A SURGERY VIEW OF BOUNDARY LINKS

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Abstract. A celebrated theorem of Kirby identifies the set of closed oriented connected 3-manifolds with the set of framed links in S^3 modulo two moves. We give a similar description for the set of knots (and more generally, boundary links) in homology 3-spheres. As an application, we define a noncommutative version of the Alexander polynomial of a boundary link. Our surgery view of boundary links is a key ingredient in a construction of a rational version of the Kontsevich integral, which is described in subsequent work.

1. Introduction

Surgery (or cut-and-paste topology) is a method of modifying a manifold to another one. Surgery was successfully used in the sixties to geometrically realize algebraic invariants of manifolds and leads, for example, to a classification of high dimensional manifolds of a fixed homotopy type.

Surgery has also been fruitfully applied to the case of embedding questions, most notably to codimension 2 embeddings, i.e., knot theory. For an excellent survey, see [LO]. This was pioneered by Levine, who used surgery to geometrically realize known knot (and link) invariants, such as the Alexander polynomial, the Alexander module, and well-known concordance invariants of knots, [Le1]. Rolfsen used surgery for similar reasons in his reader-friendly introduction to knot theory, [Rf]. The key idea behind this is the fact that knots (or rather, knot projections) can be unknotted via a sequence of crossing changes, and that a crossing change can be achieved by surgery on a ±1-framed unknot as follows:

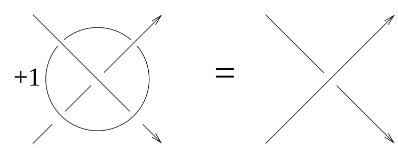

Figure 1. A crossing change can be achieved by surgery on a unit framed unknot.

Thus, every knot K in S^3 can be obtained by surgery on a framed link C in the complement $S^3 \smallsetminus \mathcal{O}$ of an unknot $\mathcal{O}$. We will call such a link C, an *untying link* for K. Observe that untying links are framed, and null homotopic in $S^3 \smallsetminus \mathcal{O}$, the interior of a solid torus. Further, their linking matrix is invertible over $\mathbb{Z}$, since surgery on them gives rise to a integral homology 3-sphere.

In the above mentioned applications of surgery theory to knot theory, one starts with a knot invariant (such as the Alexander polynomial) with a known behavior under surgery. Using this information, one can realize all possible values of the invariant by choosing suitable untying links on $S^3 \smallsetminus \mathcal{O}$ and performing surgery on them. This is exactly what Levine, Rolfsen and others did.

There is one point, though, that the above literature does not usually discuss: a knot can be untied by many different links in S^3. This is due to the fact that surgery is related to handlebodies that have different descriptions. If two framed links are related by a κ ($= \kappa_1$ or κ_2) move, then surgery on them gives rise to diffeomorphic manifolds. Here, κ_1 corresponds to adding a parallel of a link component to another, and κ_2 corresponds to adding/removing a unit-framed unlink away from a link.

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The above discussion applies in the case of knots K in *integral homology 3-spheres* M (that is, closed 3-manifolds with the same $\mathbb{Z}$-homology as S^3), and leads to the following *surgery map*:

$$\mathcal{N}(\mathcal{O})/\langle\kappa\rangle \longrightarrow \text{Knots}, \tag{1}$$

where Knots is the set of knots in homology spheres, and $\mathcal{N}(\mathcal{O})$ denotes the set of null homotopic links C with $\mathbb{Z}$-invertible linking matrix in the complement of a standard unknot $\mathcal{O}$ in S^3. The next theorem explains what we mean by the *surgery view of knots*.

Theorem 1. *The surgery map* (1) *is 1-1 and onto.*

The technique to prove this theorem was well-known in the seventies; in a sense it is a relative version of Kirby's theorem for manifolds with boundary. The reader may ask why did the classical literature not state the above result, and why is it of any use nowadays? As we mentioned already, if one is interested in *realizing* values of known knot invariants, one needs to know that the surgery map is onto. If one is interested in *constructing* a knot invariant, starting from an invariant of null homotopic links in the complement of an unknot, then one needs to know the kernel of the surgery map. This is precisely the situation that we are interested in. In forthcoming work [GK] we will use the above theorem in order to construct a new knot invariant, namely a rational version Z^{rat} of the Kontsevich integral of a knot. This work motivated us to formulate Theorem 1.

