RATIONALITY OF THE $\mathrm{SL}(2,\mathbb{C})$-REIDEMEISTER TORSION IN DIMENSION 3

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Abstract. If M is a finite volume complete hyperbolic 3-manifold with one cusp and no 2-torsion, the geometric component X_M of its $\mathrm{SL}(2,\mathbb{C})$-character variety is an affine complex curve, which is smooth at the discrete faithful representation ρ_0. Porti defined a non-abelian Reidemeister torsion in a neighborhood of ρ_0 in X_M and observed that it is an analytic map, which is the germ of a unique rational function on X_M. In the present paper we prove that (a) the torsion of a representation lies in at most quadratic extension of the invariant trace field of the representation, and (b) the existence of a polynomial relation of the torsion of a representation and the trace of the meridian or the longitude. We postulate that the coefficients of the $1/N^k$-asymptotics of the Parametrized Volume Conjecture for M are elements of the field of rational functions on X_M.

Contents

1. Introduction

1.1. **The volume of an $\mathrm{SL}(2,\mathbb{C})$-representation and the A-polynomial.** A well-known numerical invariant of a 3-dimensional finite volume hyperbolic manifold M with a cusp is its *volume*, a positive real number. A complete invariant of the hyperbolic structure of M is a discrete faithful representation of $\pi_1(M)$ into $\mathrm{PSL}(2,\mathbb{C})$ (well-defined up to conjugation) which is also a topological invariant, as follows from Mostow

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rigidity Theorem. Every $\mathrm{PSL}(2,\mathbb{C})$-representation ρ of $\pi_1(M)$ has a real-valued volume $\mathrm{Vol}(\rho)$; see [Dun99, Ch.2] and also [Fra04, FK06]. When a representation varies in a 1-parameter family ρ_t, the variation of the volume $\frac{d}{dt}\mathrm{Vol}(\rho_t)$ depends only on the restriction of ρ_t to the boundary torus ∂M. This is a general principle of Atiyah-Patodi-Singer, and in our special case it also follows from Schalfi's formula. This raises the question: which $\mathrm{PSL}(2,\mathbb{C})$-representations of ∂M extend to a representation of M? The answer is given by an algebraic condition between the eigenvalues of a meridian and longitude of ∂M. This condition is the vanishing of the so-called *A-polynomial* of M; see [CCG$^+$94]. The A-polynomial of M encodes important informations about

(a) the hyperbolic geometry of M, and determines the variation of the volume of the hyperbolic structure of M.
(b) the topology of M and more precisely about the slopes of incompressible surfaces in the knot complement, as follows from Culler-Shalen theory; see [CCG$^+$94].

More recently, the A-polynomial (or rather, its extension that includes the images of all components of the character variety) is conjecturally linked in two different ways to a *quantum knot invariant*, namely the *colored Jones polynomials* of a knot in 3-space (for a definition of the latter, which we will not use in the present paper, see [Tur88] and [GL05]):

(a) There is an A_q-polynomial in two q-commuting variables which encodes a minimal order linear q-difference equation for the sequence of colored Jones polynomials; see [GL05]. The AJ Conjecture of [Gar04] states that when $q = 1$, the A_q-polynomial coincides with the A-polynomial.
(b) There is a parametrized version of the Volume Conjecture which links the variation of the limit in the Volume Conjecture to the A-polynomial; see [GM08, GL11].

Aside from conjectures, the following result of [DG04] and [BZ05] (based on foundational work of Kronheimer-Mrowka) shows that the A-polynomial detects the unknot.

Theorem 1.1. [BZ05, DG04] *The A-polynomial of a nontrivial knot in 3-space is nontrivial.*

1.2. **The $\mathrm{SL}(2,\mathbb{C})$-character variety of M and its field of rational functions.** For historical reasons that simplify the linear algebra, it is useful to consider $\mathrm{SL}(2,\mathbb{C})$ (rather than $\mathrm{PSL}(2,\mathbb{C})$)-representations of $\pi_1(M)$. In the rest of the paper, M will denote a finite volume hyperbolic 3-manifold with one cusp, such that the homology of M contains no 2-torsion. In this case, the discrete faithful representation of M lifts to a $\mathrm{SL}(2,\mathbb{C})$-representation $\rho_0\colon \pi_1(M) \to \mathrm{SL}(2,\mathbb{C})$; see [Cul86]. To understand how the $\mathrm{SL}(2,\mathbb{C})$-representation ρ_0 of $\pi_1(M)$ varies, we consider the unique component X_M of the $\mathrm{SL}(2,\mathbb{C})$-*character variety* of M that contains ρ_0. It is well-known that X_M is an affine curve defined over $\mathbb{Q}$ and that ρ_0 is a smooth point of X_M; see [CCG$^+$94]. Moreover, the coordinate ring $\mathbb{Q}[X_M]$ is generated by tr_γ for all $\gamma \in \pi_1 M$ (see [Sha02, Prop.1.1.1]), where tr_γ is the so called *trace-function* defined by:

$$\mathrm{tr}_\gamma : X_M \longrightarrow \mathbb{C}, \qquad \mathrm{tr}_\gamma(\rho) = \mathrm{tr}(\rho(\gamma)). \tag{1}$$

Here $\mathrm{tr}(A) = \sum_i a_{ii}$ denotes the trace of a square matrix $A = (a_{ij})$. Let $\mathbb{Q}(X_M)$ denote the field of rational functions of X_M. For a detailed discussion on character varieties, the reader may consult Shalen's survey [Sha02] and also [BDRV, Sec.10] and [CCG$^+$94, Gol86].

1.3. **The Reidemeister torsion of an $\mathrm{SL}(2,\mathbb{C})$-representation.** Another important numerical invariant of a representation of a manifold is its *Reidemeister torsion*, which comes in several combinatorial or analytic flavors, see Milnor's survey [Mil66] or Turaev's monograph [Tur02] for details. Combinatorially, the Reidemeister torsion is defined in terms of ratios of determinants of matrices assigned to based, acyclic complexes, which themselves are associated with a cell decomposition of a manifold and an acyclic representation. One can define torsion for all (not necessarily acyclic) representations of a manifold as an element of a top exterior power of a twisted (co)homology group, and one can obtain a complex number after choosing a basis for the twisted (co)homology. Porti [Por97] defined a Reidemeister torsion for the adjoint representation associated to an $\mathrm{SL}(2,\mathbb{C})$-representation ρ of $\pi_1(M)$ when ρ is in a *neighborhood* U of $\rho_0 \in X_M$[1]. Such representations

[1] The referee points out that the torsion of the adjoint of an $\mathrm{SL}(2,\mathbb{C})$ representation of M depends only on the corresponding $\mathrm{PSL}(2,\mathbb{C})$ representation. This holds since the adjoint representation of $\mathrm{SL}(2,\mathbb{C})$ factors through $\mathrm{PSL}(2,\mathbb{C})$.

are not acyclic and a basis for the twisted homology (and thus the torsion) depends on an *admissible curve* γ, i.e., a simple closed curve γ in ∂M which is not nullhomologous in ∂M (see [Por97, Chap. 3] for details). Thus, the *non–abelian Reidemeister torsion* is a map:

$$\tau_\gamma \colon U \longrightarrow \mathbb{C}. \tag{2}$$

Moreover, Porti [Por97] observed that τ_γ is an analytic map, and obtained the following result.

