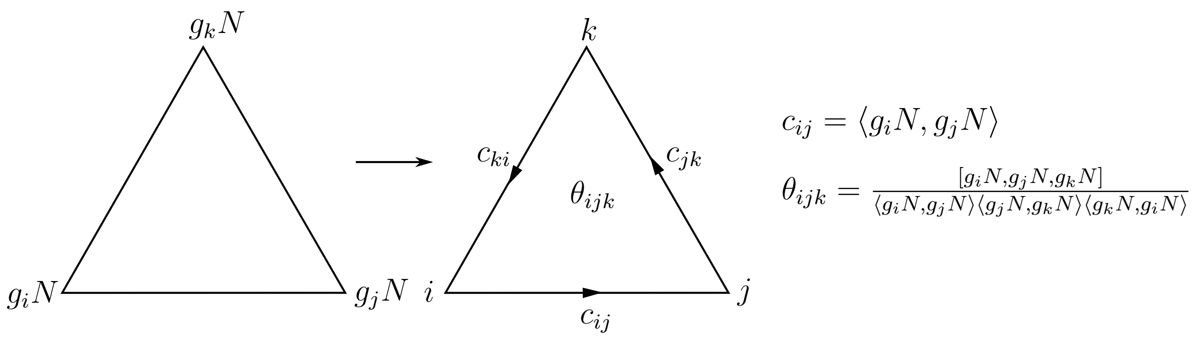

FIGURE 1. From decorated representations to Ptolemy assignments, with the bilinear and trilinear forms as in (26) and (27).

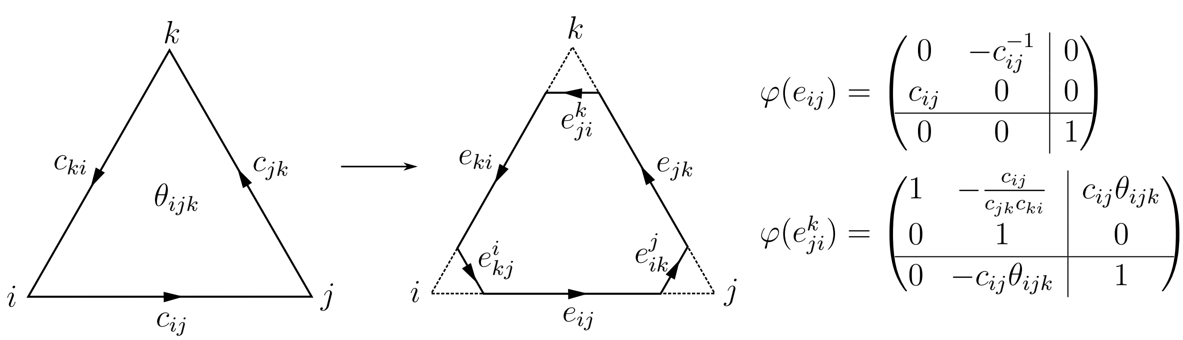

FIGURE 2. From Ptolemy assignments to natural cocycles.

As written in (4), a super-Ptolemy assignment (c,θ) satisfies linear equations in θ. It turns out that these linear equations can be written in a matrix form

$$F_c\,\theta = 0 \tag{6}$$

where F_c is a sqaure matrix whose entries are given by the Ptolemy variable c with some signs. Obviously, we are interested in the case of F_c being singular, otherwise θ should be trivial. This motivates (see e.g. Section 2.9) the definition of a 1-loop invariant

$$\delta_{\mathcal{T},c,2} = \frac{1}{c_1\cdots c_N}\left(\prod_{\Delta}\frac{1}{c(e_\Delta)}\right)\det F_c \tag{7}$$

given in terms of the determinant of F_c. What's more, it motivates the definition of a 1-loop polynomial

$$\delta_{\mathcal{T},c,2}(t) = \frac{1}{c_1\cdots c_N}\left(\prod_{\Delta}\frac{1}{c(e_\Delta)}\right)\det F_c(t) \tag{8}$$

given in terms of the determinant of a t-twisted version $F_c(t)$ of F_c (see Section 3.3). This 1-loop polynomial $\delta_{\mathcal{T},c,2}(t)$ is unchanged under Pachner 2–3 moves (see Theorem 3.4) and its value $\delta_{\mathcal{T},c,2}(1)$ at $t=1$ determines whether the $\mathrm{SL}_2(\mathbb{C})$-representation ρ of $\pi_1(M)$ corresponding to the Ptolemy assignment c admits an $\mathrm{OSp}_{2|1}(\mathbb{C})$-lift or not (see Theorem 3.6). Based on the analogy with the 1-loop polynomial $\delta_{\mathcal{T},c,3}(t)$ of [GYb], we conjecture that the 1-loop polynomial $\delta_{\mathcal{T},c,2}(t)$ equals to the $\mathbb{C}^2$-torsion polynomial $\tau_{M,\rho,2}(t)$ (see Conjecture 3.7).

2. $\mathrm{OSp}_{2|1}(\mathbb{C})$-representations of 3-manifolds

2.1. **The orthosymplectic group.** In this section we recall the definition of the orthosymplectic super-Lie group $\mathrm{OSp}_{2|1}(\mathbb{C})$. For a detailed description of super-manifolds and super-Lie groups we refer the reader to [Ber87, Man91, CR88].

Let $\mathrm{G}(\mathbb{C})$ be the Grassmann algebra over the complex numbers with unit 1 generated by ϵ_i for $i\in\mathbb{N}$:

$$\mathrm{G}(\mathbb{C}) = \mathbb{C}\langle 1,\epsilon_1,\epsilon_2,\dots \mid 1\epsilon_i=\epsilon_i=\epsilon_i 1,\ \epsilon_i\epsilon_j=-\epsilon_j\epsilon_i \text{ for all } i,j\in\mathbb{N}\rangle\,. \tag{9}$$

It is a $\mathbb{Z}/2\mathbb{Z}$-graded algebra with the unit having degree 0 and each ϵ_i having degree 1. We denote by $\mathrm{G}_0(\mathbb{C})$ and $\mathrm{G}_1(\mathbb{C})$ its even and odd part, respectively, and by $\mathrm{G}_0^*(\mathbb{C})$ the set of invertible elements in $\mathrm{G}_0(\mathbb{C})$. There is an algebra epimorphism $\sharp:\mathrm{G}(\mathbb{C})\to\mathbb{C}$ sending all ϵ_i to 0, hence an element $e\in\mathrm{G}(\mathbb{C})$ is invertible if and only if $\sharp(e)\neq 0$. We write $\sharp(e)$ simply as $e^\sharp$ and call it the body of e.

An even $n|m\times n|m$-matrix g is of the form

$$g = \left(\begin{array}{c|c} A & B \\ \hline C & D\end{array}\right) \tag{10}$$

where $A\in M_{n,n}(\mathrm{G}_0(\mathbb{C}))$, $B\in M_{n,m}(\mathrm{G}_1(\mathbb{C}))$, $C\in M_{m,n}(\mathrm{G}_1(\mathbb{C}))$, and $D\in M_{m,m}(\mathrm{G}_0(\mathbb{C}))$. The super-transpose of g is given by

$$g^{\mathrm{st}} = \left(\begin{array}{c|c} A^t & C^t \\ \hline -B^t & D^t\end{array}\right) \tag{11}$$

and the Berezinian (or super-determinant) of g is defined as

$$\mathrm{Ber}(g) = \det(A - BD^{-1}C)\det(D)^{-1} \tag{12}$$

provided that A and D are invertible.

The orthosymplectic group $\mathrm{OSp}_{2|1}(\mathbb{C})$ is the group of even $2|1 \times 2|1$-matrices g satisfying

$$g^{st}\left(\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & 0 & 0 \\ \hline 0 & 0 & -1 \end{array}\right) g = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & 0 & 0 \\ \hline 0 & 0 & -1 \end{array}\right) \tag{13}$$

and $\mathrm{Ber}(g) = 1$. Writing an even $2|1 \times 2|1$-matrix explicitly as

$$g = \left(\begin{array}{cc|c} a & b & \alpha \\ c & d & \beta \\ \hline \gamma & \delta & e \end{array}\right) \quad \text{for} \quad a, b, c, d, e \in \mathrm{G}_0(\mathbb{C}), \quad \alpha, \beta, \gamma, \delta \in \mathrm{G}_1(\mathbb{C}), \tag{14}$$

the defining equations (13) of $\mathrm{OSp}_{2|1}(\mathbb{C})$ are

$$ad - bc - \gamma\delta = e^2 + 2\alpha\beta = 1, \qquad a\beta - c\alpha - e\gamma = b\beta - d\alpha - e\delta = 0 \tag{15}$$

together with

$$\mathrm{Ber}(g) = (ad - bc)(1 - 2\alpha\beta e^{-2})e^{-1} = 1\,. \tag{16}$$

Note that these equations imply that $e^{\pm 1} = 1 \mp \gamma\delta$; in particular, $e^\sharp = 1$. Note also that the inverse of g as in (14) is given by

$$g^{-1} = \left(\begin{array}{cc|c} d & -b & \delta \\ -c & a & -\gamma \\ \hline -\beta & \alpha & e \end{array}\right). \tag{17}$$

The special linear group $\mathrm{SL}_2(\mathbb{C})$ embeds in $\mathrm{OSp}_{2|1}(\mathbb{C})$ in an obvious way:

$$\mathrm{SL}_2(\mathbb{C}) \hookrightarrow \mathrm{OSp}_{2|1}(\mathbb{C}), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left(\begin{array}{cc|c} a & b & 0 \\ c & d & 0 \\ \hline 0 & 0 & 1 \end{array}\right). \tag{18}$$

Conversely, applying the epimorphism $\sharp$ entrywise, we obtain an epimorphism

$$\mathrm{OSp}_{2|1}(\mathbb{C}) \twoheadrightarrow \mathrm{SL}_2(\mathbb{C}), \quad \left(\begin{array}{cc|c} a & b & \alpha \\ c & d & \beta \\ \hline \gamma & \delta & e \end{array}\right) \mapsto \begin{pmatrix} a^\sharp & b^\sharp \\ c^\sharp & d^\sharp \end{pmatrix}. \tag{19}$$

Abusing notation, we also denote by $\sharp$ the above epimorphism and refer to $\sharp(g) = g^\sharp$ as the body of $g \in \mathrm{OSp}_{2|1}(\mathbb{C})$. It follows from (18) and (19) that for any group G the epimorphism $\sharp$ induces a surjective map

$$\mathrm{Hom}(G, \mathrm{OSp}_{2|1}(\mathbb{C}))/_\sim \overset{\sharp}{\twoheadrightarrow} \mathrm{Hom}(G, \mathrm{SL}_2(\mathbb{C}))/_\sim \tag{20}$$

where the quotient $\sim$ is given by conjugation.