In a sense, Theorem 1 defines knot theory in terms of link theory in a solid torus. One might think that this is a step in the wrong direction. However, this view of knots is closely related to the *Homology Surgery* view of knots of Cappell-Shaneson, [CS1, CS2]. Cappell and Shaneson study the closed 3-manifold obtained by 0 surgery on a knot. This manifold has the homology of a circle but not the $\mathbb{Z}$-equivariant homology of a circle.

In [GK], we actually construct a rational invariant of F-links. Boundary links (∂-links), and their cousins, F-links are a generalization of knots which we now explain. Recall first that every knot K in a homology sphere M gives rise to a map $\phi : \pi_1(M \smallsetminus K) \to H_1(M \smallsetminus K) \cong \mathbb{Z}$ which maps a meridian to a generator of $\mathbb{Z}$. An F-*link* is a triple (M, L, ϕ) of a link L in a homology sphere M, where $\phi : \pi_1(M \smallsetminus L) \to F$ is a map that sends meridians of L to generators of the free group F. Here, F denotes the free group in generators $t_1, \dots, t_g$, where r is the number of components of L. An F-link gives rise to a map $M \smallsetminus L \to \vee^r S^1 = K(F, 1)$ and the preimage of generic points, one on each circle, gives rise to a disjoint union of surfaces Σ_i with boundary the ith component L_i of L. Thus, (M, L) is a *boundary link*.

It turns out that every boundary link has an F-structure, however not a canonical one. For a discussion of this, we refer the reader to [GL2] and [GK].

Generalizing Theorem 1, let $\mathcal{O}$ denote a standard unlink of r components in S^3, with a canonical isomorphism $\phi : \pi_1(S^3 \smallsetminus \mathcal{O}) \cong F$. If C is a null homotopic link in $S^3 \smallsetminus \mathcal{O}$ with $\mathbb{Z}$ invertible linking matrix, then after surgery it gives rise to an F-link (M, L, ϕ). The next theorem describes the *surgery view of F-links*.

Theorem 2. *The surgery map gives a 1-1 onto correspondence*

$$\mathcal{N}(\mathcal{O})/\langle\kappa\rangle \longleftrightarrow F-\text{links}.$$

As we mentioned above, we will use this theorem in [GK] in order to construct a rational version of the Kontsevich integral. At present, we will use the above theorem to define a noncommutative version of the Alexander polynomial of an F-link, as well as a matrix-valued invariant of an F-link which is essentially equivalent to the Blanchfield pairing of its free cover.

Consider a link $C \in \mathcal{N}(\mathcal{O})$. Since $C \subset X_0 = S^3 \smallsetminus \mathcal{O}$ is null homotopic, it lifts to a link $\widetilde{C}$ in the free F-cover $\widetilde{X}_0$ of X_0. Since $\widetilde{X}_0$ is contractible, it follows that we can define linking numbers between the components of $\widetilde{C}$, and thus we can define the *equivariant linking matrix* $\text{lk}(\widetilde{C})$ of C by:

$$\text{lk}(\widetilde{C})_{ij} = \left(\sum_{g \in F} \text{lk}(\widetilde{C}_i, g\,\widetilde{C}_j)\, g \right)_{ij}$$

where $\widetilde{C}_i$ denotes a lift of a component C_i of C. The above sums are finite and $\text{lk}(\widetilde{C})$ is a matrix over the group-ring $\Lambda = \mathbb{Z}[F]$ of F. The size of the matrix is the number of components of C. The equivariant linking matrix of C is a *Hermitian matrix* (that is, a matrix A over Λ is that satisfies $A = A^\star$ where $A^\star$ is the transpose of A followed by the natural involution $g \in F \to g^{-1} \in F$), defined up to a minor indeterminacy that depends on the choice of lifts $\widetilde{C}_i$ of the components of C. A different choice of lifts replaces the linking matrix $\mathrm{lk}(\widetilde{C})$ by $P\,\mathrm{lk}(\widetilde{C})P^\star$, where P is a diagonal matrix with elements of $\pm F \subset \Lambda$ along the diagonal.