Theorem 1.2. [Por97, Thm.4.1] *For every admissible curve γ, the non-abelian Reidemeister torsion $\tau_\gamma : U \longrightarrow \mathbb{C}$ is the germ of a unique element of $\mathbb{Q}(X_M)$, which is regular at ρ_0.*

In Section 3.2 we will give an independent proof of Theorem 1.2, which we need for the main results of our paper. To phrase our results, recall that the *trace field* $\mathbb{Q}(\rho)$ of an $\mathrm{SL}(2,\mathbb{C})$-representation ρ of M is the field $\mathbb{Q}(\mathrm{tr}_g(\rho)|g \in \pi_1(M))$. For an admissible curve γ, let $\{e_\gamma(\rho), e_\gamma(\rho)^{-1}\}$ denote the eigenvalues of $\rho(\gamma)$. Observe that the field $\mathbb{Q}(\rho)(e_\gamma(\rho))$ is at most a quadratic extension of the trace field of ρ. Our next theorem uses the notion of a *generic representation*, defined in Section 2. Note that this is a Zariski open condition, and that the discrete faithful representation is generic (*regular* in the language of Porti's work).

Theorem 1.3. *For every admissible curve γ and every generic representation ρ, $\tau_\gamma(\rho)$ lies in the field $\mathbb{Q}(\rho)(\epsilon_\gamma(\rho))$. In particular, $\tau_\gamma(\rho_0)$ lies in the trace field of M.*

Note that since the homology of M has no 2-torsion, the trace field of M coincides with its invariant trace field; see [NR92, Thm.2.2]. Our next theorem shows that τ_γ is an *algebraic function* of tr_γ. This follows easily from the fact that τ_γ and tr_γ are rational functions on X_M and that $\mathbb{Q}(X_M)$ has transcendence degree 1, since X_M is an affine curve defined over $\mathbb{Q}$.

Theorem 1.4. *For every admissible curve γ, there exists a polynomial $T_\gamma(\tau, y) \in \mathbb{Z}[\tau, y]$, called the T_γ-polynomial, so that*

$$T_\gamma(\tau_\gamma, \mathrm{tr}_\gamma) = 0. \tag{3}$$

Let us make some remarks regarding Theorems 1.2 and 1.4.

Remark 1.1. The dependence of the torsion function τ_γ on γ is determined by the A-polynomial; see Equation (19). Thus, T_γ is determined by T_μ and the A-polynomial of M. Moreover, if we let $\{e_\mu(\rho), e_\mu^{-1}(\rho)\}$ (resp. $\{e_\lambda(\rho), e_\lambda^{-1}(\rho)\}$) de the eigenvalues for the meridian μ (resp. longitude λ) at ρ, that is to say, if

$$e_\mu(\rho) + e_\mu^{-1}(\rho) = \mathrm{tr}_\mu(\rho) \text{ and } e_\lambda(\rho) + e_\lambda^{-1}(\rho) = \mathrm{tr}_\lambda(\rho)$$

then one has (see [Por97, Thm.4.1]):

$$\tau_\lambda = \frac{e_\mu}{e_\lambda} \cdot \frac{\partial e_\lambda}{\partial e_\mu} \cdot \tau_\mu$$

In particular, at the discrete faithful representation ρ_0, we have:

$$\tau_\lambda(\rho_0) = \mathfrak{c} \cdot \tau_\mu(\rho_0) \tag{4}$$

where $\mathfrak{c}$ is the *cusp-shape*. This holds since near ρ_0 we have $A(1 + t + O(t)^2, -1 + \mathfrak{c}\, t + O(t^2)) = 0$ where $A(M, L)$ is the A-polynomial.

Remark 1.2. Theorem 1.2 is an instance of a well-recorded phenomenon: many classical and quantum invariants of knotted 3-dimensional objects are algebraic. For a detailed discussion regarding conjectures and facts, see [Gar08]. For a quick explanation of the algebricity in dimension 3, see Section 3.1 below.

1.4. **Examples.** In this section, we illustrate Theorem 1.4 for the complement of the figure eight knot 4_1, and the complement of the 5_2 knot.

Example 1.3. Consider the complement M of the figure eight knot 4_1 with a meridian-longitude system (μ, λ). The non–abelian Reidemeister torsion (with respect to the longitude λ) on the character variety X_M is given by (see [Por97] or [Dub06]):

$$\tau_\lambda = \sqrt{17 + 4\,\mathrm{tr}_\lambda}.$$

with the convention that we choose the positive square root near the discrete faitfhul representation ρ_0 with $\operatorname{tr}_\lambda(\rho_0) = -2$ (see [Cal06, Cor.2.4]). Thus $T_\lambda(\tau_\lambda, \operatorname{tr}_\lambda) = 0$ where

$$T_\lambda(x, y) = 17 + 4y - x^2.$$

Let $\operatorname{tr}_\lambda = e_\lambda + e_\lambda^{-1}$, $\operatorname{tr}_\mu = e_\mu + e_\mu^{-1}$. The vanishing of the A-polynomial for the figure eight knot gives us the following identity (see [CCG+94]):

$$A(\epsilon_\lambda, e_\mu) = -2 + (e_\mu^4 + e_\mu^{-4}) - (e_\mu^2 + e_\mu^{-2}) - (e_\lambda + e_\lambda^{-1}).$$

Thus, we obtain:

$$\operatorname{tr}_\lambda = \operatorname{tr}_\mu^4 - 5\operatorname{tr}_\mu^2 + 2.$$

For details, see [Por97, DHY09]. On the other hand, the torsion with respect to the meridian is given by (see Equation (18)):

$$\tau_\mu = \tau_\lambda \cdot \left(\frac{\operatorname{tr}_\lambda^2 - 4}{\operatorname{tr}_\mu^2 - 4}\right)^{1/2} \cdot \frac{\partial \operatorname{tr}_\mu}{\partial \operatorname{tr}_\lambda} = \frac{1}{2}\sqrt{(\operatorname{tr}_\mu^2 - 5)(\operatorname{tr}_\mu^2 - 1)}.$$

Thus $T_\mu(\tau_\mu, \operatorname{tr}_\mu) = 0$ where

$$T_\mu(\tau, z) = -5 + 6z^2 - z^4 + 4\tau^2.$$

At the discrete faithful representation ρ_0, we have $\operatorname{tr}_\lambda(\rho_0) = -2$ (see [Cal06, Cor.2.4]) and $\operatorname{tr}_\mu(\rho_0) = \pm 2$ giving that

$$\tau_\lambda(\rho_0) = 3, \qquad \tau_\mu(\rho_0) = \frac{i\sqrt{3}}{2}.$$

On the other hand, the trace field of 4_1 is $\mathbb{Q}(x)$ where $x^2 + 3 = 0$. This confirms Theorem 1.3 for the discrete faithful representation ρ_0 of 4_1. In addition, the cusp-shape of 4_1 is $\mathfrak{c} = -2i\sqrt{3}$, confirming Equation (4).