Remark 2.1. For full generality we use the Grassmann algebra with infinitely many generators as in (9), but one may use one with finitely many generators. In particular, if we use the Grassmann algebra with one odd generator

$$\mathbb{C}\langle 1, \epsilon \mid 1\epsilon = \epsilon = \epsilon 1,\ \epsilon^2 = 0\rangle = \mathbb{C}[\epsilon]/(\epsilon^2), \tag{21}$$

then its even and odd parts are $\mathbb{C}$ and $\mathbb{C}\epsilon$ respectively, and the orthosymplectic group $\mathrm{OSp}_{2|1}(\mathbb{C})$ reduces to the special affine transformation group $\mathrm{SL}_2(\mathbb{C}) \ltimes \mathbb{C}^2$. Indeed, if there is only one odd generator, then the map

$$\mathrm{OSp}_{2|1}(\mathbb{C}) \to \mathrm{SL}_2(\mathbb{C}) \ltimes \mathbb{C}^2, \qquad \left(\begin{array}{cc|c} a & b & \alpha\epsilon \\ c & d & \beta\epsilon \\ \hline \gamma\epsilon & \delta\epsilon & 1 \end{array}\right) \mapsto \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right) \tag{22}$$

is an isomorphism. This may seem to simplify things too much but, in fact, will be sufficient for our 1-loop invariants–see Section 3 below.

2.2. **The unipotent subgroup and pairings.** Since the body $g^\sharp$ of $g \in \mathrm{OSp}_{2|1}(\mathbb{C})$ is in $\mathrm{SL}_2(\mathbb{C})$, the natural action of $\mathrm{OSp}_{2|1}(\mathbb{C})$ on $\mathrm{G}_0(\mathbb{C})^2 \oplus \mathrm{G}_1(\mathbb{C})$ restricts to an action on $A^{2|1}$. Here $A^{2|1}$ is the pre-image of $\mathbb{C}^2 \setminus \{(0,0)^t\}$ under the map

$$\mathrm{G}_0(\mathbb{C})^2 \oplus \mathrm{G}_1(\mathbb{C}) \to \mathbb{C}^2, \quad (a, b, \alpha)^t \mapsto (a^\sharp, b^\sharp)^t. \tag{23}$$

The induced action is transitive, and the stabilizer group at $(1,0,0)^t \in A^{2|1}$ is the unipotent subgroup

$$N = \left\{ \left(\begin{array}{cc|c} 1 & b & \alpha \\ 0 & 1 & 0 \\ \hline 0 & -\alpha & 1 \end{array}\right) \;\middle|\; b \in \mathrm{G}_0(\mathbb{C}),\ \alpha \in \mathrm{G}_1(\mathbb{C}) \right\} \tag{24}$$

of $\mathrm{OSp}_{2|1}(\mathbb{C})$. This induces a bijection

$$\mathrm{OSp}_{2|1}(\mathbb{C})/N \ \leftrightarrow\ A^{2|1}, \qquad gN \ \leftrightarrow\ \text{left column of } g\,. \tag{25}$$

The space $A^{2|1} \simeq \mathrm{OSp}_{2|1}(\mathbb{C})/N$ comes equipped with an even-valued bilinear pairing

$$\langle \cdot, \cdot \rangle : A^{2|1} \times A^{2|1} \to \mathrm{G}_0(\mathbb{C}), \qquad \left\langle \begin{pmatrix} a \\ b \\ \alpha \end{pmatrix}, \begin{pmatrix} c \\ d \\ \beta \end{pmatrix} \right\rangle := ad - bc - \alpha\beta \tag{26}$$

and an odd-valued trilinear pairing

$$[\cdot,\cdot,\cdot] : A^{2|1} \times A^{2|1} \times A^{2|1} \to \mathrm{G}_1(\mathbb{C}), \quad \left[\begin{pmatrix} a \\ b \\ \alpha \end{pmatrix}, \begin{pmatrix} b \\ c \\ \beta \end{pmatrix}, \begin{pmatrix} e \\ f \\ \gamma \end{pmatrix} \right] := \det \begin{pmatrix} a & b & e \\ b & c & f \\ \alpha & \beta & \gamma \end{pmatrix} - 2\alpha\beta\gamma\,. \tag{27}$$

Both pairings are skew-symmetric and $\mathrm{OSp}_{2|1}(\mathbb{C})$-invariant

$$\langle v, w \rangle = \langle gv, gw \rangle, \qquad [u, v, w] = [gu, gv, gw], \qquad g \in \mathrm{OSp}_{2|1}(\mathbb{C})\,. \tag{28}$$

2.3. **Super-Ptolemy assignments.** Let M be a compact 3-manifold with non-empty boundary and $\mathcal{T}$ be an ideal triangulation of its interior. We denote by $\mathcal{T}^1$ and $\mathcal{T}^2$ the oriented edges and the unoriented faces of $\mathcal{T}$, respectively. We first assume that $\mathcal{T}$ is ordered, i.e., each tetrahedron has a vertex-ordering respecting the face-gluing, but this condition can be relaxed; see Section 2.8 below.

Definition 2.2. A *super-Ptolemy assignment* on $\mathcal{T}$ is a pair of maps

$$c : \mathcal{T}^1 \to \mathrm{G}_0^*(\mathbb{C}), \qquad \theta : \mathcal{T}^2 \to \mathrm{G}_1(\mathbb{C}) \tag{29}$$

satisfying $c(-e) = -c(e)$ for all $e \in \mathcal{T}^1$ and

$$c_{01}c_{23} - c_{02}c_{13} + c_{03}c_{12} + c_{01}c_{03}c_{12}c_{13}c_{23}\theta_0\theta_2 = 0 \tag{30}$$

as well as

$$\begin{aligned} &E_{\Delta,f_3}: \; c_{12}\theta_0 - c_{02}\theta_1 + c_{01}\theta_2 = 0 \qquad && E_{\Delta,f_2}: \; c_{13}\theta_0 - c_{03}\theta_1 + c_{01}\theta_3 = 0 \\ &E_{\Delta,f_1}: \; c_{23}\theta_0 - c_{03}\theta_2 + c_{02}\theta_3 = 0 && E_{\Delta,f_0}: \; c_{23}\theta_1 - c_{13}\theta_2 + c_{12}\theta_3 = 0 \end{aligned} \tag{31}$$

for each tetrahedron Δ of $\mathcal{T}$. Here $c_{ij} = c(e_{ij})$ where e_{ij} is the oriented edge $[i,j]$ of Δ and $\theta_k = \theta(f_k)$ where f_k is the face of Δ opposite to the vertex k as in Figure 3.

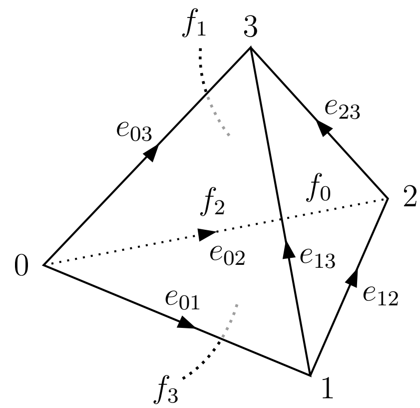

Figure 3. Edge and face labels for a tetrahedron.

Lemma 2.3. If any two of (31) together with (30) are satisfied, then so are the other two.

Proof. Multiplying the first equation in (31) by θ_0 implies that $c_{01}\theta_0\theta_2 = c_{02}\theta_0\theta_1$. Similarly, we deduce that $c_{ij}^{-1}\theta_i\theta_j$ does not depend on a choice of $i \neq j$. It follows that Equation (30) is equivalent to

$$c_{01}c_{23} - c_{02}c_{13} + c_{03}c_{12} + c_{01}c_{02}c_{03}c_{12}c_{13}c_{23}\, c_{ij}^{-1}\theta_i\theta_j = 0 \quad \text{for } i \neq j. \tag{32}$$

This implies that $(c_{01}c_{23} - c_{02}c_{13} + c_{03}c_{12})\theta_i = 0$ for all i. Then one easily checks that any three out of the four equations in (31) are linearly dependent. For instance,

$$c_{01}E_{\Delta,f_1} - c_{02}E_{\Delta,f_2} + c_{03}E_{\Delta,f_3} = 0\,. \tag{33}$$

This completes the proof. $\square$

Remark 2.4. Each equation in (31) corresponds to a face of a tetrahedron Δ. It follows that

$$\begin{aligned} E_{\Delta,f_0} &= 0 \\ -E_{\Delta,f_1} &= 0 \\ E_{\Delta,f_2} &= 0 \\ -E_{\Delta,f_3} &= 0 \end{aligned} \Leftrightarrow F_{\Delta,c}\begin{pmatrix}\theta_0\\ \theta_1\\ \theta_2\\ \theta_3\end{pmatrix} = 0 \tag{34}$$

where $F_{\Delta,c}$ is a 4×4 matrix whose rows and columns are indexed by the faces of Δ, given explicitly by

$$F_{\Delta,c} = \begin{pmatrix} 0 & c_{23} & -c_{13} & c_{12} \\ -c_{23} & 0 & c_{03} & -c_{02} \\ c_{13} & -c_{03} & 0 & c_{01} \\ -c_{12} & c_{02} & -c_{01} & 0 \end{pmatrix}. \tag{35}$$

Note that $F_{\Delta,c}$ is a skew-symmetric matrix whose (i,j)-entry for $i \neq j$ is, up to a sign, the Ptolemy variable of the edge $f_i \cap f_j$. Note also that $F_{\Delta,c}$ has rank 3, whereas its body $F^\sharp_{\Delta,c}$ (the matrix obtained by applying the epimorphism $\sharp$ to all entries of $F_{\Delta,c}$) has rank 2.

Let $P_{2|1}(\mathcal{T})$ be the set of all super-Ptolemy assignments on $\mathcal{T}$. Composing the epimorphism $\sharp : \mathrm{G}(\mathbb{C}) \to \mathbb{C}$ with a super-Ptolemy assignment (c,θ), we obtain a Ptolemy assignment $c^\sharp : \mathcal{T}^1 \to \mathbb{C}^*$ (note that θ vanishes if we apply $\sharp$), i.e., $c^\sharp$ satisfies

$$c^\sharp_{01}c^\sharp_{23} - c^\sharp_{02}c^\sharp_{13} + c^\sharp_{03}c^\sharp_{12} = 0 \tag{36}$$

for each tetrahedron of $\mathcal{T}$ [GTZ15]. On the other hand, any Ptolemy assignment on $\mathcal{T}$ forms a super-Ptolemy assignment with the trivial map $\mathcal{T}^2 \to \mathrm{G}_1(\mathbb{C})$, assigning 0 to all faces. Therefore, the epimorphism $\sharp$ induces a surjective map

$$P_{2|1}(\mathcal{T}) \overset{\sharp}{\twoheadrightarrow} P_2(\mathcal{T}) \tag{37}$$

where $P_2(\mathcal{T})$ is the set of all Ptolemy assignments on $\mathcal{T}$.

2.4. **Natural cocycles.** After truncating the ideal tetrahedra of $\mathcal{T}$, one obtains a cell decomposition $\mathring{\mathcal{T}}$ of M; see Figure 4. The 1-cells $\mathring{\mathcal{T}}^1$ of $\mathring{\mathcal{T}}$ consist of long and short edges, and the 2-cells $\mathring{\mathcal{T}}^2$ consist of hexagons (one for each face of $\mathcal{T}$) and triangles (one for each vertex of a tetrahedron of $\mathcal{T}$). Note that the triangles of $\mathring{\mathcal{T}}$ give a triangulation of the boundary ∂M of M and that the 2-skeleton of $\mathring{\mathcal{T}}$ defines a natural groupoid associated to $\mathcal{T}$, whose generators are the short and long edges of $\mathring{\mathcal{T}}$ and relations are the triangular and hexagonal faces of $\mathring{\mathcal{T}}$.

