

# Analysis Comprehensive Exam

## September 1, 2006

Complete FIVE of the SEVEN problems below. If you attempt more than five questions, specify which five should be graded.

1. (a) Prove that every sequence of real numbers either has a non-decreasing subsequence or a non-increasing subsequence.

(b) Deduce that every bounded sequence of real numbers has a convergent subsequence.

2. Let  $(f_n)$  be a sequence of nonnegative Lebesgue measurable functions on  $[0, 1]$ , and let  $(E_m)$  be a sequence of Lebesgue measurable subsets of  $[0, 1]$ .

- (a) Suppose that there is an integrable function  $f$  such that for  $n \geq 1$  and almost every  $x \in [0, 1]$ ,

$$f_n(x) \leq f(x). \quad (1)$$

Prove that

$$\limsup_{n \rightarrow \infty} \int_0^1 f_n \leq \int_0^1 \limsup_{n \rightarrow \infty} f_n. \quad (2)$$

Is the hypothesis (1) for some integrable  $f$  necessary?

- (b) Let  $E_n \subset [0, 1]$  for  $n \geq 1$  and let

$$E = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} E_m.$$

- (i) Prove that

$$\limsup_{n \rightarrow \infty} \text{meas}(E_n) \leq \text{meas}(E).$$

- (ii) Either prove that equality holds in part (i) or give an example in which strict inequality holds.

- (iii) Prove that

$$\sum_{k=1}^{\infty} \text{meas}(E_k) < \infty \Rightarrow \text{meas}(E) = 0.$$

3. (a) State a necessary and sufficient criterion for a function  $f$  on  $[0, 1]$  to be Riemann integrable. Your criterion must involve Lebesgue measure.
- (b) Let  $f : [0, 1] \rightarrow \mathbb{R}$  be Lebesgue integrable. Set  $f(x) = f(0)$  for  $x < 0$  and  $f(x) = f(1)$  for  $x > 1$ . Prove that

$$\lim_{h \rightarrow 0} \int_0^1 |f(x+h) - f(x)| dx = 0.$$

(You may assume results about approximation of Lebesgue integrable functions by continuous functions.)

- (c) Let  $g : [0, 1] \rightarrow [0, 1]$  be measurable. Prove that if  $f$  is (bounded and) Riemann integrable in  $[0, 1]$ , then

$$\lim_{h \rightarrow 0} \int_0^1 |f(x+hg(x)) - f(x)| dx = 0.$$

4. Let  $1 < p < \infty$  and  $\{f_n\}$  be a sequence of functions in  $L_p[0, 1]$ . Give a proof of, or counterexample to, the following assertions:
- (a) If  $f_n \rightarrow f$  weakly in  $L_p[0, 1]$  as  $n \rightarrow \infty$ , then there is a subsequence  $\{f_{n_k}\}$  that converges a.e. to  $f$ .
- (b) If  $f_n \rightarrow f$  in norm in  $L_p[0, 1]$  as  $n \rightarrow \infty$ , then there is a subsequence  $\{f_{n_k}\}$  that converges a.e. to  $f$ .
5. Let  $f$  be a function defined on  $\mathbb{R}^2$  with continuous second partial derivatives. Use Fubini's theorem to give an easy proof that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

(Hint: Assume it is not true, and integrate  $\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}$  over a suitable square.)

6. (a) Show that there is a bounded linear functional  $\varphi$  on  $\ell_\infty$  such that

$$\varphi(\mathbf{x}) = \lim_{i \rightarrow \infty} x_i$$

for every convergent sequence  $\mathbf{x} = (x_i)$  in  $\ell_\infty$ .

- (b) Is this linear functional unique?

7. Prove that there is no norm under which the vector space  $P$  of polynomials with real coefficients is a Banach space.