Definition 1.1. Taking into account the Kirby moves, we can define a map

$$W : \mathcal{N}(\mathcal{O})/\langle\kappa\rangle \longrightarrow \mathcal{B}(\Lambda \to \mathbb{Z})$$

where $W(C) = \mathrm{lk}(\widetilde{C})$ and where $\mathcal{B}(\Lambda \to \mathbb{Z})$ denotes the set of *simple stable congruence classes of Hermitian matrices over* Λ*, invertible over* $\mathbb{Z}$[1]. Following [GL1], we call two Hermitian matrices A, B *simply stably congruent* iff $A \oplus S_1 = P(B \oplus S_2)P^\star$ for some diagonal matrices S_1, S_2 with ± 1 entries and some matrix P which is a product of either *elementary* (i.e., one that differs from the identity matrix on a single non-diagonal entry) or *diagonal* (with entries in $\pm F \subset \Lambda$) matrices.

Let $\hat{\Lambda}$ denote the completion of the rational group-ring $\mathbb{Q}[F]$ with respect to the augmentation ideal, that is the ideal generated by $t_1 - 1$, where $t_1, \dots, t_g$ generate F. $\hat{\Lambda}$ can be identified with the ring $\mathbb{Q}[[h_1, \dots, h_g]]$ of formal power series in non-commuting variables h_i.

Definition 1.2. Let

$$\chi : \mathrm{Mat}(\hat{\Lambda} \to \mathbb{Z}) \longrightarrow \hat{\Lambda}$$

be the invariant on the set $\mathrm{Mat}(\hat{\Lambda} \to \mathbb{Z})$ *of matrices* W *over* $\hat{\Lambda}$*, nonsingular over* $\mathbb{Z}$, defined by

$$\chi(W) = \mathrm{tr}\log'(W)$$

where

$$\log'(W) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \mathrm{tr}\left((W(\epsilon W)^{-1} - I)^n\right)$$

and $\epsilon : \Lambda \to \mathbb{Z}$.

In a sense, χ is a logarithmic determinant. In our case, we want to define χ on the quotient $\mathcal{B}(\Lambda \to \mathbb{Z})$ of the set of hermitian matrices over Λ which are invertible over $\mathbb{Z}$. Unfortunately, χ does not descend to a $\hat{\Lambda}$-valued invariant on the set $\mathcal{B}(\Lambda \to \mathbb{Z})$, but it does descend to a $\hat{\Lambda}/(\text{cyclic})$-valued invariant, where $\hat{\Lambda}/(\text{cyclic})$ is the abelian group quotient of the abelian group $\hat{\Lambda}$ modulo the subgroup $xy - yx$ for $x, y \in \hat{\Lambda}$. Equivalently, $\hat{\Lambda}/(\text{cyclic})$ can be described as the free abelian group generated by cyclic words in h_i variables.

Our next result shows that the χ invariant is a noncommutative analogue of the Alexander polynomial of an F-link. As a motivation, recall that the Alexander polynomial of a knot measures the order of the first homology of the free abelian cover of its complement. Given an F-link (M, L, ϕ), we can consider the free cover $\widetilde{X}$ of its complement $X = M \smallsetminus L$, and its homology $H_1(\widetilde{X})$, which is a module over the group-ring Λ. Since the ring Λ is not commutative (when L is not a knot), the notion of a torsion module does not make sense. However, in [GL2], Levine and the first author defined an invariant χ_Δ of F-links with values in $\hat{\Lambda}/(\text{cyclic})$, the abelian group of formal power series (with rational coefficients) in noncommutative variables, modulo cyclic permutation. The invariant χ_Δ is a reformulation of an invariant defined earlier by Farber, [Fa], and in the case of knots equals, up to a multiple, to the logarithm of the Alexander polynomial. Thus, χ_Δ can be thought as a *noncommutative torsion* invariant of F-links.