Example 1.4. We will repeat the previous example for the twist knot 5_2. The non–abelian Reidemeister torsion (with respect to the longitude λ) for 5_2 is given by (see [DHY09]):

$$\tau_\lambda = (-10\operatorname{tr}_\mu^2 + 21) + \left(5\operatorname{tr}_\mu^4 - 27\operatorname{tr}_\mu^2 + 35\right) u + \left(7 - 5\operatorname{tr}_\mu^2\right) u^2,$$

where u satisfies the polynomial equation

$$(2\operatorname{tr}_\mu^2 - 7) - \left(\operatorname{tr}_\mu^4 - 7\operatorname{tr}_\mu^2 + 14\right) u + \left(2\operatorname{tr}_\mu^2 - 7\right) u^2 - u^3 = 0.$$

Eliminating u from the above equations, it follows that $T_\lambda(\tau_\lambda, \operatorname{tr}_\mu) = 0$ where

$$T_\lambda(x, y) = x^3 + x^2(35 - 26y^2 + 5y^4) + x(294 - 280y^2 + 83y^4 - 10y^6) + 343 + 196y^2 - 126y^4 + 20y^6$$

We choose the branch of u such that at the discrete faithful representation, u_0 satisfies the equation

$$1 - 2u_0 + u_0^2 - u_0^3 = 0, \qquad u_0 = 0.21508\ldots - 1.30714\ldots i$$

which coincides with the Riley polynomial of 5_2; see [MR03]. The invariant trace field of 5_2 is the cubic subfield $\mathbb{Q}(\alpha)$ of the complex numbers given by:

$$\alpha^3 - \alpha^2 + 1 = 0, \qquad \alpha = 0.877439\ldots - 0.744862\ldots i$$

and the cusp shape $\mathfrak{c}$ is given by:

$$\mathfrak{c} = 4\alpha - 6 = -2.49024\ldots - 2.97945\ldots i$$

which is related with the the root of the Riley polynomial by:

$$u_0 = \frac{4}{-\mathfrak{c} - 2}$$

The above equation agrees with [DHY09, Eqn.(3.9)] up to the mirror image of 5_2. It follows that at the discrete faithful representation ρ_0, $\tau_\lambda(\rho_0)$ is the root of the equation

$$\tau_\lambda(\rho_0)^3 + 11\tau_\lambda(\rho_0)^2 - 138\tau_\lambda(\rho_0) + 391 = 0, \qquad \tau_\lambda(\rho_0) = 4.11623\ldots - 1.84036\ldots i$$

and in terms of the invariant trace field, is given by:

$$\tau_\lambda(\rho_0) = -6\alpha^2 + 13\alpha - 6$$

Equation (4) and the above discussion imply that:

$$\tau_\mu(\rho_0) = \frac{\tau_\lambda(\rho_0)}{\mathfrak{c}} = 1 - \frac{3}{2}\alpha = -0.316158\ldots + 1.11729\ldots i$$

Notice that $-2\tau_\mu(\rho_0) = 3\alpha - 2$ is a prime of norm -23. In fact, the invariant trace field $\mathbb{Q}(\alpha)$ has discriminant -23 and 23 ramifies as:

$$-23 = (3\alpha - 2)^2(3\alpha + 1)$$

where $3\alpha - 2$ and $3\alpha + 1$ are the primes above 23. The above discussion confirms Theorem 1.3 for the discrete faithful representation.

1.5. **Problems.** In this section we list a few problems and future directions.

Problem 1.5. Is the T_λ-polynomial of a hyperbolic knot nontrivial?

Remark 1.6. The volume and the Reidemeister torsion appear as the *classical* and *semiclassical* limit in a parametrized version of the Volume Conjecture; see for example [GM08]. Physics arguments suggest that the non-commutative A-polynomial and the Reidemeister torsion is determined by the A-polynomial and the volume of the manifold alone. However, computations with twist knots suggest that the A and T_λ-polynomials seem to be independent from each other. Perhaps this discrepancy can be explained by the difference between on-shell and off-shell physics computations.

Let us now formulate a speculation regarding the *Parametrized Volume Conjecture* of Gukov-Murakami and Le-Garoufalidis; see [GM08, GL11]. If K is a knot in S^3, let $J_{K,N}(q) \in \mathbb{Q}[q^{\pm 1}]$ denote the quantum group invariant of K colored by the N-dimensional irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$, and normalized to be 1 at the unknot. For fixed $\alpha \in \mathbb{C}$, the Parametrized Volume Conjecture studies the asymptotics of the sequence $(J_{K,N}(e^{\alpha/N}))$ for $N = 1, 2, \ldots$. For suitable α near $2\pi i$, and for hyperbolic knots K, one expects an asymptotic expansion of the form

$$J_{K,N}(e^{\alpha/N}) \sim e^{\frac{N \mathrm{CS}(\rho_\alpha)}{2\pi i}} N^{3/2} c_0(\alpha) \left(1 + \sum_{k=1}^\infty \frac{c_k(\alpha)}{N^k}\right)$$

where $\rho_\alpha \in X_M$ denotes a representation near ρ_0 with $\mathrm{tr}_\mu(\rho_\alpha) = e^\alpha + e^{-\alpha}$; see [DGLZ09, GL11].

Problem 1.7. For every k, and with suitable normalization, show that $c_k(\alpha)$ are germs of unique elements of the field $\mathbb{Q}(X_M)$.

Conjecture 1.8. Show that

$$c_0(0) = (2\tau_\mu(\rho_0))^{-1/2} \tag{5}$$

H. Murakami has proven the above conjecture for the 4_1 knot (see [Mur]), and unpublished computations of the second author and D. Zagier have numerically verified the above conjecture for the 5_2 and the $(-2, 3, 7)$ pretzel knot. The details will appear in forthcoming work.

Our next problem concerns the extension of Theorem 1.2 to simple complex Lie groups $G_\mathbb{C}$, rather than $\mathrm{SL}(2,\mathbb{C})$. Physics arguments regarding the 1-loop computation of *perturbative Chern-Simons theory* suggest that an extension of Theorem 1.2 to arbitrary complex simple groups $G_\mathbb{C}$ is possible. It is reasonable to expect that an extension of the non abelian Reidemeister torsion is possible (see for example [BH07, BH08]), and that Theorem 1.2 extends.

Problem 1.9. Extend Theorem 1.2 to arbitrary simple complex Lie groups $G_\mathbb{C}$.

2. The character variety of hyperbolic 3-dimensional manifolds

2.1. **Four favors of the character variety, après Dunfield.** The careful reader may observe that the volume function is defined for $\mathrm{PSL}(2,\mathbb{C})$ representations of a 1-cusped hyperbolic manifold M, whereas the Reidemeister torsion is defined for $\mathrm{SL}(2,\mathbb{C})$-representations of M. Our proof of Theorem 1.2 requires a new variant of a representation, the so-called *augmented representation* that comes in two flavors: the $\mathrm{PSL}(2,\mathbb{C})$ and the $\mathrm{SL}(2,\mathbb{C})$ one. For an excellent discussion, we refer the reader to [Dun99, Sec.2-3] and [BDRV, Sec.10]. Much of the results of this section the second author learnt from N. Dunfield, whom we thank for his guidance. Naturally, we are responsible for any comprehension errors.

Let us define the four versions of the character variety of M. Let $R(M, \mathrm{SL}(2,\mathbb{C}))$ denote the set of all homomorphisms of $\pi_1(M)$ into $\mathrm{SL}(2,\mathbb{C})$ and let $X_{M,\mathrm{SL}(2,\mathbb{C})}$ be the set of *characters* of $\pi_1(M)$ into $\mathrm{SL}(2,\mathbb{C})$ — which is in a sense the algebro-geometric quotient $R(M, \mathrm{SL}(2,\mathbb{C}))//\mathrm{SL}(2,\mathbb{C})$, where $\mathrm{SL}(2,\mathbb{C})$ acts by conjugation (see [Sha02]). The *character* $\chi_\rho\colon \pi_1(M) \to \mathbb{C}$ associated to the representation ρ is defined by $\chi_\rho(g) = \mathrm{tr}(\rho(g))$, for all $g \in \pi_1(M)$. For irreducible representations, two representations are conjugate (in $\mathrm{SL}(2,\mathbb{C})$) if, and only if, they have the same character (see [CCG$^+$94] or [Sha02]). It is easy to see that $R(M, \mathrm{SL}(2,\mathbb{C}))$ and $X_{M,\mathrm{SL}(2,\mathbb{C})}$ are affine varieties defined over $\mathbb{Q}$.

