Comprehensive Exam, Spring 1999

REAL ANALYSIS

Answer five out of the following seven problems. If you work more than five problems, specify which five problems you would like graded, otherwise the first five answered will be graded.

1. Let f be a real-valued function on $(-\infty, \infty)$ which at every point is continuous from the right. Prove that f is Borel measurable.

2. a. Let $f_n: [0,1] \to [0, \infty)$ be a sequence of continuous functions such that $f_n(x) \geq f_{n+1}(x)$ for all $x \in [0,1]$. Define $f(x) = \lim_{n\to\infty} f_n(x)$. Prove that there exists a $t \in [0,1]$ such that $f(t) = \sup_{x\in[0,1]} f(x)$.

b. Either prove or find a counterexample: $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 f(x)\,dx$.

c. Determine whether the conclusion of part a remains valid if we assume only that for each $x \in [0,1]$, there exists N_x such that $f_n(x) \geq f_{n+1}(x)$ for all $n \geq N_x$.

3. Suppose that $(X, \|\cdot\|)$ is a complete seminormed space and let Φ be a collection of continuous seminorms on X. Suppose that for each $a \in X$, the set $\{\rho(a) : \rho \in \Phi\}$ is bounded. Prove that $\{\|\rho\| : \rho \in \Phi\}$ is bounded, where $\|\rho\| = \sup\{\rho(x) : \|x\| \leq 1\}$.

Remark: A function $\rho: X \to \mathbf{R}$ is a seminorm if

(i) $\rho(x) \geq 0$ for every $x \in X$,
(ii) $\rho(cx) = |c|\rho(x)$ for every $x \in X$ and $c \in \mathbf{R}$, and
(iii) $\rho(x+y) \leq \rho(x) + \rho(y)$ for every $x,\ y \in X$.

4. Prove that if f and g are bounded and are in ordinary L^1 of $(-\infty, \infty)$, then

$$x \mapsto \int_{-\infty}^{\infty} f(x+y)\,g(y)\,dy$$

defines a continuous function of x. Hint: The continuous functions with compact support are dense in L^1.

5. Let S be a linear subspace of $C[0,1]$ which is closed as a subspace of $L^2[0,1]$.

a. Show that S is a closed subspace of $C[0,1]$ (equipped with the L^∞ norm).

b. Show that the L^2 and L^∞ norms *restricted to* S are equivalent.

c. Show that for each $y \in [0,1]$ there is a function $k_y \in L^2[0,1]$ such that

$$\forall f \in S, \quad f(y) = \int_0^1 k_y(x)\, f(x)\, dx.$$

d. Let $f_n, f \in S$. Show that if $f_n \to f$ weakly in $L^2[0,1]$, then $f_n(y) \to f(y)$ for each $y \in [0,1]$.

6. Let f be a real-valued function define on some interval in the real line. Recall the definitions of the upper and lower derivatives at $c \in I$:

$$\underline{D}f(c) = \liminf_{x \to c} \left(\frac{f(x) - f(c)}{x - c} \right) \quad \text{and} \quad \overline{D}f(c) = \limsup_{x \to c} \left(\frac{f(x) - f(c)}{x - c} \right).$$

Prove that if both $\underline{D}f(c)$ and $\overline{D}f(c)$ are finite, then f is continuous at c.

7. Let $f \in L^1(E)$, where $E \subset \mathbf{R}$ has finite Lebesgue measure. Prove that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if A is a measurable subset of E with Lebesgue measure less than δ, then $\int_A |f(x)|\, dx < \varepsilon$.