The next result gives an independent definition of the noncommutative torsion of an F-link, and of its Blanchfield form.

Theorem 3. (a) *There exists an invariant* W *of* F*-links with values in* $\mathcal{B}(\Lambda \to \mathbb{Z})$.
(b) $H_1(\widetilde{X})$ *is a* Λ*-module with presentation* W.
(c) *The Blanchfield pairing* [Dv]

$$H_1(\widetilde{X}, \mathbb{Z}) \otimes H_1(\widetilde{X}, \mathbb{Z}) \to \Lambda_{\mathrm{loc}}/\Lambda$$

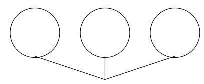
is given by $W^{-1} \bmod \Lambda$*, where* Λ_{loc} *is the Cohn localization of* Λ*,* [FV].
(d) $\chi = \chi_\Delta$.

[1]By invertible over $\mathbb{Z}$, we mean that the image of a matrix which is defined over Λ under the map $\Lambda \to \mathbb{Z}$ is invertible over $\mathbb{Z}$.

Thus, the W invariant of an F-link determines its Blanchfield form. Conversely, using various exact sequences of algebraic surgery, one can show that the Blanchfield form determines W, as was communicated to us by D. Sheiham. Since this would take us far afield, we prefer to postpone this to a future publication.

2. PROOFS

2.1. **Beginning the proof of Theorem 2.** Fix once and for all a *based unlink* $\mathcal{O}$ of g components in S^3. A based unlink consists of an unlink $\mathcal{O}$ in S^3, together with arcs from a base point in $S^3 \smallsetminus \mathcal{O}$ to each of the component of the unlink, as shown below (for $g = 3$):



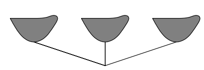
Let $\mathcal{N}(\mathcal{O})$ denote the set of *null homotopic links L with $\mathbb{Z}$-invertible linking matrix* in the complement of $\mathcal{O}$.

Surgery on an element C of $\mathcal{N}(\mathcal{O})$ transforms $(S^3, \mathcal{O})$ to a F-link (M, L). Indeed, since C is null homotopic the natural map $\pi_1(S^3 \smallsetminus \mathcal{O}) \to F$ gives rise to a map $\pi_1(M \smallsetminus L) \to F$. Alternatively, one can construct disjoint Seifert surfaces for each component of L by tubing the disjoint disks that $\mathcal{O}$ bounds, which is possible, since each component of C is null homotopic. Since the linking matrix of C is invertible over $\mathbb{Z}$, M is an integral homology 3-sphere. Let $\overset{\kappa}{\sim}$ denote the equivalence relation on $\mathcal{N}(\mathcal{O})$ generated by the moves of *handle-slide* κ_1 (i.e., adding a parallel of a link component to another component) and *stabilization* κ_2 (i.e., adding to a link an unknot away from the link with framing ± 1). It is well-known that κ-equivalence preserves surgery, thus we get a surgery map

$$\mathcal{N}(\mathcal{O})/\langle \kappa \rangle \longrightarrow F-\text{links}.$$

Now, we will show that the surgery map is onto. Given a F-link (M, L, ϕ) choose disjoint Seifert surfaces Σ for its components and choose a framed link C in M such that $M_C = S^3$. C might intersect Σ, however by a small isotopy (which does not change the property that $M_C = S^3$) we may assume that C is disjoint from Σ. Consider the image Σ' of Σ in $S^3 = M_C$. By reversing the C surgery on M, consider the image C' of a parallel of C in $S^3 = M_C$ such that $M = S^3_{C'}$. By doing crossing changes among the bands of Σ', translated in terms of surgery on a unit-framed link $C'' \subset S^3 \smallsetminus \Sigma'$, we can assume that each component of Σ' lies in a ball disjoint from the other components and further that the boundary of Σ' is an unlink. The link $C' \cup C''$ is null homotopic in the complement of the unlink $\mathcal{O} = \partial \Sigma'$ since it is disjoint from Σ', and since the map $\phi : \pi_1(S^3 \smallsetminus \mathcal{O}) \cong F$ is given geometrically by intersecting a loop in $S^3 \smallsetminus \mathcal{O}$ with Σ' and recording the corresponding element of the free group.