Let $\overline{R}(M, \mathrm{SL}(2,\mathbb{C}))$ denote the subvariety of $R(M, \mathrm{SL}(2,\mathbb{C})) \times P^1(\mathbb{C})$ consisting of pairs (ρ, z) where z is a fixed point of $\rho(\pi_1(\partial M))$. Let $\overline{X}_{M,\mathrm{SL}(2,\mathbb{C})}$ denote the algebro-geometric quotient of $\overline{R}(M, \mathrm{SL}(2,\mathbb{C}))$ under the diagonal action of $\mathrm{SL}(2,\mathbb{C})$ by conjugation and Möbius transformations respectively. We will call elements $(\rho, z) \in \overline{R}(M, \mathrm{SL}(2,\mathbb{C}))$ *augmented representations*. Their images in the augmented character variety $\overline{X}(M, \mathrm{SL}(2,\mathbb{C}))$ will be called *augmented characters* and will be denoted by square brackets $[(\rho, z)]$. Likewise, replacing $\mathrm{SL}(2,\mathbb{C})$ by $\mathrm{PSL}(2,\mathbb{C})$, we can define the character variety $X_{M,\mathrm{PSL}(2,C)}$ and its augmented version $\overline{X}_{M,\mathrm{PSL}(2,\mathbb{C})}$.

The advantage of the augmented character variety $\overline{X}_{M,\mathrm{SL}(2,\mathbb{C})}$ is that given $\gamma \in \pi_1(\partial M)$ there is a regular function e_γ which sends $[(\rho, z)]$ to the eigenvalue of $\rho(\gamma)$ corresponding to z. In contrast, in $X_{M,\mathrm{SL}(2,\mathbb{C})}$ only the trace $e_\gamma + e_\gamma^{-1}$ of $\rho(\gamma)$ is well-defined. Likewise, in $\overline{X}_{M,\mathrm{PSL}(2,C)}$ (resp. $X_{M,\mathrm{SL}(2,\mathbb{C})}$) only e_γ^2 (resp. $e_\gamma^2 + e_\gamma^{-2}$) is defined.

From now on, we will restrict to a geometric component of the $\mathrm{PSL}(2,\mathbb{C})$ character variety of M and its lifts. The four character varieties associated to M fit in a commutative diagram

$$
\begin{array}{ccc}
\overline{X}_{M,\mathrm{SL}(2,\mathbb{C})} & \longrightarrow & \overline{X}_{M,\mathrm{PSL}(2,\mathbb{C})} \\
\downarrow & & \downarrow \\
X_{M,\mathrm{SL}(2,\mathbb{C})} & \longrightarrow & X_{M,\mathrm{PSL}(2,\mathbb{C})}
\end{array}
\tag{6}
$$

where the vertical maps are forgetful maps $[(\rho, z)] \longrightarrow [\rho] = \chi_\rho$ and the horizontal maps are induced by the projection $\mathrm{SL}(2,\mathbb{C}) \longrightarrow \mathrm{PSL}(2,\mathbb{C})$. The vertical maps are generically 2:1 at the geometric components. The horizontal maps are discussed in [Dun99, Cor.3.2].

The notation X_M of Section 1 matches the notation $X_M = X_{M,\mathrm{SL}(2,\mathbb{C})}$ of this section.

The next lemma describes the coordinates rings of the four versions of the character variety.

Lemma 2.1. (1) The coordinate ring of $X_{M,\mathrm{SL}(2,\mathbb{C})}$ is generated by tr_g for all $g \in \pi_1(M)$.
(2) The coordinate ring of $X_{M,\mathrm{PSL}(2,\mathbb{C})}$ is generated by tr_g^2 for all $g \in \pi_1(M)$.
(3) The coordinate ring of $\overline{X}_{M,\mathrm{SL}(2,\mathbb{C})}$ is generated by tr_g for all $g \in \pi_1(M)$ and by e_γ for $\gamma \in \pi_1(\partial M)$.
(4) The coordinate ring of $\overline{X}_{M,\mathrm{PSL}(2,\mathbb{C})}$ is generated by tr_g^2 for all $g \in \pi_1(M)$ and by e_γ^2 for $\gamma \in \pi_1(\partial M)$.

The commutative diagram (6) gives an inclusion of fields of rational functions:

$$
\begin{array}{ccc}
\mathbb{Q}(\overline{X}_{M,\mathrm{SL}(2,\mathbb{C})}) & \hookleftarrow & \mathbb{Q}(\overline{X}_{M,\mathrm{PSL}(2,\mathbb{C})}) \\
\uparrow & & \uparrow \\
\mathbb{Q}(X_{M,\mathrm{SL}(2,\mathbb{C})}) & \hookleftarrow & \mathbb{Q}(X_{M,\mathrm{PSL}(2,\mathbb{C})})
\end{array}
\tag{7}
$$

where the vertical field extensions are of degree 2.

2.2. **The coefficient field of augmented representations.** A crucial part in our proof of Theorem 1.2 is the choice of a coefficient field of an $\mathrm{SL}(2,\mathbb{C})$-representation of $\pi_1(M)$. In this section, we show that the notion of an augmented representation fits well with the choice of a coefficient field.

First, let us describe the problem. Given a subgroup Γ of $\mathrm{SL}(2,\mathbb{C})$, we can define its *trace field* $\mathbb{Q}(\Gamma)$ (resp. its *coefficient field* $E(\Gamma)$) by $\mathbb{Q}(\mathrm{tr}(A) \,|\, A \in \Gamma)$ (resp. the field generated over $\mathbb{Q}$ by the entries of all elements A of Γ). The trace field but *not* the coefficient field of Γ is obviously invariant under conjugation of Γ in $\mathrm{SL}(2,\mathbb{C})$. In general, it is not possible to choose a conjugate of Γ to be a subgroup of $\mathrm{SL}(2,\mathbb{Q}(\Gamma))$. The following lemma shows that this is possible after passing to at most quadratic extension of the trace field.

Lemma 2.2. ([Mac83, Prop. 3.3][MR03, Cor. 3.2.4]) If Γ is non-elementary, then Γ is conjugate to $\mathrm{SL}(2,K)$ where $K = \mathbb{Q}(\Gamma)(e)$ is an extension of degree $[K : \mathbb{Q}(\Gamma)] \leq 2$, and e can be chosen to be an eigenvalue of a loxodromic element of Γ.

For the definition of a *non-elementary* subgroup of $\mathrm{SL}(2,\mathbb{C})$ and of a *loxodromic* element, see [Mac83, MR03]. The proof of Lemma 2.2 uses the theory of 4-dimensional *quaternion algebras*.

We want to apply Lemma 2.2 to a representation $\rho \in R(M, \mathrm{SL}(2,\mathbb{C}))$. Recall that the discrete faithful representation ρ_0 of $\pi_1(M)$ is non-elementary, and that the subset of characters of elementary representations in the geometric component $X_{M,\mathrm{SL}(2,C)}$ is Zariski closed, and therefore, finite; see [MR03].

Given a representation $\rho \in R(M, \mathrm{SL}(2,\mathbb{C}))$, let $\mathbb{Q}(\rho)$ and $E(\rho)$ denote the *trace field* and the *coefficient field* of the subgroup $\rho(\pi_1(M)) \subset \mathrm{SL}(2,\mathbb{C})$ respectively. Likewise, if $(\rho, z) \in \overline{R}(M, \mathrm{SL}(2,\mathbb{C}))$ is an augmented representation, let $\mathbb{Q}(\rho, z)$ denote the field generated over $\mathbb{Q}$ by $\mathrm{tr}_g(\rho)$ for $g \in \pi_1(M)$ and e_γ for $\gamma \in \pi_1(\partial M)$. Similarly, we define the coefficient field $E(\rho, z)$ associated to the augmented representation (ρ, z).