Definition 2.5. A *natural* $(\mathrm{OSp}_{2|1}(\mathbb{C}), N)$*-cocycle* or simply *natural cocycle* on $\mathring{\mathcal{T}}$ is a map $\varphi : \mathring{\mathcal{T}}^1 \to \mathrm{OSp}_{2|1}(\mathbb{C})$ of the form

$$\varphi(\text{short}) = \left(\begin{array}{cc|c} 1 & a & \theta \\ 0 & 1 & 0 \\ \hline 0 & -\theta & 1 \end{array}\right), \qquad \varphi(\text{long}) = \left(\begin{array}{cc|c} 0 & -b^{-1} & 0 \\ b & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \tag{38}$$

that maps the hexagons and the triangles to the identity. In other words, a natural cocycle is an $\mathrm{OSp}_{2|1}(\mathbb{C})$-representation of the groupoid of $\mathcal{T}$ whose generators have the form (38).

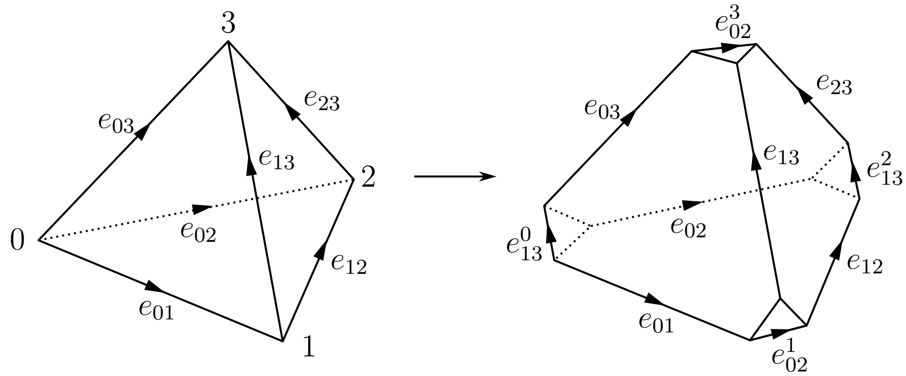

Figure 4. Truncating an ideal tetrahedron.

Given a natural cocycle φ, let us denote

$$\varphi_0(\text{short}) = \varphi(\text{short})_{1,2} \in \mathrm{G}_0(\mathbb{C}), \qquad \varphi_0(\text{long}) = \varphi(\text{long})_{2,1} \in \mathrm{G}_0^*(\mathbb{C}),$$
$$\varphi_1(\text{short}) = \varphi(\text{short})_{1,3} \in \mathrm{G}_1(\mathbb{C})\,.$$

We now express the cocycle condition for φ in terms of conditions on φ_0 and φ_1. It follows from (17) that $\varphi(-e) = \varphi(e)^{-1}$ for $e \in \mathring{\mathcal{T}}^1$ if and only if $\varphi_i(-e) = -\varphi_i(e)$ for $i = 0, 1$.

Lemma 2.6. φ satisfies the cocycle condition for a hexagon if and only if

$$\varphi_0(e^k_{ji}) = -\frac{\varphi_0(e_{ij})}{\varphi_0(e_{jk})\varphi_0(e_{ki})} \tag{39}$$

for all cyclic permutations (i, j, k) of $(0, 1, 2)$ and

$$\frac{\varphi_1(e^2_{10})}{\varphi_0(e_{01})} = \frac{\varphi_1(e^0_{21})}{\varphi_0(e_{12})} = \frac{\varphi_1(e^1_{02})}{\varphi_0(e_{20})} \tag{40}$$

where e_{ij} and e^k_{ij} denote the edges of the hexagon as in Figure 5.

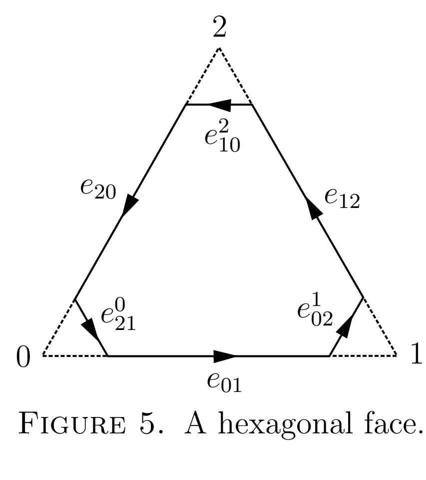

Figure 5. A hexagonal face.

Proof. The proof follows from a straightforward computation for the cocycle condition, i.e., comparing the entries of $\varphi(e^0_{21})\varphi(e_{01})\varphi(e^1_{02}) = \varphi(e_{20})^{-1}\varphi(e^2_{10})^{-1}\varphi(e_{12})^{-1}$. $\square$

Let $\theta \in \mathrm{G}_1(\mathbb{C})$ be the odd element given in Equation (40). It follows from Lemma 2.6 that the φ_0 and φ_1-values on the short edges are determined by the φ_0-values on the long edges with θ. More precisely, we have

$$\varphi(e_{ji}^k) = \left(\begin{array}{cc|c} 1 & -\frac{\varphi_0(e_{ij})}{\varphi_0(e_{jk})\varphi_0(e_{ki})} & \varphi_0(e_{ij})\theta \\ 0 & 1 & 0 \\ \hline 0 & -\varphi_0(e_{ij})\theta & 1 \end{array}\right) \tag{41}$$

for any cyclic permutations (i,j,k) of $(0,1,2)$. Then identifying each long edge of $\mathring{\mathcal{T}}$ with an edge of $\mathcal{T}$ naturally and placing the odd element θ to the corresponding hexagonal face of $\mathring{\mathcal{T}}$, or equivalently, to the face of $\mathcal{T}$, we deduce that a natural cocycle φ is determined by two maps

$$c : \mathcal{T}^1 \to \mathrm{G}_0^*(\mathbb{C}), \quad \theta : \mathcal{T}^2 \to \mathrm{G}_1(\mathbb{C})\,.$$

Note that c is the restriction of φ_0 to the long edges.

Lemma 2.7. φ satisfies the cocycle condition for all triangular faces of $\mathring{\mathcal{T}}$ if and only if the pair (c,θ) defined above is a super-Ptolemy assignment, i.e., satisfies (30) and (31) for all tetrahedra of $\mathcal{T}$.

Proof. Labeling the vertices of a tetrahedron with $\{0,1,2,3\}$ and using the same notation as in Lemma 2.6, the cocycle condition for the triangular faces has the form $\varphi(e^i_{jl}) = \varphi(e^i_{jk})\varphi(e^i_{kl})$. For instance, the triangular face near the vertex 0 gives $\varphi(e^0_{31}) = \varphi(e^0_{32})\varphi(e^0_{21})$:

$$\left(\begin{array}{cc|c} 1 & -\frac{c_{13}}{c_{30}c_{01}} & c_{13}\theta_2 \\ 0 & 1 & 0 \\ \hline 0 & -c_{13}\theta_2 & 1 \end{array}\right) = \left(\begin{array}{cc|c} 1 & -\frac{c_{23}}{c_{30}c_{02}} & c_{23}\theta_1 \\ 0 & 1 & 0 \\ \hline 0 & -c_{23}\theta_1 & 1 \end{array}\right) \left(\begin{array}{cc|c} 1 & -\frac{c_{12}}{c_{20}c_{01}} & c_{12}\theta_3 \\ 0 & 1 & 0 \\ \hline 0 & -c_{12}\theta_3 & 1 \end{array}\right). \tag{42}$$

Comparing the entries of the above equation, we obtain (30) and the first equation in (31). We obtain the other three equations of (31) similarly from the other triangular faces. ☐

Lemmas 2.6 and 2.7 show that there is a one-to-one correspondence

$$P_{2|1}(\mathcal{T}) \xleftrightarrow{1-1} \left\{ \begin{array}{c} \text{Natural } (\mathrm{OSp}_{2|1}(\mathbb{C}), N)\text{-} \\ \text{cocycles on } \mathring{\mathcal{T}} \end{array} \right\}. \tag{43}$$

This verifies one bijection in the fundamental correspondence (5) and its explicit formula given in Figure 2.

Remark 2.8. Applying the epimorphism $\sharp : \mathrm{OSp}_{2|1}(\mathbb{C}) \to \mathrm{SL}_2(\mathbb{C})$ to both sides of (43), the correspondence (43) reduces to the bijection between $P_2(\mathcal{T})$ and the set of natural

$(\mathrm{SL}_2(\mathbb{C}), N_2)$-cocycles [GGZ15b, Sec.1.2]. Namely, there is a commutative diagram

$$\begin{tikzcd} P_{2|1}(\mathcal{T}) \arrow[r, leftrightarrow, "1-1"] \arrow[d, twoheadrightarrow, "\sharp"] & \left\{\begin{matrix}\text{Natural } (\mathrm{OSp}_{2|1}(\mathbb{C}), N)\text{-} \\ \text{cocycles on } \mathring{\mathcal{T}}\end{matrix}\right\} \arrow[d, twoheadrightarrow, "\sharp"] \\ P_2(\mathcal{T}) \arrow[r, leftrightarrow, "1-1"] & \left\{\begin{matrix}\text{Natural } (\mathrm{SL}_2(\mathbb{C}), N_2)\text{-} \\ \text{cocycles on } \mathring{\mathcal{T}}\end{matrix}\right\} \end{tikzcd} \tag{44}$$

2.5. **Decorations.** Let $\widetilde{M}$ be the universal cover of M and $\widetilde{\mathcal{T}}$ be the ideal triangulation of its interior induced from $\mathcal{T}$. We denote by $\widetilde{\mathcal{T}}^0$ the set of (ideal) vertices of $\widetilde{\mathcal{T}}$. We will use similar notations for $\mathring{\mathcal{T}}$.

Definition 2.9. (a) An $(\mathrm{OSp}_{2|1}(\mathbb{C}), N)$-*representation* is an $\mathrm{OSp}_{2|1}(\mathbb{C})$-representation ρ of $\pi_1(M)$ such that $\rho(\pi_1(\partial M)) \subset N$ up to conjugation.
(b) A *decoration* of an $(\mathrm{OSp}_{2|1}(\mathbb{C}), N)$-representation ρ is a map $D : \widetilde{\mathcal{T}}^0 \to \mathrm{OSp}_{2|1}(\mathbb{C})/N$ such that

$$D(\gamma \cdot v) = \rho(\gamma)D(v) \quad \text{ for } \gamma \in \pi_1(M) \text{ and } v \in \widetilde{\mathcal{T}}^0 . \tag{45}$$

(c) A decoration D is called *generic* if $\langle D(v_0), D(v_1)\rangle^\sharp \neq 0$ for all vertices v_0 and v_1 joined by an edge of $\widetilde{\mathcal{T}}$. Here we use the identification (25) and the bilinear pairing (26).

In what follows, by a generically decorated representation we mean an $(\mathrm{OSp}_{2|1}(\mathbb{C}), N)$-representation with a generic decoration. For simplicity we identify a generically decorated representation (ρ, D) with $(g\rho g^{-1}, gD)$ for all $g \in \mathrm{OSp}_{2|1}(\mathbb{C})$. Note that if D is a (generic) decoration of ρ, then gD is a (generic) decoration of $g\rho g^{-1}$.