The linking matrix of $C' \cup C''$ is invertible over $\mathbb{Z}$ since $S^3_{C' \cup C''} = M$, an integral homology 3-sphere. Now perform an isotopy and choose arcs γ from a base point to each of the components of Σ' in such a way that $\Sigma' \cup \gamma$ intersects a ball as follows (for $g = 3$):



By construction, $C' \cup C'' \in \mathcal{N}(\mathcal{O})$ maps to (M, L, ϕ).

Next, we need to show that the surgery map is 1-1; we will do this in the next section.

2.2. **A relative version of Fenn-Rourke's theorem.** The goal of this independent section is to show a relative version of Fenn-Rourke's theorem [FR, Theorem 6]. We begin with some notation. Given a 3-manifold M (possibly with nonempty boundary) and a framed link L in its interior, surgery produces a 3-manifold M_L whose boundary is canonically identified with that of M and a cobordism W_L with boundary $M \cup (\partial M \times I) \cup M_L$.

Theorem 4. *If M is a 3-manifold with boundary, L_1, L_2 two framed links in its interior, then L_1 and L_2 are κ-equivalent iff there exist*

(a) $h : M_{L_1} \to M_{L_2}$, *a diffeo rel. boundary.*
(b) *An isomorphism* $\iota : \pi_1(W_{L_1}) \longrightarrow \pi_1(W_{L_2})$.

(c) *A commutative diagram* (Δ)

$$\begin{array}{ccc} \pi_1(M_{L_1}) & \xrightarrow{h_*} & \pi_1(M_{L_2}) \\ \downarrow & & \downarrow \\ \pi_1(W_{L_1}) & \overset{\iota}{\dashrightarrow} & \pi_1(W_{L_2}) \\ & \nwarrow \quad \nearrow & \\ & \pi_1(M) & \end{array}$$

such that $\eta(\Delta) = 0 \in H_4(\pi)$.

Here, $\eta(\Delta) \in H_4(\pi)$ is the homology class of $\rho(W)$, where $\rho : W \to K(\pi, 1)$, $W = W_{L_1} \cup_\partial W_{L_2}$ (where the boundaries are identified via the identity on M, $\partial M \times I$ and h on the top piece) and $\pi = \pi_1(W_{L_1}) = \pi_1(W_{L_2})$. Note that W is a closed 4-manifold.

The proof of Theorem 4 follows mainly from Fenn-Rourke [FR], with some simplifications due to Roberts, [Rb]. We begin with the following lemma from [FR], which reformulates the vanishing of the condition $\eta(\Delta) = 0$.

Lemma 2.1. [FR, Lemma 9] *Assume that we are in the situation of Theorem 4 and that there exists* h, ι *making* Δ *commute. Then* $\eta(\Delta) = 0$ *iff* $W \#_r \mathbb{C}P^2 \#_s \overline{\mathbb{C}P^2} = \partial \Omega^5$ *for some* r, s *and closed 5-manifold* Ω *such that the diagram*

$$\begin{array}{ccc} \pi_1(W_{L_1}) & \xrightarrow{\iota} & \pi_1(W_{L_2}) \\ & {\scriptstyle (j_1)_*} \searrow \quad \swarrow {\scriptstyle (j_2)_*} & \\ & \pi_1(\Omega^5) & \end{array}$$

commutes and $(j_1)_*, (j_2)_*$ *are split injections.*

Proof. (of Theorem 4) If L_1 and L_2 are κ-equivalent, then it follows from the "only if" statement of Lemma 2.1 that $\eta(\Delta) = 0$ since the induced homeomorphisms on π_1 do not change.