The next lemma follows from Lemma 2.2 and the above discussion.

Lemma 2.3. (1) If $\rho \in R(M, \mathrm{SL}(2,\mathbb{C}))$ is generic (i.e., non-elementary) then a conjugate of ρ is defined over a quadratic extension of $\mathbb{Q}(\rho)$.

(2) If $(\rho, z) \in \overline{R}(M, \mathrm{SL}(2,\mathbb{C}))$ is generic (i.e., non-elementary) then there exists $N \in \mathrm{SL}(2,\mathbb{C})$ so that $N^{-1}(\rho, z)N$ is defined over $E(\rho, z)$.

An alternative version of the above Lemma is possible; see Lemma 2.6 below.

2.3. **Augmented representations and the shape field.** There is an alternative description of the field $\mathbb{Q}(\overline{X}_{M,\mathrm{PSL}(2,\mathbb{C})})$ in terms of shape parameters of ideal triangulations of M, which is useful in applications. For completeness, we discuss it in this section and the next. Let us first describe $\overline{X}_{M,\mathrm{PSL}(2,\mathbb{C})}$ in terms of *pseudo-developing maps*, discussed in detail in [Dun99, Sec.2.5]. Given $\rho \in R_{M,\mathrm{PSL}(2,\mathbb{C})}$, consider a ρ-equivariant map $\widetilde{M} \longrightarrow \mathbb{H}^3$, where $\mathbb{H}^3$ denotes the 3-dimensional hyperbolic space. Since ∂M is a 2-torus, it lifts to a disjoint collection of planes $\mathbb{R}^2$ in the universal cover $\widetilde{M}$. Let $\overline{M}$ denote the space obtained by cutting $\widetilde{M}$ along these planes, and crushing them into points. Set-theoretically, the set $\overline{M} \setminus \widetilde{M}$ of ideal points is in 1-1 correspondence with the *cusps* of M in $\mathbb{H}^3$, i.e., with the coset $\pi_1(M)/\pi_1(\partial M)$. An augmented representation $(\rho, z) \in \overline{R}_{M,\mathrm{PSL}(2,\mathbb{C})}$ gives a $\pi_1(M)$-equivariant map

$$D_{(\rho,z)} : \overline{M} \longrightarrow \overline{\mathbb{H}}^3$$

where $\overline{\mathbb{H}}^3 = \mathbb{H}^3 \cup \mathbb{CP}^1$ is the compactification of hyperbolic space by adding a sphere $\mathbb{CP}^1$ at infinity. Such a map is a pseudo-developing map in [Dun99, Sec.2.5]. An augmented character $[(\rho, z)] \in \overline{X}_{M,\mathrm{PSL}(2,\mathbb{C})}$ does not have a unique pseudo-developing map, however every two are homotopic relative to $\mathbb{CP}^1$, for example using a straight line homotopy $tf(x) + (1-t)g(x)$ in $\mathbb{H}^3$. Thus, there is a well-defined map:

$$\overline{X}_{M,\mathrm{PSL}(2,\mathbb{C})} \longrightarrow \{\text{Pseudo-developing maps of M, modulo homotopy rel boundary}\} \tag{8}$$

Consider a 4-tuple of distinct points $(A, B, C, D) \in (\overline{M} \setminus \widetilde{M})^4$, and an augmented character $[(\rho, z)] \in \overline{X}_{M,\mathrm{PSL}(2,\mathbb{C})}$. Then, $D_{[(\rho,z)]}$ sends A, B, C, D to four points A', B', C', D' in $\mathbb{C} \cup \{\infty\} = \mathbb{CP}^1 = \partial\mathbb{H}^3$, and consider their *cross-ratio*

$$cr_{A,B,C,D}[(\rho, z)] = \frac{(A' - D')(B' - C')}{(A' - C')(B' - D')}.$$

If A', B', C', D' are distinct, then $cr_{A,B,C,D}[(\rho, z)] \in \mathbb{C}$, else $cr_{A,B,C,D}[(\rho, z)]$ is undefined. This gives a rational map

$$cr_{A,B,C,D} : \overline{X}_{M,\mathrm{PSL}(2,\mathbb{C})} \longrightarrow \mathbb{C}.$$

Let $\mathbb{Q}_M^{\mathrm{dev}}$ denote the field over $\mathbb{Q}$ generated by $cr_{A,B,C,D}$ for all 4-tuples of distinct points of $\overline{M} \setminus \widetilde{M}$.

Lemma 2.4. We have

$$\mathbb{Q}_M^{\mathrm{dev}} = \mathbb{Q}(\overline{X}_{M,\mathrm{PSL}(2,\mathbb{C})}).$$

The proof will be given in the next section.

2.4. **Ideal triangulations and the gluing equations variety.** A convenient way to construct the unique hyperbolic structure on M, and its small incomplete hyperbolic deformations is using an *ideal triangulation* $\mathcal{T} = (\mathcal{T}_1, \dots, \mathcal{T}_s)$ of M which recovers the complete hyperbolic structure. For a detailed description of ideal triangulations, see [BP92] and also [BDRV, App.10]. An ideal triangulation $\mathcal{T}$ which is compatible with the discrete faithful representation has nondegenerate shape parameters $z_j \in \mathbb{C} \setminus \{0, 1\}$ for $j = 1, \dots, s$. Such a triangulation always exists; for example subdivide the canonical Epstein-Penner decomposition of M by adding ideal triangles; see [EP88, BP92, PP00]. Once we choose shape parameters for each ideal tetrahedron, one can use them to give a hyperbolic metric (in general incomplete) in the universal cover $\widetilde{M}$, once a compatibility condition along the edges of $\mathcal{T}$ is satisfied. This compatibility condition defines the so-called *Gluing Equations variety* $\mathcal{G}(\mathcal{T})$. In the appendix of [BDRV], Dunfield describes a map

$$\mathcal{G}(\mathcal{T}) \longrightarrow \overline{R}_{M,\mathrm{PSL}(2,\mathbb{C})} \tag{9}$$

which projects to an injection

$$\mathcal{G}(\mathcal{T}) \longrightarrow \overline{X}_{M,\mathrm{PSL}(2,\mathbb{C})} \tag{10}$$

Consider the field $\mathbb{Q}(z_1, \dots, z_s)$ over $\mathbb{Q}$ generated by the shape parameters $z_1, \dots, z_s$. A priori, $\mathbb{Q}(z_1, \dots, z_r)$ depends on M. The next lemma describes the fields of rational functions of augmented representations in terms of the shape field.

Lemma 2.5. (a) We have

$$\mathbb{Q}(\overline{X}_{M,\mathrm{PSL}(2,\mathbb{C})}) = \mathbb{Q}(z_1, \dots, z_s) \tag{11}$$

and

$$\mathbb{Q}(\overline{X}_{M,\mathrm{SL}(2,\mathbb{C})}) = \mathbb{Q}(z_1, \dots, z_s, e_\lambda, e_\mu) \tag{12}$$

(b) If the image of $(z_1, \dots, z_s) \in \mathcal{G}(\mathcal{T})$ is $[(\rho, z)] \in \overline{R}_{M,\mathrm{PSL}(2,\mathbb{C})}$ under the map (9), then the trace field (resp. coefficient field) of an $\mathrm{SL}(2, \mathbb{C})$ lift of $[(\rho, z)]$ is $\mathbb{Q}(z_1, \dots, z_s)$ (resp. $\mathbb{Q}(z_1, \dots, z_s, e_\lambda, e_\mu)$).