Lemma 2.10. For N-cosets gN and hN with $\langle gN, hN\rangle^\sharp \neq 0$ there is a unique pair of coset-representatives $g' \in gN$ and $h' \in hN$ such that $(g')^{-1}h'$ is of the form

$$(g')^{-1}h' = \left(\begin{array}{cc|c} 0 & -c^{-1} & 0 \\ c & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) . \tag{46}$$

Moreover, $c = \langle gN, hN\rangle$.

Proof. We may assume that $gN = N$ and hN corresponds to $(a, c, \gamma)^t \in A^{2|1}$ with $c^\sharp \neq 0$. Then a straightforward computation

$$\left(\begin{array}{cc|c} 1 & f & \epsilon \\ 0 & 1 & 0 \\ \hline 0 & -\epsilon & 1 \end{array}\right)^{-1} \left(\begin{array}{cc|c} a & b & \alpha \\ c & d & \beta \\ \hline \gamma & \delta & e \end{array}\right) = \left(\begin{array}{cc|c} a - cf - \epsilon\gamma & b - df - \epsilon\delta & \alpha - e\epsilon \\ c & d & \beta \\ \hline c\epsilon + \gamma & d\epsilon + \delta & \epsilon\beta + e \end{array}\right) \tag{47}$$

shows that the matrix in the right-hand side is of the form (46) only if $\epsilon = -\gamma/c$, $f = a/c$, and $d = \beta = 0$. Then it follows from the defining equations of $\mathrm{OSp}_{2|1}(\mathbb{C})$ that $\delta = 0$, $e = 1$, $b = -1/c$, and $\alpha = -\gamma/c$. This proves that the desired pair of coset-representatives exists uniquely. □

For a generically decorated representation (ρ, D) Lemma 2.10 implies that there exists a unique map

$$\psi : \tilde{\mathring{\mathcal{T}}}^0 \to \mathrm{OSp}_{2|1}(\mathbb{C}) \tag{48}$$

such that

- $\psi(v) \in D(w)$ if v is in the boundary component of $\widetilde{M}$ corresponding to $w \in \widetilde{\mathcal{T}}^0$;
- $\psi(v_0)^{-1}\psi(v_1)$ is a matrix of the form (46) if v_0 and v_1 are joined by an long edge.

It follows from (45) that $\psi(\gamma \cdot v) = \rho(\gamma)\psi(v)$ for $\gamma \in \pi_1(M)$, hence $\psi(\gamma \cdot v_0)^{-1}\psi(\gamma \cdot v_1) = \psi(v_0)^{-1}\psi(v_1)$ for any vertices v_0 and v_1. Therefore if we define

$$\varphi : \mathring{\mathcal{T}}^1 \to \mathrm{OSp}_{2|1}(\mathbb{C}), \quad \varphi(e) := \psi(v_0)^{-1}\psi(v_1) \tag{49}$$

for any lift $[v_0, v_1]$ of an edge $e \in \mathring{\mathcal{T}}^1$, then φ is well-defined and by definition is a natural cocycle. This construction induces a one-to-one correspondence

$$\begin{Bmatrix} \text{Natural } (\mathrm{OSp}_{2|1}(\mathbb{C}), N)\text{-} \\ \text{cocycles on } \mathring{\mathcal{T}} \end{Bmatrix} \overset{1-1}{\longleftrightarrow} \begin{Bmatrix} \text{Generically decorated} \\ (\mathrm{OSp}_{2|1}(\mathbb{C}), N)\text{-reps on } M \end{Bmatrix}. \tag{50}$$

Remark 2.11. Applying the epimoprhism $\sharp : \mathrm{OSp}_{2|1}(\mathbb{C}) \to \mathrm{SL}_2(\mathbb{C})$ to both sides of (50), the correspondence (50) reduces to the bijection between natural $(\mathrm{SL}_2(\mathbb{C}), N_2)$-cocycles and generically decorated $(\mathrm{SL}_2(\mathbb{C}), N_2)$-representations; see [GGZ15b, Sec 1.2].

Combining the correspondences (43) and (50), we obtain

$$\begin{Bmatrix} \text{Generically decorated} \\ (\mathrm{OSp}_{2|1}(\mathbb{C}), N)\text{-reps on } M \end{Bmatrix} \overset{1-1}{\longleftrightarrow} P_{2|1}(\mathcal{T}) \tag{51}$$

given explicitly on edges $e_{ij} \in \mathcal{T}^1$ and faces $f_{ijk} \in \mathcal{T}^2$ by

$$c(e_{ij}) = \langle g_i N, g_j N \rangle, \qquad \theta(f_{ijk}) = \frac{[g_i N, g_j N, g_k N]}{\langle g_i N, g_j N \rangle \langle g_j N, g_k N \rangle \langle g_k N, g_i N \rangle}. \tag{52}$$

Here we use the identification (25) and the bilinear and trilinear pairings (26) and (27). This verifies the first bijection in the fundamental correspondence (5) and the explicit formula given in Figure 1.

The next theorem is a direct consequence of the fundamental correspondence (5) (see also the diagram (44)).

Theorem 2.12. *There is a map* $P_{2|1}(\mathcal{T}) \to \mathrm{Hom}(\pi_1(M), \mathrm{OSp}_{2|1}(\mathbb{C}))/_{\sim}$ *which fits into a commutative diagram*

$$\begin{tikzcd} P_{2|1}(\mathcal{T}) \arrow[r] \arrow[d, two heads, "\sharp"] & \mathrm{Hom}(\pi_1(M), \mathrm{OSp}_{2|1}(\mathbb{C}))/_{\sim} \arrow[d, two heads, "\sharp"] \\ P_2(\mathcal{T}) \arrow[r] & \mathrm{Hom}(\pi_1(M), \mathrm{SL}_2(\mathbb{C}))/_{\sim} \end{tikzcd} \tag{53}$$

and whose image is the set of all conjugacy classes of $(\mathrm{OSp}_{2|1}(\mathbb{C}), N)$*-representations admitting a generic decoration.*

2.6. **Action on $P_{2|1}(\mathcal{T})$.** Let h be the the number of (ideal) vertices of $\mathcal{T}$. There is an action of $\mathrm{G}_0^*(\mathbb{C})^h$ on $P_{2|1}(\mathcal{T})$:

$$\mathrm{G}_0^*(\mathbb{C})^h \times P_{2|1}(\mathcal{T}) \to P_{2|1}(\mathcal{T}), \quad (x,(c,\theta)) \mapsto x\cdot(c,\theta) = (x\cdot c,\ x\cdot\theta) \tag{54}$$

where $x\cdot c$ and $x\cdot\theta$ are defined as follows. Regarding that $x=(x_1,\dots,x_h)$ is assigned to the vertices of $\mathcal{T}$,

$$x\cdot c : \mathcal{T}^1 \to \mathrm{G}_0^*(\mathbb{C}), \quad e \mapsto x_i x_j c(e) \tag{55}$$

where x_i and x_j are assigned to the vertices of e, and

$$x\cdot\theta : \mathcal{T}^2 \to \mathrm{G}_1(\mathbb{C}), \quad f \mapsto (x_i x_j x_k)^{-1}\theta(f) \tag{56}$$

where x_i, x_j, and x_k are assigned to the vertices of f. One easily checks that $x\cdot(c,\theta)$ satisfies Equations (30) and (31), i.e., $x\cdot(c,\theta)\in P_{2|1}(\mathcal{T})$. This action reduces to the $(\mathbb{C}^*)^h$-action on $P_2(\mathcal{T})$ described in [GGZ15b, §4] if we forget θ and restrict x to $(\mathbb{C}^*)^h$.

Theorem 2.13. *The super-Ptolemy assignments (c,θ) and $x\cdot(c,\theta)$ determine up to conjugation the same representation.*

Proof. Let (ρ, D) be a generically decorated representation corresponding to $(c,\theta)\in P_{2|1}(\mathcal{T})$. Regarding $x=(x_1,\dots,x_h)$ is assigned to the vertices of $\mathcal{T}$, we define

$$x\cdot D : \widetilde{\mathcal{T}}^0 \to \mathrm{OSp}_{2|1}(\mathbb{C})/N, \quad v \mapsto x_i D(v) \tag{57}$$

if v is a lift of the i-th vertex of $\mathcal{T}$. Here we use the identification $\mathrm{OSp}_{2|1}(\mathbb{C})/N \simeq A^{2|1}$, hence the scalar multiplication is well-defined. Then $x\cdot D$ is also a generic decoration of ρ, and Equation (52) implies that $(\rho, x\cdot D)$ corresponds to $(x\cdot c, x\cdot\theta)$. □

Remark 2.14. For $k\in\mathrm{G}_0^*(\mathbb{C})$ we have $(k,\dots,k)\cdot(c,\theta) = (k^2c, k^{-3}\theta)$. In particular, (c,θ) and $(c,-\theta)$ determine the same representation up to conjugation.

2.7. **(m,l)-deformation.** One can generalize super-Ptolemy assignments and all arguments that we used in previous sections to $\mathrm{OSp}_{2|1}(\mathbb{C})$-representations that may not $(\mathrm{OSp}_{2|1}(\mathbb{C}), N)$. This can be done by considering the natural action of $\mathrm{OSp}_{2|1}(\mathbb{C})$ on $A^{2|1}/\mathrm{G}_0^*(\mathbb{C})$, instead of $A^{2|1}$, where the quotient is given by identifying $(a,b,\alpha)^t$ and $c(a,b,\alpha)^t$ for all $c\in\mathrm{G}_0^*(\mathbb{C})$. The stabilizer group of this action at $[(1,0,0)^t]$ is

$$B = \left\{ \left(\begin{array}{cc|c} a & b & \alpha \\ 0 & a^{-1} & 0 \\ \hline 0 & -a^{-1}\alpha & 1 \end{array}\right) \;\middle|\; a\in\mathrm{G}_0^*(\mathbb{C}),\ b\in\mathrm{G}_0(\mathbb{C}),\ \alpha\in\mathrm{G}_1(\mathbb{C}) \right\}. \tag{58}$$

We fix a cocycle σ that assigns an element of $\mathrm{G}_0^*(\mathbb{C})$ to each short edge of $\mathring{\mathcal{T}}$, and consider representations $\rho : \pi_1(M) \to \mathrm{OSp}_{2|1}(\mathbb{C})$ satisfying up to conjugation

$$\rho(\gamma) = \left(\begin{array}{cc|c} \sigma(\gamma) & * & * \\ 0 & \sigma(\gamma)^{-1} & 0 \\ \hline 0 & * & 1 \end{array}\right) \quad \text{for all } \gamma\in\pi_1(\partial M). \tag{59}$$

Here we use the same notation σ for the cocycle and for the morphism $\pi_1(\partial M)\to\mathrm{G}_0^*(\mathbb{C})$ induced from it; hopefully this will cause no confusion. As a generalization of Definitions 2.2, 2.5, and 2.9, we define:

Definition 2.15. A *σ-deformed super-Ptolemy assignment* on $\mathcal{T}$ is a pair of maps

$$c : \mathcal{T}^1 \to \mathrm{G}_0^*(\mathbb{C}), \qquad \theta : \mathcal{T}^2 \to \mathrm{G}_1(\mathbb{C}) \tag{60}$$

satisfying $c(-e) = -c(e)$ for all $e \in \mathcal{T}^1$ and

$$c_{01}c_{23} - \frac{\sigma_{03}^2\sigma_{12}^3}{\sigma_{03}^1\sigma_{12}^0}c_{02}c_{13} + \frac{\sigma_{02}^3\sigma_{13}^2}{\sigma_{02}^1\sigma_{13}^0}c_{03}c_{12} + \frac{\sigma_{02}^3}{\sigma_{02}^1}c_{01}c_{03}c_{12}c_{13}c_{23}\theta_0\theta_2 = 0 \tag{61}$$

as well as

$$\begin{aligned} E_{\Delta,f_3} &: \frac{\sigma_{23}^1}{\sigma_{23}^0}c_{12}\theta_0 - c_{02}\theta_1 + \frac{\sigma_{12}^0}{\sigma_{12}^3}c_{01}\theta_2 = 0 \qquad & E_{\Delta,f_2} &: c_{13}\theta_0 - \frac{\sigma_{01}^3}{\sigma_{01}^2}c_{03}\theta_1 + \frac{\sigma_{12}^0}{\sigma_{12}^3}c_{01}\theta_3 = 0 \\ E_{\Delta,f_1} &: \frac{\sigma_{03}^1}{\sigma_{03}^2}c_{23}\theta_0 - \frac{\sigma_{01}^3}{\sigma_{01}^2}c_{03}\theta_2 + c_{02}\theta_3 = 0 \qquad & E_{\Delta,f_0} &: \frac{\sigma_{03}^1}{\sigma_{03}^2}c_{23}\theta_1 - c_{13}\theta_2 + \frac{\sigma_{23}^1}{\sigma_{23}^0}c_{12}\theta_3 = 0 \end{aligned} \tag{62}$$

for each tetrahedron Δ of $\mathcal{T}$. Here $\sigma_{jk}^i \in \mathrm{G}_0^*(\mathbb{C})$ is the element assigned by σ at the short edge that is near to the vertex i and parallel to the edge $[j,k]$; see Figure 5.

Definition 2.16. A *σ-deformed natural cocycle* on $\mathring{\mathcal{T}}$ is map $\varphi : \mathring{\mathcal{T}}^1 \to \mathrm{OSp}_{2|1}(\mathbb{C})$ of the form

$$\varphi(\text{short}) = \left(\begin{array}{cc|c} \sigma(\text{short}) & a & \theta \\ 0 & \sigma(\text{short})^{-1} & 0 \\ \hline 0 & -\sigma(\text{short})^{-1}\theta & 1 \end{array}\right), \qquad \varphi(\text{long}) = \left(\begin{array}{cc|c} 0 & -b^{-1} & 0 \\ b & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \tag{63}$$

that maps the hexagons and the triangles to the identity. In other words, a natural cocycle is an $\mathrm{OSp}_{2|1}(\mathbb{C})$-representation of the groupoid of $\mathcal{T}$ whose generators have the form (63).

Definition 2.17. (a) An $(\mathrm{OSp}_{2|1}(\mathbb{C}), B)$*-representation* is an $\mathrm{OSp}_{2|1}(\mathbb{C})$-representation ρ of $\pi_1(M)$ such that $\rho(\pi_1(\partial M)) \subset B$ up to conjugation.
(b) A *decoration* of an $(\mathrm{OSp}_{2|1}(\mathbb{C}), B)$-representation ρ is a map $D : \widetilde{\mathcal{T}}^0 \to \mathrm{OSp}_{2|1}(\mathbb{C})/B$ satisfying (45).
(c) A decoration D is called *generic* if $\langle D(v_0), D(v_1)\rangle^\sharp \neq 0$ for all vertices v_0 and v_1 joined by an edge of $\widetilde{\mathcal{T}}$. Note that the condition makes sense, even though $\langle D(v_0), D(v_1)\rangle$ can be only defined up to $\mathrm{G}_0^*(\mathbb{C})$.

Repeating the same arguments in Sections 2.3–2.5 (see also [Yoo19, §2]), we obtain

$$\left\{\begin{array}{c}\text{Generically decorated}\\ (\mathrm{OSp}_{2|1}(\mathbb{C}), B)\text{-reps on } M \\ \text{satisfying (59)}\end{array}\right\} \xleftrightarrow{1-1} P_{2|1}^\sigma(\mathcal{T}) \xleftrightarrow{1-1} \left\{\begin{array}{c}\sigma\text{-deformed natural}\\ \text{cocycles on } \mathring{\mathcal{T}}\end{array}\right\} \tag{64}$$

where $P_{2|1}^\sigma(\mathcal{T})$ is the set of all σ-deformed super-Ptolemy assignments on $\mathcal{T}$. We note that:
1. The same argument used in Lemma 2.3 shows that any three of (62) are linearly dependent.
2. Composing the epimorphism $\sharp : \mathrm{G}(\mathbb{C}) \to \mathbb{C}$ with $(c,\theta) \in P_{2|1}^\sigma(\mathcal{T})$, we obtain a $\sigma^\sharp$-deformed Ptolemy assignment $c^\sharp : \mathcal{T}^1 \to \mathbb{C}^*$, i.e., $c^\sharp$ satisfies

$$c_{01}^\sharp c_{23}^\sharp - \frac{(\sigma_{03}^2)^\sharp(\sigma_{12}^3)^\sharp}{(\sigma_{03}^1)^\sharp(\sigma_{12}^0)^\sharp}c_{02}^\sharp c_{13}^\sharp + \frac{(\sigma_{02}^3)^\sharp(\sigma_{13}^2)^\sharp}{(\sigma_{02}^1)^\sharp(\sigma_{13}^0)^\sharp}c_{03}^\sharp c_{12} c_{01}^\sharp = 0 \tag{65}$$

for each tetrahedron of $\mathcal{T}$ [Yoo19]. This defines a map

$$P^{\sigma}_{2|1}(\mathcal{T}) \xrightarrow{\sharp} P^{\sigma^\sharp}_{2}(\mathcal{T}) \tag{66}$$

where $P^{\sigma^\sharp}_{2}(\mathcal{T})$ is the set of all $\sigma^\sharp$-deformed Ptolemy assignments on $\mathcal{T}$. This map is surjective if $\sigma^\sharp = \sigma$, i.e. σ takes values in $\mathbb{C}^*$.

3. The σ-deformed natural cocycle φ corresponding to $(c,\theta) \in P^{\sigma}_{2|1}(\mathcal{T})$ is explicitly given by

$$\varphi(e^k_{ji}) = \left(\begin{array}{cc|c} \sigma^k_{ji} & -\frac{\sigma^j_{ik}}{\sigma^i_{kj}}\frac{c_{ij}}{c_{jk}c_{ki}} & c_{ij}\theta/\sigma^i_{kj} \\ 0 & 1/\sigma^k_{ji} & 0 \\ \hline 0 & -c_{ij}\theta/(\sigma^i_{kj}\sigma^k_{ji}) & 1 \end{array}\right), \quad \varphi(e_{ij}) = \left(\begin{array}{cc|c} 0 & -c^{-1}_{ij} & 0 \\ c_{ij} & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \tag{67}$$

for Figure 5 where (i,j,k) is a cyclic permutation of $(0,1,2)$.

4. There is a $\mathrm{G}^*_0(\mathbb{C})^h$-action on $P^{\sigma}_{2|1}(\mathcal{T})$ defined in the same way as that on $P_{2|1}(\mathcal{T})$; see Section 2.6. Also, Theorem 2.13 holds for $(c,\theta) \in P^{\sigma}_{2|1}(\mathcal{T})$: the σ-deformed super-Ptolemy assignments (c,θ) and $x \cdot (c,\theta)$ determine the same representation up to conjugation.

The next theorem is a direct consequence of the correspondence (64).

Theorem 2.18. *There is a map $P^{\sigma}_{2|1}(\mathcal{T}) \to \mathrm{Hom}(\pi_1(M), \mathrm{OSp}_{2|1}(\mathbb{C}))/_\sim$ which fits into*

$$\begin{tikzcd} P^{\sigma}_{2|1}(\mathcal{T}) \arrow[r] \arrow[d, "\sharp"] & \mathrm{Hom}(\pi_1(M), \mathrm{OSp}_{2|1}(\mathbb{C}))/_\sim \arrow[d, twoheadrightarrow, "\sharp"] \\ P^{\sigma^\sharp}_{2}(\mathcal{T}) \arrow[r] & \mathrm{Hom}(\pi_1(M), \mathrm{SL}_2(\mathbb{C}))/_\sim \end{tikzcd} \tag{68}$$

and whose image is the set of all conjugacy classes of $(\mathrm{OSp}_{2|1}(\mathbb{C}), B)$-representations admitting a generic decoration and satisfying (59).

2.8. **Concrete triangulations.** In the previous Section 2.7 we defined the super-Ptolemy assignements for ordered triangulations. In this section we discuss how to define these assignments for concrete triangulations, that is, for triangulations where each tetrahedron comes with a bijection of its vertices with those of the standard 3-simplex. Concrete triangulations were used in [GGZ15a, Sec.2] to define the gluing equations of $\mathrm{PGL}_n(\mathbb{C})$ representations of $\pi_1(M)$, as well as in [GGZ15b, Sec.2.1] to define the Ptolemy variety, with or without an obstruction class. Note that all the triangulations in `SnapPy` and `Regina` are concrete [CDGW, Bur].

Recall that the Ptolemy variables are assigned to oriented edges and satisfy the relation that reversing the orientation of an edge reverses the value of the Ptolemy variable. In a concrete triangulation, several edges of tetrahedra before face-pairings are identified with each other, and this identification introduces signs when the edges are identified in an orientation-reversed way. For a concrete example, see Examples 3.1.1 and 3.1.2 of [GGZ15b], for the case of the trivial obstruction class, and Example 3.2.1 of [GGZ15b] for a non-trivial obstruction class.

We now discuss the presence of these signs in a systematic way. A face-pairing of an ordered triangulation glues two faces preserving vertex-orderings, hence it preserves the orientations of the faces induced from vertex-orderings. Therefore, it is sufficient for ordered triangulations to consider faces with vertex-order in counterclockwise as in Figure 5. However, as concrete triangulations do not have such a property, both faces with vertex-order in clockwise and ones with counterclockwise appear. Therefore, we consider both sides of a face with each side having one odd element, i.e., we assign one odd element to each oriented face. This seems to double the number of odd elements, but in fact, two odd elements θ and θ' assigned to the front and back sides of a face (as in Figure 6) determine each other.