So, assume that the algebraic conditions of Theorem 4 are satisfied for L_1, L_2. We will show that L_1 is κ-equivalent to L_2.

The first step is to arrange via κ-moves that W_{L_1} and W_{L_2} diffeomorphic. The proof of [FR, p.9] (where they assume $\partial M = \phi$) using surgery works here, too.

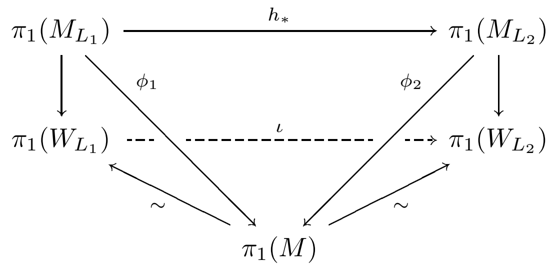
The next step is to use Cerf theory. Namely, consider Morse functions f_i realizing the cobordisms W_{L_i} for $i = 1, 2$. Since W_{L_1} is diffeomorphic to W_{L_2}, f_1 and f_2 are related by a one parameter family of smooth functions f_t for $1 \leq t \leq 2$. The arguments of Roberts [Rb, Stage 2, p.3] in this case imply that L_1 is κ-equivalent to L_2. □

Remark 2.2. If L is a null homotopic link in M, then W_L is homotopic to a one point union of M and some number of 2-spheres S^2. In particular, $\pi_1(M) \simeq \pi_1(W_L)$.

End of the Proof of Theorem 2. We will apply Theorem 4 to $M = S^3 \smallsetminus \mathcal{O}$. Consider (L_i, ϕ_i) two surgery presentations of the same F-link for $i = 1, 2$. Then, we have a homotopy commutative diagram

$$\begin{array}{ccc} M_{L_1} & \xrightarrow{h} & M_{L_2} \\ {\scriptstyle \phi_1} \downarrow & & \downarrow {\scriptstyle \phi_2} \\ M & = & M \end{array}$$

where $\phi_i : M_{L_i} \to M$ are splitting maps. It follows by Remark 2.2 that a map ι exists such that the following diagram (Δ) commutes



The obstruction $\eta(\Delta)$ lies in $H_4(F)$. However, $K(F,1) = \vee S^1$, thus $H_4(F) = H_4(\vee S^1) = 0$. Thus, Theorem 4 implies that L_1 is κ-equivalent to L_2. □

2.3. **A matrix-valued invariant of F-links.** The goal of this section is to prove Theorem 3. Combining the map $\mathcal{N}(\mathcal{O})/\langle\kappa\rangle \longrightarrow \mathcal{B}(\Lambda \to \mathbb{Z})$ of Definition 1.1 with the surgery map of Theorem 2, gives rise to the map W of Theorem 3 (a).

Recall the χ-invariant of Definition 1.2.

Proposition 2.3. (a) *For matrices A, B over $\hat{\Lambda}$, nonsingular over $\mathbb{Z}$, we have in $\hat{\Lambda}/(\text{cyclic})$ that*

$$\chi(AB) = \chi(A) + \chi(B) \quad \textit{and} \quad \chi(A \oplus B) = \chi(A) + \chi(B).$$

(b) *χ descends to a $\hat{\Lambda}/(\text{cyclic})$-valued invariant of the set $\mathcal{B}(\Lambda \to \mathbb{Z})$.*
(c) *For A as above, $\chi(A) = \chi(A\,\epsilon(A^{-1}))$ where $\epsilon : \Lambda \to \mathbb{Z}$.*
(d) *If A is a matrix with integer entries invertible over $\mathbb{Z}$, then $\chi(A) = 0$.*