Proof. The shape parameters z_j , for $j = 1, \dots, s$, are coordinate functions on the curve $\mathcal{G}(\mathcal{T})$. In addition, the squares e_λ^2 and e_μ^2 of the eigenvalues of a meridian-longitude pair (λ, μ) of ∂M are rational functions of the shape parameters z_j. Since the map in Equation (10) is an inclusion of a curve into another, it follows that their fields of rational functions are equal. This proves Equation (11). Equation (12) follows from Lemma 2.3 and the fact that $e_\lambda^2, e_\mu^2 \in \mathbb{Q}(z_1, \dots, z_s)$. This proves part (a). Part (b) follows from [MR03, Cor.3.2.4]. □

Proof. (of Lemma 2.4) It follows by applying verbatim the proof of [MR03, Lem.5.5.2]. □

Let us end this section with an alternative version of Lemma 2.3 using shape fields. Recall from [Dun99, Sec.2] that the map in Equation (9) can be defined as follows. Fix a solution $(z_1, \dots, z_s)$ of the Gluing Equations of $\mathcal{T}$. Lift $\mathcal{T}$ to an ideal triangulation of $\widetilde{M}$, and then map the lift of one ideal tetrahedron to a fixed ideal tetrahedron of $\mathbb{H}^3$ of the same shape, and then use $\pi_1(M)$-equivariance to send every other ideal tetrahedron to an appropriate ideal tetrahedron of $\mathbb{H}^3$, using face-pairings. There is a consistency condition, which is satisfied since we are using a solution to the Gluing Equations. This defines a developing map and a corresponding $\mathrm{PSL}(2,\mathbb{C})$-representation ρ. In [BDRV, App. 10], Dunfield describes how to define not only a representation in $\mathrm{PSL}(2,\mathbb{C})$, but also an augmented one (ρ, z).

The combinatorial structure of $\mathcal{T}$ gives a presentation of $\Pi = \pi_1(M)$ in terms of *face-pairings*:

$$\Pi = \langle g_1, \dots, g_s \mid r_1, \dots, r_{s-1} \rangle . \tag{13}$$

Each generator of Π is represented by a path in the 1-skeleton of the dual triangulation of $\mathcal{T}$; see [MR03, Chap. 5] or [Rat06, Ch.11]. The entries of $\rho(g_j)$, for $j = 1, \dots, s$, are given by face-pairings, and are explicit matrices with entries in $\mathbb{Q}(z_1, \dots, z_s)$; see [MR03, Chap. 5]. The above discussion proves the following version of Lemma 2.3.

Lemma 2.6. (1) The image of the map in Equation (9) is defined over $\mathbb{Q}(z_1, \dots, z_s)$.
(2) Generically, a lift of the image of the map in Equation (9) to $\overline{R}(M, \mathrm{SL}(2,\mathbb{C}))$ is defined over $\mathbb{Q}(z_1, \dots, z_s, e_\lambda, e_\mu)$.

3. The non-abelian Reidemeister torsion

3.1. **An explanation of the rationality of the Reidemeister torsion in dimension 3.** Before we prove the rationality of the torsion stated in Theorem 1.2, let us give the main idea which is rather simple, and defer the technical details for the next section.

The starting point is a hyperbolic manifold M with one cusp. The character variety $\overline{X}_{M,\mathrm{SL}(2,\mathbb{C})}$ depends only on $\pi_1(M)$ but we view it in a specific birational equivalent way by using a combinatorial decomposition of M into ideal tetrahedra. Every such manifold is obtained by a combinatorial face-pairing of a finite collection $\mathcal{T}$ of nondegenerate (but perhaps flat, or negatively oriented) ideal tetrahedra $\mathcal{T}_1, \dots, \mathcal{T}_s$. The hyperbolic shape of a nondegenerate ideal tetrahedron is determined by a complex number $z \in \mathbb{C} \setminus \{0, 1\}$, up to the action of a finite group of order 6. The discrete faithful representation ρ_0 assigns hyperbolic shapes z_j to the tetrahedra $\mathcal{T}_j$ for $j = 1, \dots, s$. As we already observe, these shapes satisfy the so-called Gluing Equations, which is a collection of polynomial equations in z_j and $1 - z_j$ to make the metric match along the edges of the ideal tetrahedra. The Gluing Equations define a variety $\mathcal{G}(\mathcal{T})$ which of course depends on $\mathcal{T}$. When the discrete faithful representation ρ_0 slightly deforms in ρ_t (i.e., bends, in the language of Thurston) this causes the shapes z_j of $\mathcal{T}_j$ to deform to $z_j(t)$. For small enough t, the new shapes still satisfy the Gluing Equations. Consequently, for every t, the shapes $z_j(t)$, for $j = 1, \dots, s$, are algebraically dependent, and so is any algebraic function of the shapes.

In the case of the A-polynomial, the squares $e_\lambda(t)^2$ and $e_\mu(t)^2$ of the eigenvalues $e_\lambda(t)$ and $e_\mu(t)$ of a meridian-longitude pair of $T^2 = \partial M$ are rational functions in $z_j(t)$ (in fact, monomials in $z_j(t)$ and $1 - z_j(t)$ with integer exponents), thus $(e_\lambda(t), e_\mu(t))$ are algebraically dependent. This dependence defines the A-polynomial.

In the case of Reidemeister torsion and Theorem 1.2, the torsion $\tau_\mu(\rho_t)$ of the relevant chain complex is defined over the field $\mathbb{Q}(z_1(t), \dots, z_s(t), e_\lambda(t), e_\mu(t))$. In other words all matrices that compute the torsion (and thus the ratios of their determinants) have entries in the field $\mathbb{Q}(z_1(t), \dots, z_s(t), e_\lambda(t), e_\mu(t))$.

3.2. **Proof of Theorem 1.2.** In this section, we will prove Theorem 1.2. Let M be a one-cusp finite-volume complete hyperbolic 3-manifold. Choose an ideal triangulation $\mathcal{T} = (\mathcal{T}_1, \dots, \mathcal{T}_s)$ compatible with the discrete faithful representation of M as described above, and let $(z_1, \dots, z_s)$ denote the shape parameters of $\mathcal{T}$. Let E denote the following field:

$$\mathbb{K} = \mathbb{Q}(z_1, \dots, z_s, e_\lambda, e_\mu) = \mathbb{Q}(\overline{X}_{M,\mathrm{SL}(2,\mathbb{C})})$$

where the last equality follows from Lemma 2.5.

Let J denote an open interval in $\mathbb{R}$ that contains 0, and consider a 1-parameter family $t \in J \mapsto z(t) = (z_1(t), \dots, z_s(t)) \in \mathcal{G}(\mathcal{T})$ of solutions of the Gluing Equations, with image $(\rho'_t, z'_t) \in \overline{R}(M, \mathrm{PSL}(2,\mathbb{C}))$ under the map in Equation (9) and with lift $(\rho_t, z_t) \in \overline{R}(M, \mathrm{SL}(2,\mathbb{C}))$ where ρ_0 is a lift to $\mathrm{SL}(2,\mathbb{C})$ of the discrete faithful representation of M. Fix γ an essential curve in the boundary torus ∂M.

We will explain how to define the Reidemeister torsion $\tau_\gamma(\rho_t)$ (for complete definitions the reader can refer to Porti's monograph [Por97] and to Turaev's book [Tur02]), and why it coincides with the evaluation of an element of $\mathbb{K}$ at ρ_t.