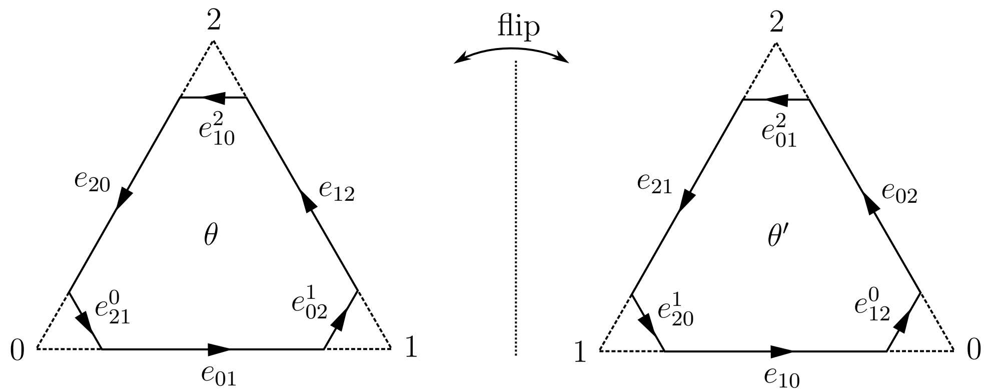

FIGURE 6. Front/back sides of a face

Recall (67) that for any cyclic permutation (i, j, k) of $(0, 1, 2)$ we have

$$\varphi(e^k_{ji}) = \left(\begin{array}{cc|c} \sigma^k_{ji} & -\frac{\sigma^j_{ik}}{\sigma^i_{kj}}\frac{c_{ij}}{c_{jk}c_{ki}} & c_{ij}\theta/\sigma^i_{kj} \\ 0 & 1/\sigma^k_{ji} & 0 \\ \hline 0 & -c_{ij}\theta/(\sigma^i_{kj}\sigma^k_{ji}) & 1 \end{array}\right). \tag{69}$$

Applying the same formula to the back side of the face, we obtain

$$\varphi(e^k_{ij}) = \left(\begin{array}{cc|c} \sigma^k_{ij} & -\frac{\sigma^i_{jk}}{\sigma^j_{ki}}\frac{c_{ji}}{c_{ik}c_{kj}} & c_{ji}\theta'/\sigma^j_{ki} \\ 0 & 1/\sigma^k_{ij} & 0 \\ \hline 0 & -c_{ji}\theta'/(\sigma^j_{ki}\sigma^k_{ij}) & 1 \end{array}\right). \tag{70}$$

Note that $c_{ij} = -c_{ji}$ and $\sigma^i_{jk} = 1/\sigma^i_{kj}$. Then a straightforward computation shows that $\varphi(e^k_{ji})\varphi(e^k_{ij}) = I$ if and only if

$$\theta' = \sigma^k_{ij}\sigma^i_{jk}\sigma^j_{ki}\theta\,. \tag{71}$$

This shows that super-Ptolemy assignments on a concrete triangulation are described by the same equations (61) and (62) but some of θ_i may be replaced by θ'_i. Note that $\theta_0, \dots, \theta_3$ in (62) are assigned to the sides of $f_0, \dots, f_3$ that face front. Note also that if a face-pairing preserves the orientation of the faces induced from the vertex-orderings, then only one of θ_i or θ'_i appears in the face equations, and otherwise, both θ_i and θ'_i appear.

2.9. **Example: the 4_1 knot.** Let $\mathcal{T}$ be the standard ideal triangulation of the knot complement of 4_1 obtained by the face-pairings of two ordered ideal tetrahedra Δ_1 and Δ_2 with edges e_1 and e_2 and with faces $f_1, \dots, f_4$. See Figure 7. We choose a cocycle σ on the short edges for $m, \ell \in \mathrm{G}_0^*(\mathbb{C})$ as follows (see [Yoo19, Ex.2.8]):

$$\sigma(s_2) = \sigma(s_5) = \sigma(s_8) = \sigma(s_{11}) = m, \quad \sigma(s_6) = \sigma(s_9) = \sigma(s_{12}) = m^{-1} \tag{72}$$

$$\sigma(s_4) = \sigma(s_7) = \sigma(s_{10}) = 1, \quad \sigma(s_1) = \ell^{-1}m^{-2}, \quad \sigma(s_3) = \ell m \tag{73}$$

where $s_1, \ldots, s_{12}$ are the short edges of $\mathring{\mathcal{T}}$ as in Figure 7. Note that the morphism induced by σ sends the meridian and canonical longitude of the knot to m and ℓ, respectively.

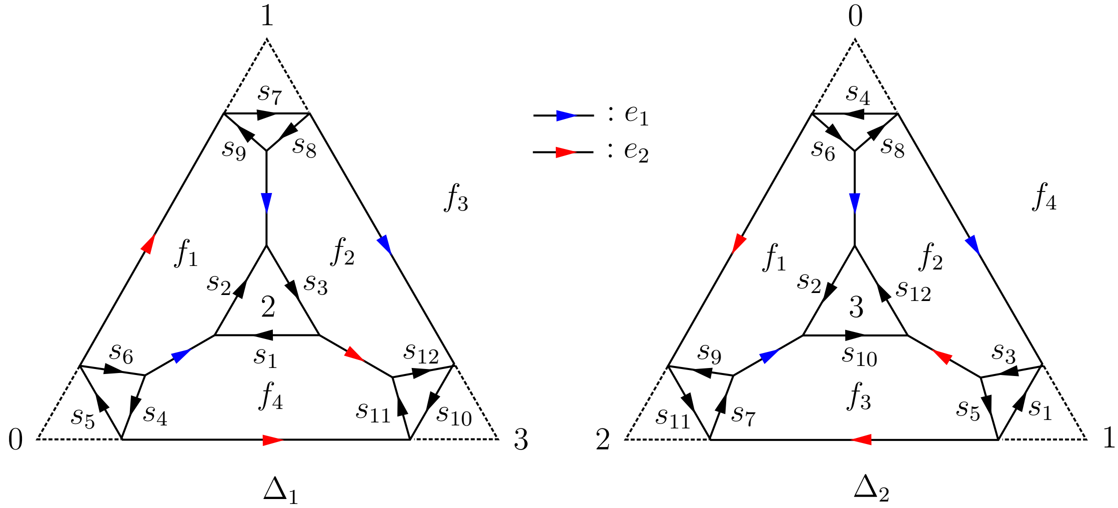

FIGURE 7. The knot complement of 4_1.

A σ-deformed super-Ptolemy assignments is a pair of maps $c : \{e_1, e_2\} \to \mathrm{G}_0^*(\mathbb{C})$ and $\theta : \{f_1, \ldots, f_4\} \to \mathrm{G}_1(\mathbb{C})$ satisfying

$$\begin{aligned} c_2^2 - \ell m^4 c_1^2 + lm^2 c_1 c_2 + m^2 c_1^3 c_2^2 \theta_2 \theta_3 = 0 \\ c_1^2 - \ell^{-1} c_2^2 + \ell^{-1} c_1 c_2 + \ell^{-1} m^{-1} c_1^3 c_2^2 \theta_3 \theta_2 = 0 \end{aligned} \tag{74}$$

and

$$\begin{aligned} E_{\Delta_1, f_4} : &\quad \ell^{-1} m^{-2} c_2 \theta_2 - m^{-1} c_2 \theta_3 + c_1 \theta_1 = 0 \\ E_{\Delta_1, f_3} : &\quad c_1 \theta_2 - m^{-1} c_2 \theta_4 + m^{-2} c_2 \theta_1 = 0 \\ E_{\Delta_2, f_4} : &\quad c_2 \theta_3 - c_2 \theta_1 + c_1 \theta_2 = 0 \\ E_{\Delta_2, f_3} : &\quad \ell c_1 \theta_1 - c_2 \theta_2 + c_2 \theta_4 = 0 \end{aligned} \tag{75}$$

where $c_i := c(e_i)$ and $\theta_i := \theta(f_i)$. Writing the equations in (75) in a matrix form, we have

$$\begin{pmatrix} m^{-2} c_2 & c_1 & 0 & -m^{-1} c_2 \\ c_1 & \ell^{-1} m^{-2} c_2 & -m^{-1} c_2 & 0 \\ -c_2 & c_1 & c_2 & 0 \\ \ell c_1 & -c_2 & 0 & c_2 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{76}$$

We are interested in the case of the above 4×4-matrix F being singular, as all θ_i should be zero, otherwise. One computes that if $\det F = 0$, then the kernel F is a free $\mathrm{G}_1(\mathbb{C})$-module of rank 1:

$$(\theta_1, \theta_2, \theta_3, \theta_4) = \eta \left(\frac{c_1 + \frac{1}{\ell m} c_2}{m c_1 - c_2}, 1, -\frac{m c_1^2 + \frac{1}{\ell m} c_2^2}{c_2 (m c_1 - c_2)}, \frac{\ell c_1^2 + (m + \frac{1}{m}) c_1 c_2 - c_2^2}{c_2 (m c_1 - c_2)} \right), \quad \eta \in \mathrm{G}_1(\mathbb{C}).$$

It follows that either $\det F = 0$ or not, we have $\theta_i\theta_j = 0$ for any i, j and thus Equation (74) is simplified to

$$\begin{aligned} c_2^2 - \ell m^4 c_1^2 + l m^2 c_1 c_2 &= 0 \\ c_1^2 - \ell^{-1} c_2^2 + \ell^{-1} c_1 c_2 &= 0 \end{aligned} \tag{77}$$

with

$$\det F = 2c_1^1 c_2^3\, m^{-2}(m + m^{-1} - 1)\,. \tag{78}$$

This shows that a σ-deformed Ptolemy assignment (c, θ) with $\theta \neq 0$ exists if and only if $m + m^{-1} - 1 = 0$. For instance, we restrict m and ℓ to complex numbers, then

$$m = \frac{1 \pm \sqrt{-3}}{2}, \quad \ell = -1, \quad (c_1, c_2) = k\left(\frac{1 \mp \sqrt{-3}}{2}, 1\right) \quad \text{for } k \in \mathrm{G}_0^*(\mathbb{C})\,. \tag{79}$$

Equation (78) agrees with the fact that the $\mathbb{C}^2$-torsion of the knot 4_1 is $2(m + m^{-1} - 1)$, as shown by Kitano [Kit94].

3. 1-loop and $\mathbb{C}^2$-torsion polynomials

3.1. **The face-matrix of an ideal triangulation.** In this section we define the 1-loop invariant, the 1-loop polynomial, and their (m, l)-deformed version from an ideal triangulation, and conjecture that it agrees with the corresponding version of the torsion polynomial. We will give the definition of the 1-loop invariant in its three flavors in separate sections below.

As mentioned in Remark 2.1, in this section, we use the Grassmann algebra with one odd generator; its even and odd part are isomorphic to $\mathbb{C}$, and the product of any two odd elements is zero. Then Equation (30) reduces to the ordinary Ptolemy equation

$$c_{01}c_{23} - c_{02}c_{13} + c_{03}c_{12} = 0\,. \tag{80}$$

Therefore, a super-Ptolemy assignment (c, θ) on an ideal triangulation $\mathcal{T}$ is given by a pair of a Ptolemy assignment $c : \mathcal{T}^1 \to \mathbb{C}^*$ with a map $\theta : \mathcal{T}^2 \to \mathbb{C}$ satisfying Equation (31) for each tetrahedron of $\mathcal{T}$. If $\mathcal{T}$ has N tetrahedra, then it has N edges and $2N$ faces. Hence a super-Ptolemy assignment on $\mathcal{T}$ is represented by a tuple $(c_1, \dots, c_N)$ of non-zero complex numbers satisfying the Ptolemy equation (80) for each tetrahedron and a tuple $(\theta_1, \dots, \theta_{2N})$ of complex numbers satisfying four linear equations (31) for each tetrahedron. We call these linear equations face-equations and write them in matrix form as

$$\begin{pmatrix} F_c^0 \\ F_c^1 \\ F_c^2 \\ F_c^3 \end{pmatrix} \theta = 0, \qquad \theta = (\theta_1, \dots, \theta_{2N})^t \tag{81}$$

where F_c^k for $k = 0, 1, 2, 3$ are $N \times 2N$ matrices whose rows and columns are indexed by the tetrahedra and the faces of $\mathcal{T}$, respectively. However, it was shown in Lemma 2.3 that at each tetrahedron any three of the linear equations (31) are dependent. As an equation in (31) corresponds to a face of a tetrahedron Δ, a choice of two equations in (31) is characterized by a common edge e_Δ of the two corresponding faces. Choosing an edge e_Δ for every tetrahedron Δ, we create a $2N \times 2N$ matrix F_c, called a face-matrix, so that the face-equations for θ take the form

$$F_c\,\theta = 0\,. \tag{82}$$

Note that F_c is a trimmed version of the $4N \times 2N$ matrix of Equation (81) and that entries of F_c are linear forms on c (in fact, the nonzero entries at up to sign, equal the value of c on an edge, see Equation (35)).