Proof. (b) follows from (a). For (a), let $\mathcal{R}$ denote the ring of square matrices with entries in $\hat{\Lambda}$, and $[\mathcal{R}, \mathcal{R}]$ denote the subgroup of it generated by matrices of the form $AB - BA$ where $A, B \in \mathcal{R}$. Given two matrices $A, B \in \mathcal{R}$, with augmentations $A_0 = \epsilon A$ and $B_0 = \epsilon B$, we claim that

$$\log'(AB) = \log'(A) + \log'(B) \mod [\mathcal{R}, \mathcal{R}]. \tag{2}$$

Indeed, if $A_0 = B_0 = I$, then this follows from the Baker-Cambell-Hausdorff formula, since $\log'(AB) = \log(AB)$. For the general case, we have modulo $[\mathcal{R}, \mathcal{R}]$, that

$$\begin{aligned} \log'(AB) &= \log(AB(A_0B_0)^{-1}) = \log(ABB_0^{-1}A^{-1}AA_0^{-1}) \\ &= \log(ABB_0^{-1}A^{-1}) + \log(AA_0^{-1}) \\ &= A\log(BB_0^{-1})A^{-1} + \log(AA_0^{-1}) = \log'(B) + \log'(A). \end{aligned}$$

Since

$$\operatorname{tr}(AB) = \operatorname{tr}(BA) \in \hat{\Lambda}/(\text{cyclic}),$$

after taking traces, Equation (2) implies (a).
(c) follows immediately from the definition of χ and the fact that $\epsilon(A\,\epsilon A^{-1}) = I$.
(d) follows from that fact that if A is invertible over $\mathbb{Z}$, then $A = \epsilon A$, which implies $\log'(A) = 0$, thus $\chi(A) = \operatorname{tr}\log'(A) = 0$. □

We will also denote by χ the $\hat{\Lambda}/(\text{cyclic})$-valued invariant of F-links defined by

$$F - \text{links} \cong \mathcal{N}(\mathcal{O})/\langle\kappa\rangle \to \mathcal{B}(\Lambda \to \mathbb{Z}) \to \hat{\Lambda}/(\text{cyclic}).$$

Proof. (of Theorem 3) Consider $C \in \mathcal{N}(\mathcal{O})$ and let (M, L, ϕ) denote the corresponding F-link. Let W denote the equivariant linking matrix of C, and $\widetilde{X}$ (resp. $\widetilde{X}_0$) denote the free cover of $X = M \smallsetminus L$ (resp. $X_0 = S^3 \smallsetminus \mathcal{O}$) given by ϕ. Note that $\widetilde{X}$ is obtained from $\widetilde{X}_0$ by surgery on $\widetilde{C}$.

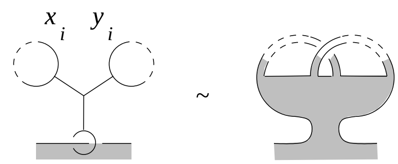
Part (b) follows from classical surgery arguments presented in the case of knots by Levine in [Le1, Le2] and adapted without difficulty in the case of F-links; see also [GL3, Lemma 2.4]. For the case of knots, Lemma 12.2 and the discussion of p.46-47 in [Le2], imply that the Blanchfield pairing is given by W^{-1} modulo Λ. This discussion can be generalized without change to the case of F-links, and proves part (c).

For part (d), observe that the χ_Δ and the χ invariant of [GL2] are defined, respectively, via the Seifert surface and the surgery view of boundary links. The two views can be related using the so-called Y-view of knots presented in [GGP, Section 6.4] and extended without difficulty to the case of boundary links. Consider a disjoint union Σ of Seifert surfaces in S^3 which we think of as a disjoint union of embedded disks with pairs of bands attached in an alternating way along each disk.