The 2-skeleton of the combinatorial dual W to $\mathcal{T}$ is a 2-dimensional CW-complex which is a spine of M; see [BP92]. Mostow rigidity Theorem implies that every homotopy equivalence of M is homotopic to a homeomorphism (even to an isometry), and Chapman's theorem concludes that every homotopy equivalence of M is simple; [Coh73]. Thus, W is simple homotopy equivalent to M, and we can use W to compute $\tau_\gamma(\rho_t)$. The ideas of the definition of the non-abelian torsion $\tau_\gamma(\rho_t)$ are the following:

(a) Consider the universal cover $\widetilde{W}$ of W and the integral chain complex $C_*(\widetilde{W};\mathbb{Z})$ of $\widetilde{W}$ for $* = 0,1,2$. The fundamental group $\Pi = \pi_1(W) = \pi_1(M)$ acts on $\widetilde{W}$ by covering transformations. This action turns the complex $C_*(\widetilde{W};\mathbb{Z})$ into a $\mathbb{Z}[\Pi]$-module. The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ also can be viewed as a $\mathbb{Z}[\Pi]$-module by using the composition $Ad \circ \rho_t$, where Ad denotes the adjoint representation of $\mathfrak{sl}_2(\mathbb{C})$. We let $\mathfrak{sl}_2(\mathbb{C})_{\rho_t}$ denote this $\mathbb{Z}[\Pi]$-module. The *twisted chain complex* of W is the $\mathbb{C}$-vector space:

$$C_*^{\rho_t} = C_*(\widetilde{W};\mathbb{Z}) \otimes_{\mathbb{Z}[\Pi]} \mathfrak{sl}_2(\mathbb{C})_{\rho_t}. \tag{14}$$

(b) The twisted chain complex $C_*^{\rho_t}$ computes the so-called *twisted homology* of W which is denoted by $H_*^{\rho_t}$. The betti numbers of $H_*^{\rho_t}$ are given by (because ρ_t lies in a neighborhood of the discrete and faithful representation and thus is generic, or regular in Porti's language, see [Por97, Chap. 3]):

$$\dim_{\mathbb{C}}(H_0^{\rho_t}) = 0, \qquad \dim_{\mathbb{C}}(H_1^{\rho_t}) = 1, \qquad \dim_{\mathbb{C}}(H_2^{\rho_t}) = 1.$$

(c) For $i = 1,2$ construct elements $\mathbf{h}_i^t$ in $C_i^{\rho_t}$, which project to bases of the twisted homology groups $H_i^{\rho_t}$.

(d) Then, the torsion $\tau_\gamma(\rho_t)$ is an explicit ratio of determinants; see [Dub06] or [Por97, Chap. 3] and Equation (17) below.

We now give the details of the definition of the non-abelian Reidemeister torsion and prove Theorem 1.2. To clarify the presentation, suppose that V_t is a 1-parameter family of $\mathbb{C}$-vector spaces for $t \in J$. We will say that V_t is defined over $\mathbb{K}$ if there exists a vector space $V_{\mathbb{K}}$ over $\mathbb{Q}$ such that $V_t = (V_{\mathbb{K}} \otimes_{\mathbb{Q}} E(\rho_t, z_t)) \otimes_{\mathbb{Q}} \mathbb{C}$ for all $t \in J$, where $E(\rho_t, z_t)$ is the coefficient field of (ρ_t, z_t), defined in Section 2.2. Likewise, a 1-parameter family of $\mathbb{C}$-linear transformations $T_t \in \mathrm{Hom}_{\mathbb{C}}(V_t, W_t)$ is defined over $\mathbb{K}$ if $T \in \mathrm{Hom}_{\mathbb{Q}}(V_{\mathbb{K}}, W_{\mathbb{K}}) \otimes_{\mathbb{Q}} \mathbb{C}$. In concrete terms, a 1-parameter family of matrices (resp. vectors) is defined over E if its entries (resp. coordinates) lie in $\mathbb{K}$.

Lemma 2.6 implies the following.

Claim 3.1. The 1-parameter family (ρ_t, z_t) $(t \in J)$ is defined over $\mathbb{K}$.

Consider the presentation Π in Equation (13) of $\pi_1(M)$ given by face-pairings. A coordinate description of the chain complex $C_*^{\rho_t}$ is given by (see [Dub06])

$$0 \longrightarrow \mathfrak{sl}_2(\mathbb{C})^{s-1} \xrightarrow{d_2^{\rho_t}} \mathfrak{sl}_2(\mathbb{C})^{s} \xrightarrow{d_1^{\rho_t}} \mathfrak{sl}_2(\mathbb{C}) \longrightarrow 0$$

for $* = 0,1,2$ where the boundary operators are given by

$$d_1^{\rho_t}(x_1,\dots,x_s) = \sum_{j=1}^{s}(1-g_j)\circ x_j, \text{ and } d_2^{\rho_t}(x_1,\dots,x_{s-1}) = \left(\sum_{j=1}^{s-1}\frac{\partial r_j}{\partial g_k}\circ x_j\right)_{1\leqslant k\leqslant s}.$$

Here $g \circ x = Ad_{\rho_t(g)}(x)$ and $\frac{\partial r_j}{\partial g_k}$ denotes the *Fox derivative* of r_j with respect to g_k. The above description of $C_*^{\rho_t}$ and Claim 3.1 imply the following.

Claim 3.2. The 1-parameter family $C_*^{\rho_t}$ $(t \in J)$ is defined over $\mathbb{K}$.

Next, we construct a 1-parameter family of basing elements $\mathbf{h}_i^t$ for $i = 1,2$ and show that it is defined over $\mathbb{K}$. Let $\left\{e_1^{(i)},\dots,e_{n_i}^{(i)}\right\}$ be the set of i-dimensional cells of W. We lift them to the universal cover and we choose an arbitrary order and an arbitrary orientation for the cells $\left\{\tilde{e}_1^{(i)},\dots,\tilde{e}_{n_i}^{(i)}\right\}$. If $\mathcal{B} = \{\mathbf{a},\mathbf{b},\mathbf{c}\}$ is an orthonormal basis of $\mathfrak{sl}_2(\mathbb{C})$, then we consider the corresponding (geometric) basis over $\mathbb{C}$:

$$\mathbf{c}_{\mathcal{B}}^i = \left\{\tilde{e}_1^{(i)}\otimes\mathbf{a}, \tilde{e}_1^{(i)}\otimes\mathbf{b}, \tilde{e}_1^{(i)}\otimes\mathbf{c}, \dots, \tilde{e}_{n_i}^{(i)}\otimes\mathbf{a}, \tilde{e}_{n_i}^{(i)}\otimes\mathbf{b}, \tilde{e}_{n_i}^{(i)}\otimes\mathbf{c}\right\}$$

of $C_i^{\rho_t}$. We fix a generator P^{ρ_t} of $H_0^{\rho_t}(\partial M) \subset C_0^{\rho_t}$ i.e., $P^{\rho_t} \in \mathfrak{sl}_2(\mathbb{C})$ is such that $Ad_{\rho_t(g)}(P^{\rho_t}) = P^{\rho_t}$ for all $g \in \pi_1(\partial M)$.

Claim 3.3. The 1-parameter family P^{ρ_t} $(t \in J)$ is defined over $\mathbb{K}$.

