Algebra Comprehensive Exam
August 25, 2006

Complete FIVE of the EIGHT problems below. If you attempt more than five questions, specify which five should be graded.

1. Find all finite simple groups having a subgroup of index 3.

2. Let p be a prime number, let k be a positive integer, and suppose that G is a group of order p^k acting on a finite set S. Show that the number of elements of S fixed by every $g \in G$ is congruent to $|S|$ (the cardinality of S) modulo p.

3. If p and q are primes, prove that a group of order p^2q cannot be simple.

4. Give examples (with proof) of commutative ring R with identity such that:

   (i) R has exactly 10 ideals.

   (ii) R has exactly 10 maximal ideals.

5. Suppose R is a ring with identity having p^2 elements for some prime number p. Prove that R is commutative.

6. Let K be a field, and let $f, g \in K[x]$ be polynomials with f irreducible over K. If h is any irreducible polynomial dividing $f \circ g = f(g(x))$, prove that $\deg(f) \mid \deg(h)$.

7. Suppose A, B are commuting $n \times n$ matrices over the field $\mathbf{C}$ of complex numbers. Prove that A, B have a common eigenvector.

8. Let M be a 3×3 matrix with integer entries and $\det(M) = -1$. Assume that every real eigenvalue of M is rational. What are the possibilities for the minimal polynomial of M?