

# Open problems III: Entropy of orthogonal polynomials

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Assume that  $\{p_n\}$  is a sequence of orthonormal polynomials with respect to a unit positive weight  $\omega$  on  $\mathbb{R}$ . The Boltzmann-Shannon entropy of these polynomials is defined as

$$E_n \stackrel{\text{def}}{=} - \int p_n^2(x) \log(p_n^2(x)) \omega(x) dx .$$

There are two mathematical problems of absolutely different nature related with these quantities:

- (i) Explicit formula or at least a stable numerical algorithm for computation of  $E_n$  for any reasonable  $n \in \mathbb{N}$ ;
- (ii) The asymptotic analysis of  $E_n$  as  $n \rightarrow \infty$ .

Regarding (i), general explicit formulas for  $E_n$  have been obtained for the Gegenbauer polynomials with index  $\lambda = 0$  (Chebyshev),  $\lambda = 1$  (Chebyshev of the second kind) and  $\lambda = 2$  only. In particular, it has been shown in [4, 7] that for Chebyshev polynomials,  $E_n = \log(2) - 1$ , for all  $n \in \mathbb{N}$ . Moreover, as it follows from [1], for the unit weight  $\omega$  on  $[-1, 1]$ ,  $\log(2) - 1$  is the asymptotically maximum value that  $E_n$  can achieve.

**Question 1** *Does the property  $E_n \equiv \text{const}$  characterize the family of Chebyshev polynomials? The answer is “yes” in the Bernstein-Szegő class (see [1]), but what about, say the whole Szegő class?*

**Question 2** *Is there any other family of polynomials for which  $E_n$  has a compact closed expression?*

In [3] a stable algorithm for computing  $E_n$  for a unit weight  $\omega$  on  $[-1, 1]$  has been found. It uses the coefficients of the three-terms recurrent relation for the  $p_n$ 's as the only input, in the spirit of the well-known procedure for the computation of the gaussian quadratures.

**Question 3** *Is it possible to extend this method (or to find an alternative one) for the case of the unbounded interval of orthogonality?*

Problem (ii) is closely related to the strong (or at least,  $L^{2+\epsilon}$ ) asymptotics of the polynomials. In [5] it was shown that under rather general conditions,

$$E_n = \int_{a_{-n}}^{a_n} \frac{\log \omega(x)}{\sqrt{(a_n - x)(x - a_{-n})}} dx (1 + o(1)) , \quad n \rightarrow \infty , \quad (1)$$

where  $a_{\pm n}$  are the well-known Mhaskar-Rakhmanov-Saff numbers.

**Question 4** The Pollaczek polynomials  $p_n(x; \lambda, a, b)$  are orthogonal on  $[-1, 1]$  with respect to the unit weight

$$\omega(x; \lambda, a, b) = \frac{2^{2\lambda} (\lambda + a)}{2\pi \Gamma(2\lambda)} (1 - x^2)^{\lambda-1/2} e^{(2 \arccos x - \pi) \frac{ax+b}{\sqrt{1-x^2}}} \left| \Gamma\left(\lambda + i \frac{ax+b}{\sqrt{1-x^2}}\right) \right|^2.$$

They also satisfy the three-term recurrence relation

$$x p_n(x; \lambda, a, b) = a_{n+1} p_{n+1}(x; \lambda, a, b) + b_n p_n(x; \lambda, a, b) + a_n p_{n-1}(x; \lambda, a, b),$$

with

$$a_n = \frac{1}{2} \sqrt{\frac{n(n+2\lambda-1)}{(n+\lambda+a)(n+\lambda+a-1)}}, \quad b_n = -\frac{b}{n+\lambda+a}.$$

As it follows from [1], for these polynomials  $E_n$  diverges. However, they do not satisfy the conditions of [5]. Is formula (1) still valid?

If  $\mu$  and  $\nu$  are two Borel (generally speaking, real signed) measures on  $\mathbb{C}$ , we denote by

$$I[\nu, \mu] = - \iint \ln |z - t| d\nu(t) d\mu(z)$$

their mutual energy. With each polynomial

$$p_n(z) = \gamma_n \prod_{j=1}^n (z - \zeta_j^{(n)})$$

we can associate naturally two probability measures:

$$\lambda_n = \frac{1}{n} \sum_{j=1}^n \delta_{\zeta_j^{(n)}} \quad \text{and} \quad d\nu_n(x) = p_n^2(x) d\mu(x).$$

Both measures are standard objects of study in the analytic theory of orthogonal polynomials. For instance, the normalized zero counting measure  $\lambda_n$  is closely connected with the  $n$ -th root asymptotics of  $p_n$ , and as was shown by Rakhmanov in his pioneering work [6],  $\nu_n$  is associated with the behavior of the ratio  $p_{n+1}/p_n$  as  $n \rightarrow \infty$ .

A nice link between the entropy and the logarithmic potential theory is established by the following identity: if  $\gamma_n > 0$  is the leading coefficient of  $p_n$ , then

$$E_n = -2 \log(\gamma_n) + 2n I[\lambda_n, \nu_n]. \quad (2)$$

Hence, if we know the asymptotic behavior of the entropy, it gives us information about the mutual energy  $I[\lambda_n, \nu_n]$ , and viceversa.

Assume that  $\omega$  is a weight on  $[-1, 1]$ , strictly positive a. e. on the interval. Then both  $\lambda_n$  and  $\nu_n$  tend (in a weak-star sense) to the equilibrium measure of the interval,  $\mu$ , and it is not surprising at all to find out that  $I[\lambda_n, \nu_n] \rightarrow I[\mu, \mu] = \log(2)$ . But what about the next term

of the asymptotics of the energy, does it depend on the weight  $\omega$ ? From the results of [1] it follows that for a large subset of the Szegő class (for instance, for weights satisfying that

$$\sup_n \int_{-1}^1 (p_n^2(x))^{1+\varepsilon} \omega(x) dx < \infty$$

for a  $\varepsilon > 0$ ), we have

$$I[\lambda_n, \nu_n] = \log(2) - \frac{1}{2n} + o\left(\frac{1}{n}\right), \quad n \rightarrow \infty. \quad (3)$$

**Question 5** *What is the explanation of this “universal” behavior of the second term? Is the constant  $1/2$  appearing there the logarithmic capacity of the support of the measure of orthogonality? Is there any direct proof of (3) which does not use (2)? Is (3) still valid in a larger class of weights  $\omega$ ?*

Finally, if  $p_n$  is now a sequence of polynomials orthonormal with respect to a varying weight,

$$\int_{\Delta} p_n(x) p_k(x) \omega^n(x) dx = \delta_{nk}, \quad k = 0, 1, \dots, n,$$

the asymptotic behavior of  $E_n$  is described in terms of the equilibrium measure in the external field  $Q(x) = -\log(\omega(x))$ : if  $K$  is its support, and  $\mu_K$  is the Robin distribution on  $K$ , then

$$E_n = -2n \int_{\Delta} Q d\mu_K + o(n), \quad n \rightarrow \infty. \quad (4)$$

This formula was established in [2] only for the case when  $\omega$  is the Jacobi or Laguerre weight.

**Question 6** *Is this result valid in a more general setting, say when  $K$  is an interval?*

## References

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