Proof. Observe that P^{ρ_t} is a generator of the intersection

$$\ker(Ad_{\rho_t(\mu)} - \mathbf{1}) \cap \ker(Ad_{\rho_t(\lambda)} - \mathbf{1}).$$

Since this family of vector spaces and linear maps is defined over $\mathbb{K}$ (by Claim 3.2), the result follows. □

The canonical inclusion $j \colon \partial M \to M$ induces (see [Por97, Corollary 3.23]) an isomorphism

$$j_* \colon H_2^{\rho_t}(\partial M) \to H_2^{\rho_t}(M) \simeq H_2^{\rho_t}(W) = \ker d_2^{\rho_t} \subset C_2^{\rho_t}.$$

Moreover, one can prove that (see [Por97, Proposition 3.18])

$$H_2^{\rho_t}(\partial M) \cong H_2(\partial M; \mathbb{Z}) \otimes \mathbb{C}.$$

More precisely, let $[\![\partial M]\!] \in H_2(\partial M; \mathbb{Z})$ be the fundamental class induced by the orientation of ∂M, one has $H_2^{\rho_t}(\partial M) = \mathbb{C}\,[[\![\partial M]\!] \otimes P_t^{\rho}]$. The *reference generator* of $H_2^{\rho_t}(M)$ is defined by

$$\mathbf{h}_2^t = j_*([[\![\partial M]\!] \otimes P^{\rho_t}]) \in C_2^{\rho_t}. \tag{15}$$

Claim 3.3 implies that

Claim 3.4. The 1-parameter family $\mathbf{h}_2^t$ $(t \in J)$ is defined over $\mathbb{K}$.

Since ρ_t is near ρ_0 and γ is admissible, the inclusion $\iota \colon \gamma \longrightarrow M$ induces (see [Por97, Definition 3.21]) an *isomorphism*

$$\iota^* \colon H_1^{\rho_t}(\gamma) \to H_1^{\rho_t}(M) \simeq H_1^{\rho_t}(W) = \ker d_1^{\rho_t} / \mathrm{im}\, d_2^{\rho_t}.$$

The *reference generator* of the first twisted homology group $H_1^{\rho_t}(M)$ is defined by

$$\mathbf{h}_1^t = \iota_* \left([[\![\gamma]\!] \otimes P_t^{\rho}]\right) \in C_1^{\rho_t}. \tag{16}$$

Claim 3.3 implies that:

Claim 3.5. The 1-parameter family $\mathbf{h}_1^t$ $(t \in J)$ is defined over $\mathbb{K}$.

Using the bases described above, the non-abelian Reidemeister torsion of the 1-parameter family ρ_t is defined by:

$$\tau_\gamma(\rho_t) = \mathrm{Tor}(C_*^{\rho_t}(W; \mathfrak{sl}_2(\mathbb{C})_{\rho_t}), \mathbf{c}_{\mathcal{B}}^*, \mathbf{h}_t^*) \in \mathbb{C}^*. \tag{17}$$

The torsion $\tau_\gamma(\rho_t)$ is an invariant of M which is *well defined up to a sign*. Moreover, if ρ_t and $\tilde{\rho}_t$ are two 1-parameter family of representations which pointwise have the same character then $\tau_\gamma(\rho_t) = \tau_\gamma(\tilde{\rho}_t)$. Finally, one can observe that $\tau_\gamma(\rho_t)$ does not depend on the choice of the invariant vector P^{ρ_t} (see [Dub06]).

The above discussion implies that

Claim 3.6. For every essential curve $\gamma \in \partial M$, the 1-parameter family $\tau_\gamma(\rho_t)$ $(t \in J)$ is defined over $\mathbb{K}$.

In other words, there exist $\hat{\tau}_\gamma \in \mathbb{Q}(\overline{X}_{M,\mathrm{SL}(2,\mathbb{C})})$ such that for (ρ_t, z) near (ρ_0, z_0) we have $\tau_\gamma(\rho) = \hat{\tau}_\gamma(\rho_t, z)$. Since the left hand side does not depend on z, it follows from Section 2.1 that $\hat{\tau}_\gamma \in \mathbb{Q}(X_{M,\mathrm{SL}(2,\mathbb{C})})$. This concludes the proof of Theorem 1.2. □

3.3. **Proof of Theorems 1.3 and 1.4.** The proof of Theorem 1.2 implies that for every admissible curve γ, the torsion function τ_γ is the germ of an element of $\mathbb{Q}(\overline{X}_{M,\mathrm{SL}(2,\mathbb{C})})$. Theorem 1.3 follows from Theorem 1.2 and Lemmas 2.3 and 2.5.

Theorem 1.4 follows from the fact that $\overline{X}_{M,\mathrm{SL}(2,\mathbb{C})}$ is an affine complex curve, and its field of rational functions has transcendence degree 1. In addition, τ_γ and tr_γ are rational functions on $\overline{X}_{M,\mathrm{SL}(2,\mathbb{C})}$.

3.4. **The dependence of the Reidemeister torsion on the admissible curve and the A-polynomial.** In this section, we discuss the dependence of the non-abelian Reidemeister torsion on the admissible curve. Although this discussion is independent of the proof of Theorem 1.2, it might be useful in other contexts. Recall that the non-abelian Reidemeister torsion is defined in terms of the twisted chain complex in Equation (14) which is not acyclic. Thus, it requires the choice of distinguished bases $\mathbf{h}_i$ for $i = 1, 2$. Such bases can be chosen once an *admissible curve* $\gamma \in \partial M$ is chosen; see [Por97, Chap. 3]. Porti proves that for every homotopically non-trivial curve γ in ∂M, the discrete and faithful representation ρ_0 is γ-*regular*. The same holds for representations ρ near ρ_0. A well-known application of Thurston's *Hyperbolic Dehn Surgery Theorem* implies that $\rho_0 \in X_M$ is a smooth point of X_M and that a neighborhood U of ρ_0 is parametrized by the polynomial function tr_γ; see for example [NZ85] and [Por97, Cor. 3.28]. Choose a meridian-longitude pair (μ, λ) in ∂M, set $\mathrm{tr}_\mu(\rho_t) = e_\mu + e_\mu^{-1}$, $\mathrm{tr}_\lambda(\rho) = e_\lambda + e_\lambda^{-1}$, and consider the A-polynomial $A_M = A_M(e_\mu, e_\lambda) \in \mathbb{Z}[e_\mu^{\pm 1}, e_\lambda^{\pm 1}]$ of M. For a detailed discussion on the A-polynomial of M and its relation to the various views of the character, see the appendix of [BDRV].

With the above notation, Porti proves that the dependence of the torsion on the admissible curve γ is controlled by the A-polynomial. More precisely, one has [Por97, Cor. 4.9, Prop. 4.7]:

$$\tau_\mu = \tau_\lambda \cdot \left(\frac{\mathrm{tr}_\lambda^2 - 4}{\mathrm{tr}_\mu^2 - 4} \right)^{1/2} \cdot \frac{\partial \,\mathrm{tr}_\mu}{\partial \,\mathrm{tr}_\lambda} \tag{18}$$

$$= \tau_\lambda \cdot (\mathrm{res}^* \circ (\Delta^*)^{-1}) \left(\frac{e_\lambda}{e_\mu} \frac{\partial A_M / \partial e_\lambda}{\partial A_M / \partial e_\mu} \right), \tag{19}$$

where $\mathrm{res}^* \colon X_{M,\mathrm{SL}(2,\mathbb{C})} \to X_{\partial M,\mathrm{SL}(2,\mathbb{C})}$ is the restriction-map induced by the usual inclusion $\partial M \hookrightarrow M$, and Δ^* works has follows on the trace field

$$\Delta^*(\mathrm{tr}_\gamma) = e_\gamma + {e_\gamma}^{-1}.$$

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