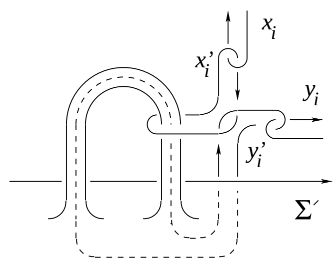
Consider an additional link L' in $S^3 \smallsetminus \Sigma$ such that its linking matrix C satisfies $\det(C) = \pm 1$ and such that the linking number between the cores of the bands and L' vanishes. With respect to a choice of orientation of 1-cycles corresponding to the cores of the bands, a Seifert matrix of Σ is given by

$$A = \begin{pmatrix} L^{xx} & L^{xy} + I \\ L^{yx} & L^{yy} \end{pmatrix}$$

Start from a surgery presentation of (S^3, Σ) in terms of *clovers with three leaves*, as was explained in [GGP, Section 6.4] and summarized in the following figure:



Surgery on a clover with three leaves can be described in terms of surgery on a six component link L'''. It was observed by the second author in [Kr, Figure 3.1] that L''' can be simplified via Kirby moves to a four component link L'' disjoint from a surface Σ' shown as follows (where $\{x_i, y_i\}$ correspond to the same link in the figure above and below):



The equivariant linking matrix of (a based representative of) $L'' \cup L'$ is given by $W \oplus C$ where W is the following matrix in $\{x_i, y_i, x_i', y_i'\}$ basis:

$$W = \begin{pmatrix} L^{xx} & L^{xy} & I & 0 \\ L^{xy} & L^{yy} & 0 & I \\ I & 0 & 0 & xI \\ 0 & I & \bar{x}I & 0 \end{pmatrix} = \begin{pmatrix} L & I \\ I & B \end{pmatrix}$$

where $x = t - 1$, $\bar{x} = \bar{t} - 1$, $\bar{t} = t^{-1}$ and

$$B = \begin{pmatrix} 0 & xI \\ \bar{x}I & 0 \end{pmatrix}.$$

Furthermore,

$$WW(1)^{-1} = \begin{pmatrix} L & I \\ I & B \end{pmatrix} \begin{pmatrix} 0 & I \\ I & -L \end{pmatrix} = \begin{pmatrix} I & 0 \\ B & I - BL \end{pmatrix}.$$

thus, by part (c) of Proposition $\chi(W) = \chi(I - BL)$.

It is easy to see that every F-link has a surgery presentation $L'' \cup L'$ as above by adapting the onto part of the proof of Theorem 2. We can now compute as follows:

$$\begin{aligned} \chi(W \oplus C) &= \chi(W) + \chi(C) && \text{by Proposition 2.3(a)} \\ &= \chi(W) && \text{by Proposition 2.3(d), since } C \text{ is invertible} \\ &= \chi(WW(1)^{-1}) && \text{by Proposition 2.3(c)} \\ &= \chi(I - BL) && \text{by the above calculation} \end{aligned}$$

Using the notation of [GL2], let $Z = A(A - A')^{-1}$. Since $(A-A')^{-1} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ it follows that

$$L + \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} = A = Z \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \quad \text{thus} \quad L = Z \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} - \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}.$$

Substituting for L, we obtain that

$$\begin{aligned} I - BL &= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} - B \left(Z \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} - \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \right) \\ &= \begin{pmatrix} I & 0 \\ 0 & \bar{t}I \end{pmatrix} - \begin{pmatrix} 0 & xI \\ \bar{x}I & 0 \end{pmatrix} Z \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \\ &= \left(\begin{pmatrix} 0 & -I \\ \bar{t}I & 0 \end{pmatrix} - \begin{pmatrix} 0 & xI \\ \bar{x}I & 0 \end{pmatrix} Z \right) \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -I \\ \bar{t}I & 0 \end{pmatrix} (I + XZ) \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \end{aligned}$$

where $X = \begin{pmatrix} xI & 0 \\ 0 & xI \end{pmatrix}$. Proposition 2.3 implies that

$$\begin{aligned} \chi(I - BL) &= \chi \begin{pmatrix} 0 & -I \\ \bar{t}I & 0 \end{pmatrix} + \chi(I + XZ) + \chi \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \\ &= \chi \begin{pmatrix} 0 & -I \\ \bar{t}I & 0 \end{pmatrix} + \chi(I + XZ) \\ &= \chi_\Delta(A) \qquad \text{by the definition of } \chi_\Delta \end{aligned}$$

as required. □

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