3.2. **1-loop invariant.** We now have all the ingredients to define the 1-loop invariant.

Definition 3.1. For a Ptolemy assignment c on $\mathcal{T}$ we define the 1-loop invariant by

$$\delta_{\mathcal{T},c,2} := \frac{1}{c_1 \cdots c_N} \left(\prod_{\Delta} \frac{1}{c(e_\Delta)} \right) \det F_c \,. \tag{83}$$

Note that $\delta_{\mathcal{T},c,2}$ is a degree 0-function of c, i.e., invariant under scaling each c_i to kc_i for $k \in \mathbb{C}^*$. It turns out that $\delta_{\mathcal{T},c,2}$ does not depend on the choice of edge e_Δ and is invariant under 2–3 Pachner moves. This follows from the specialization of Lemma 3.3 and Theorem 3.4 at $t = 1$ below.

We conjecture that the 1-loop invariant $\delta_{\mathcal{T},c,2}$ agrees with the $\mathbb{C}^2$-torsion $\tau_{M,\rho,2}$ of ρ (also called the Reidemeister torsion associated to ρ) up to sign, where $\rho : \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$ is a representation associated to the Ptolemy assignment c. This is the specialization at $t = 1$ of Conjecture 3.7 below.

3.3. **1-loop polynomial.** In this section we upgrade our 1-loop invariant to the 1-loop polynomial.

We assume that M has an infinite cyclic cover $\widetilde{M}$ and denote by $\widetilde{\mathcal{T}}$ the ideal triangulation of $\widetilde{M}$ induced from $\mathcal{T}$. We identify the deck transformation group of $\widetilde{M}$ with $\{t^k \,|\, k \in \mathbb{Z}\}$ for a formal variable t and choose a lift of every cell of $\mathcal{T}$ to $\widetilde{\mathcal{T}}$. Then every cell of $\widetilde{\mathcal{T}}$ is uniquely represented by a cell of $\mathcal{T}$ with a monomial in t; for instance, a face of $\widetilde{\mathcal{T}}$ is represented by $t^k \cdot f$ for a face f of $\mathcal{T}$ and $k \in \mathbb{Z}$. Recall that a face-equation is of the form

$$c_\alpha \, \theta(f_0) + c_\beta \, \theta(f_1) + c_\gamma \, \theta(f_2) = 0 \tag{84}$$

where f_0, f_1, and f_2 are three faces of a tetrahedron Δ. Since the lift of Δ has three faces $t^{k_i} \cdot \widetilde{f_i}$ for some $k_i \in \mathbb{Z}$ $(i = 0, 1, 2)$, we can formally modify Equation (84) as

$$c_\alpha t^{k_0} \, \theta(f_0) + c_\beta t^{k_1} \, \theta(f_1) + c_\gamma \, t^{k_2} \, \theta(f_2) = 0 \,. \tag{85}$$

The effect of this insertion of monomials in t leads to a twisted face-matrix $F_c(t)$.

Definition 3.2. For a Ptolemy assignment c on $\mathcal{T}$ we define the 1-loop polynomial by

$$\delta_{\mathcal{T},c,2}(t) := \frac{1}{c_1 \cdots c_N} \left(\prod_{\Delta} \frac{1}{c(e_\Delta)} \right) \det F_c(t) \,. \tag{86}$$

It is clear that $\delta_{\mathcal{T},c,2} = \delta_{\mathcal{T},c,2}(1)$. In addition, $\delta_{\mathcal{T},c,2}(t)$ determines the 1-loop invariant $\delta_{\mathcal{T}^{(n)},c,2}$ of all cyclic n-covers $M^{(n)}$ of M. This follows by arguments similar to the ones presented in [GYb] (for the 1-loop polynomial $\delta_{\mathcal{T},c,3}(t)$) and will not be repeated here.

Lemma 3.3. The 1-loop polynomial $\delta_{\mathcal{T},c,2}(t)$ does not depend on the choice of edge e_Δ.

Proof. It suffices to compare two different edge-choices for one tetrahedron Δ. Comparing $e_\Delta = [0,1]$ and $e_\Delta = [0,2]$, we have (cf. Equation (33))

$$\begin{pmatrix} c_{02} & -c_{03} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_{13} & -c_{03} & 0 & c_{01} \\ c_{12} & -c_{02} & c_{01} & 0 \end{pmatrix} = \begin{pmatrix} c_{01} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_{23} & 0 & -c_{03} & c_{02} \\ c_{12} & -c_{02} & c_{01} & 0 \end{pmatrix} . \tag{87}$$

The insertion of monomials in t affects on both sides of (87), but it is given by multiplying the same diagonal matrix (with diagonal in monomials in t) on the right. This proves that $\delta_{\mathcal{T},c,2}(t)$ is unchanged. $\square$

Theorem 3.4. *The 1-loop polynomial $\delta_{\mathcal{T},c,2}(t)$ is invariant under 2–3 Pachner moves.*

Proof. Suppose that $\mathcal{T}$ has two tetrahedra $[0,2,3,4]$ and $[1,2,3,4]$ with a common face $[2,3,4]$ as in Figure 8. Let $\mathcal{T}'$ denote the ideal triangulation obtained by replacing these two tetrahedra by $[0,1,2,3]$, $[0,1,3,4]$, and $[0,1,2,4]$.

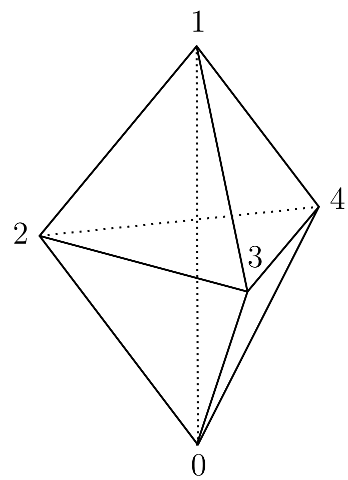

FIGURE 8. A 2–3 Pachner move.

Recall that the face equation of a face $[i,j,k]$ of a tetrahedron $[i,j,k,l]$ with $i<j<k$ is given by

$$E^l_{ijk} : c_{ij}\theta_{ijl} - c_{ik}\theta_{ikl} + c_{jk}\theta_{jkl} = 0 \,. \tag{88}$$

For $\Delta = [0,2,3,4]$ and $[1,2,3,4]$ we choose $e_\Delta = [2,4]$. Then the face-matrix $F_{\mathcal{T},c}$ of $\mathcal{T}$ contains a submatrix

$$\begin{array}{c|c|c|} & \theta_{234} & \text{others} \\ \hline E^3_{024} & c_{24} & R^3_{024} \\ E^0_{234} & & R^0_{234} \\ E^3_{124} & c_{24} & R^3_{124} \\ E^1_{234} & & R^1_{234} \end{array} \tag{89}$$

where R^l_{ijk} is the row of E^l_{ijk} except the θ_{234}-entry. Using an elementary row operation, we can modify the face-matrix without changing its determinant as

$$\begin{array}{c|c|c|}
 & \theta_{234} & \text{others} \\
\hline
E^3_{024} & & R^3_{024} - R^3_{124} \\
E^0_{234} & & R^0_{234} \\
E^3_{124} & c_{24} & R^3_{124} \\
E^1_{234} & & R^1_{234}
\end{array} \tag{90}$$

For $\Delta = [0,1,2,3]$ (resp., $[0,1,3,4]$ and $[0,1,2,4]$) we choose $e_\Delta = [2,3]$ (resp., $[3,4]$ and $[2,4]$). Then the face-matrix $F_{\mathcal{T}',c}$ of $\mathcal{T}'$ contains a submatrix

$$\begin{array}{c|c|c|c|c|}
 & \theta_{012} & \theta_{013} & \theta_{014} & \text{others} \\
\hline
E^0_{123} & c_{12} & -c_{13} & & R^0_{123} \\
E^1_{023} & c_{02} & -c_{03} & & R^1_{023} \\
E^0_{134} & & c_{13} & -c_{14} & R^0_{134} \\
E^1_{034} & & c_{03} & -c_{04} & R^1_{034} \\
E^0_{124} & c_{12} & & -c_{14} & R^0_{124} \\
E^1_{024} & c_{02} & & -c_{04} & R^1_{024}
\end{array} \tag{91}$$

Preserving the determinant, we apply elementary row operations to obtain:

$$\begin{array}{c|c|c|c|c|}
 & \theta_{012} & \theta_{013} & \theta_{014} & \text{other 6 faces} \\
\hline
E^0_{123} & & & & R^0_{123} - \frac{c_{12}}{c_{02}} R^1_{023} - \frac{c_{23}}{c_{02}}\frac{c_{04}}{c_{34}} R^0_{134} + \frac{c_{23}}{c_{02}}\frac{c_{14}}{c_{34}} R^1_{034} \\
E^1_{023} & c_{02} & -c_{03} & & R^1_{023} \\
E^0_{134} & & -\frac{c_{34}}{c_{04}} c_{01} & & R^0_{134} - \frac{c_{14}}{c_{04}} R^1_{034} \\
E^1_{034} & & c_{03} & -c_{04} & R^1_{034} \\
E^0_{124} & & & & R^0_{124} - R^0_{134} - R^0_{123} \\
E^1_{024} & & & & R^1_{024} - R^1_{034} - R^1_{023}
\end{array} \tag{92}$$

On the other hand, one computes that

$$\begin{aligned}
R^0_{234} &= R^0_{124} - R^0_{134} - R^0_{123} \\
R^1_{234} &= R^1_{024} - R^1_{034} - R^1_{023} \\
R^3_{024} - R^3_{124} &= \frac{c_{02}}{c_{23}} R^0_{123} - \frac{c_{12}}{c_{23}} R^1_{023} - \frac{c_{04}}{c_{34}} R^0_{134} + \frac{c_{14}}{c_{34}} R^1_{034}
\end{aligned} \tag{93}$$

It follows that (92) is equal to

$$
\begin{array}{c|c|c|c|c|}
 & \theta_{012} & \theta_{013} & \theta_{014} & \text{others} \\
\hline
E^0_{123} & & & & \frac{c_{23}}{c_{02}}(R^3_{024}-R^3_{124}) \\
E^1_{023} & c_{02} & -c_{03} & & R^1_{023} \\
E^0_{134} & & -\frac{c_{34}}{c_{04}}c_{01} & & R^0_{134}-\frac{c_{14}}{c_{04}}R^1_{034} \\
E^1_{034} & & c_{03} & -c_{04} & R^1_{034} \\
E^0_{124} & & & & R^0_{234} \\
E^1_{024} & & & & R^1_{234}
\end{array}
\tag{94}
$$

Comparing (90) and (94), we have

$$\det F_{\mathcal{T},c} = \frac{c_{24}}{(c_{01}c_{23}c_{34})} \det F_{\mathcal{T}',c}\,. \tag{95}$$

One easily checks that the monomial factor in the right-hand side agrees with the difference coming from the monomial term $c_1 \cdots c_N \prod c(e_\Delta)$ in (83). This proves that $\delta_{\mathcal{T},c,2} = \delta_{\mathcal{T}',c,2}$. As the effect of the insertion of monomials in t is separated from the above computation, this also proves that $\delta_{\mathcal{T},c,2}(t) = \delta_{\mathcal{T}',c,2}(t)$. □

Remark 3.5. The proof of the above theorem contains the behavior of the Ptolemy variety under Pachner 2–3 moves. This can be used to show that the determinant of the Jacobian of the Ptolemy variety, suitable normalized, is invariant under 2–3 Pachner moves, and conjecturally equal to the 1-loop invariant defined in [DG13]; see [Yoo].

Theorem 3.6. *A Ptolemy assignment c on $\mathcal{T}$ lifts to a super-Ptolemy assignment (c,θ) with $\theta \neq 0$ if and only if $\delta_{\mathcal{T},c,2}(1) = 0$.*

Proof. It is clear that if $\delta_{\mathcal{T},c,2}(1) \neq 0$, or equivalently, if $F_c = F_c(1)$ is non-singular, then θ should be zero. Conversely, if $\delta_{\mathcal{T},c,2}(1) = 0$, then there is a nonzero vector $v \in \mathbb{C}^{2N}$ with $F_c\, v = 0$, and for any $\eta \neq 0 \in \mathbb{C}$ the pair $(c, \eta v)$ is a super-Ptolemy assignment. □

Conjecture 3.7. The 1-loop polynomial $\delta_{\mathcal{T},c,2}(t)$ agrees with the $\mathbb{C}^2$-torsion polynomial $\tau_{M,\rho,2}(t)$ of ρ up to multiplying signs and monomials in t. Here $\rho : \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$ is a representation associated to the Ptolemy assignment c.

3.4. **(m,l)-deformation.** In this section, we deform the 1-loop invariant as well as the 1-loop polynomial.

We fix a cocycle σ that assigns a non-zero complex number to each short edge of $\mathring{\mathcal{T}}$. Recall Equation (62) that the face-equations in (31) admit a deformation according to σ. As in Section 3.1, we choose an edge e_Δ for each tetrahedron Δ of $\mathcal{T}$ so that we can choose two face-equations in (62). We then create a $2N \times 2N$ matrix F^σ_c so that the chosen face-equations for $\theta = (\theta_1, \ldots, \theta_{2N})^t$ take the form

$$F^\sigma_c\, \theta = 0\,. \tag{96}$$

Definition 3.8. For a σ-deformed Ptolemy assignment c on $\mathcal{T}$ we define the 1-loop invariant by

$$\delta_{\mathcal{T},c,2} := \frac{1}{c_1 \cdots c_N} \left(\prod_\Delta \frac{1}{c^\sigma(e_\Delta)} \right) \det F^\sigma_c \tag{97}$$

where $c^\sigma(e_\Delta)$ is the value of c on the edge e_Δ times its σ-coefficient in (62):

$$\begin{array}{c|c|c|c|c|c|c} e_\Delta & [0,1] & [0,2] & [0,3] & [1,2] & [1,3] & [2,3] \\ \hline c^\sigma(e_\Delta) & \frac{\sigma^0_{21}}{\sigma^3_{12}} c_{01} & c_{02} & \frac{\sigma^3_{01}}{\sigma^2_{01}} c_{03} & \frac{\sigma^1_{23}}{\sigma^0_{23}} c_{12} & c_{13} & \frac{\sigma^1_{03}}{\sigma^2_{03}} c_{23} \end{array} \tag{98}$$

As explained in Section 3.3, if an infinite cyclic cover of M is given, we twist the matrix F^σ_c to obtain $F^\sigma_c(t)$ by inserting monomials in t.

Definition 3.9. For a σ-deformed Ptolemy assignment c on $\mathcal{T}$ we define the 1-loop polynomial as

$$\delta_{\mathcal{T},c,2}(t) := \frac{1}{c_1 \cdots c_N} \left(\prod_\Delta \frac{1}{c^\sigma(e_\Delta)} \right) \det F^\sigma_c(t) \,. \tag{99}$$

Repeating the same computation given in Sections 3.2 and 3.3, we obtain:
1. A σ-deformed Ptolemy assignment c on $\mathcal{T}$ lifts to a super-Ptolemy assignment (c, θ) with $\theta \neq 0$ if and only if $\delta_{\mathcal{T},c,2} = 0$.
2. The 1-loop polynomial $\delta_{\mathcal{T},c,2}(t)$ does not depend on the choice of edge e_Δ and is invariant under 2–3 Pachner moves up to scalar multiplication by non-zero complex numbers.

Conjecture 3.10. The 1-loop polynomial $\delta_{\mathcal{T},c,2}(t)$ agrees with the $\mathbb{C}^2$-torsion polynomial $\tau_{M,\rho,2}(t)$ of ρ up to multiplying non-zero complex numbers and monomials in t. Here $\rho : \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$ is a representation associated to the σ-deformed Ptolemy assignment c.

When M is hyperbolic and the representation ρ in the above conjecture is a preferred lift of the geometric representation, then we simply denote the 1-loop polynomial $\delta_{\mathcal{T},c,2}(t)$ by $\delta_{\mathcal{T},2}(t)$ and the $\mathbb{C}^2$-torsion polynomial $\tau_{M,\rho,2}(t)$ by $\tau_{M,2}(t)$. In this case, Conjecture 3.10 reads

$$\delta_{\mathcal{T},2}(t) \overset{?}{=} \tau_{M,2}(t) \tag{100}$$

up to multiplying signs and monomials in t (see Remark 3.11 below).

Remark 3.11. The scalar-multiplication ambiguity in Conjecture 3.10 is given by some products of complex numbers that σ assigns to short edges. In particular, if σ takes values in $\{\pm 1\}$ (for instance, if M is hyperbolic and ρ is a lift of the geometric representation), then we can replace this scalar-multiplication ambiguity by sign-ambiguity as in Conjecture 3.7.

3.5. **Example: the 4_1 knot continued.** In this section we verify Conjecture 3.10 for the 4_1 knot. With the notation of Section 2.9, we have $e_{\Delta_1} = [0,3]$ and $e_{\Delta_2} = [1,2]$, hence

$$c^\sigma(e_{\Delta_1}) = \frac{\sigma(s_{10})^{-1}}{\sigma(s_2)} c_2 = \frac{c_2}{m}, \qquad c^\sigma(e_{\Delta_2}) = \frac{\sigma(s_5)^{-1}}{\sigma(s_6)} c_2 = c_2 \tag{101}$$

As explained in [GYb, Sec.3.1], we insert monomials in t to the matrix in Equation (76) to obtain

$$F^\sigma_c(t) = \begin{pmatrix} m^{-2}c_2 & c_1 & 0 & -m^{-1}c_2 \\ c_1 & \ell^{-1}m^{-2}c_2 & -m^{-1}c_2 & 0 \\ -c_2 & c_1 & c_2 t & 0 \\ \ell c_1 & -c_2 & 0 & c_2 t \end{pmatrix} . \tag{102}$$

Then we obtain

$$\delta_{\mathcal{T},c,2}(t) = \frac{1}{c_1 c_2}\frac{m}{c_2^2}\det F_c^\sigma(t) = m^{-1}(t^2 - 2(m+m^{-1})t + 1) \tag{103}$$

which agrees with the $\mathbb{C}^2$-torsion polynomial of 4_1 up to m^{-1}.
In particular, for $m = 1$ (and $l = -1$) we obtain the $\mathbb{C}^2$-torsion polynomial

$$t^2 - 4t + 1 \tag{104}$$

for an $\mathrm{SL}_2(\mathbb{C})$-lift of the geometric representation of the 4_1 knot, in agreement with `SnapPy`

```
snappy.Manifold('4_1').hyperbolic_SLN_torsion(2)
a^2 - 4.000000000000000000000000000*a + 0.9999999999999999999999999999
```

4. Further discussion

In this paper and in our prior work [GYb], we introduced 1-loop polynomials $\delta_{\mathcal{T},2}(t)$ and $\delta_{\mathcal{T},3}(t)$ determined, respectively, by the twisted face-matrix and twisted NZ-matrix of an ideal triangulation $\mathcal{T}$ of a 3-manifold M, and conjectured to be equal to the $\tau_{M,2}(t)$ and $\tau_{M,3}(t)$ torsion polynomials.

In this section we explain briefly how to derive 1-loop polynomials that conjecturally equal to the torsion polynomials $\tau_{M,n}(t)$ for all $n \geq 2$. Recall that the latter are a sequence of polynomials for $n \geq 2$ associated to a cusped hyperbolic 3-manifold, and determined by the homology of the infinite cyclic cover of M twisted by the $(n-1)$-rst symmetric power of an $\mathrm{SL}_2(\mathbb{C})$-lift of the geometric representation.

The torsion polynomials are closely related to the adjoint reprensentation of $\mathrm{PGL}_n(\mathbb{C})$ which decomposes

$$\mathrm{Ad}(\mathrm{PGL}_n(\mathbb{C})) = \oplus_{i=1}^{n-1}\mathbb{C}^{2i+1} \tag{105}$$

into *odd* dimensional representations of $\mathrm{SL}_2(\mathbb{C})$. This decomposition is not special to $\mathrm{PGL}_n(\mathbb{C})$, indeed every complex semisimple Lie group G has a canonical principal $\mathrm{SL}_2(\mathbb{C})$ subgroup, and decomposing the adjoint representation of G as an $\mathrm{SL}_2(\mathbb{C})$-representation

$$\mathrm{Ad}(G) = \oplus_{i=1}^{r}\mathbb{C}^{2e_i+1} \tag{106}$$

one obtains only *odd* dimensional irreducible representations of $\mathrm{SL}_2(\mathbb{C})$, where e_i are the exponents of G [Kos59]. Using the above decomposition (105), one can define for each $n \geq 2$ the product

$$\prod_{i=1}^{n-1}\tau_{M,2i+1}(t) \tag{107}$$

from which one can extract the odd torsion polynomials $\tau_{M,\mathrm{odd}}(t)$, see e.g. [Por, Sec.5].

Using the fact that $\mathrm{PGL}_n(\mathbb{C})$-representations can be described by gluing equations associated to $\mathrm{PGL}_n(\mathbb{C})$-type Neumann–Zagier matrices [GGZ15a], if one twists these matrices by considering their lifts to an infinite cyclic cover as was done in [GYb], one can define a 1-loop polynomial $\delta_{\mathcal{T},\mathrm{PGL}_n}(t)$ which would factor as $\prod_{i=1}^{n-1}\delta_{M,2i+1}(t)$ and would conjecturally equal to the polynomial (107). Doing so, one can obtain the odd 1-loop polynomials that conjecturally equal to the corresponding odd torsion polynomials.

Likewise, an extension of the decomposition (106) to high-dimensional orthosymplectic groups, together with a construction of Neumann–Zagier matrices that describe representations of 3-manifold groups to orthosymplectic groups, along with their twisted version would determine even 1-loop polynomials that conjecturally equal to the corresponding even torsion polynomials.

Acknowledgments. S.G. wishes to thank Nathan Dunfield for enlightening conversations. S.Y. wishes to thank Teruaki Kitano and Joan Porti for helpful conversations.

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