Some open problems

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1. It is well known [4, 5] that under suitable conditions on $W(x) = \exp(-Q(x))$, there exists, for every integer $n \geq 1$, a unique probability measure $\mu_{W,n}$, supported on $[-1,1]$ that maximizes

$$\int\int \log |W(a_n x)W(a_n t)(x-t)|\, d\nu(x) d\nu(t)$$

among all compactly supported probability measures ν supported on $\mathbb{R}$, where a_u is defined by

$$u = \frac{2}{\pi}\int_0^1 a_u t Q'(a_u t)\frac{dt}{\sqrt{1-t^2}}, \qquad u > 0.$$

It will be interesting to prove the following analogue of a theorem of Erdős, Kroó, and Szabados [2].
Let $x_{k,n}$ be distinct points on $\mathbb{R}$, W be a weight function such that the measures $\mu_{W,n}$ are supported on $[-1,1]$. The following are equivalent.
(a) *To every f with $Wf \in C_0(\mathbb{R})$ and $\epsilon > 0$, there exists a sequence of polynomials $r_n \in \Pi_{n(1+\epsilon)}$ such that $r_n(x_{k,n}) = f(x_{k,n})$ for $k = 1, \cdots, n$, and $\|(f - r_n)W\|_{\infty,\mathbb{R}} \to 0$ as $n \to \infty$.*
(b) *We have*

$$\limsup_{n\to\infty} \frac{\#\{k \ : \ x_{k,n}/a_n \in I_n\}}{n\mu_{W,n}(I_n)} \leq 1 \qquad (1)$$

for every sequence of intervals $I_n \subseteq [-1,1]$ for which $\lim_{n\to\infty} n\mu_{W,n}(I_n) \to \infty$, and

$$\liminf_{n\to\infty} n\mu_{W,n}([x_{k+1,n}/a_n, x_{k,n}/a_n]) > 0, \qquad 1 \leq k \leq n. \qquad (2)$$

It is worth mentioning here that the location and distribution of node systems $\{x_{k,n}\}$ that provide a "good" interpolation process has been studied by Szabados [6], Damelin [1], and Vertesi [8, 7].

2. Let $q \geq 1$ be an integer, $\mathbb{S}^q$ be the unit sphere embedded in the Euclidean space $\mathbb{R}^{q+1}$, and μ be the volume element of $\mathbb{S}^q$. We are interested in quadrature formulas of the form

$$\sum_{\xi \in \mathcal{C}} w_\xi f(\xi) \approx \int_{\mathbb{S}^q} f(\mathbf{x}) d\mu(\mathbf{x}),$$

where $\mathcal{C}$ is a finite set of points on $\mathbb{S}^q$, w_ξ are positive numbers, and the formula is required to be exact for spherical polynomials of degree as high as possible. The highest degree of polynomials for which the formula is exact will be called the order of the formula. The formula will be called interpolatory if P is any spherical polynomial of degree at most $|\mathcal{C}|$, and $P(\xi) = 0$ for each $\xi \in \mathcal{C}$, then $P \equiv 0$. Numerical experiments in [3] suggest that if $\mathcal{C}$ is the Saff-Kuijlaars system then one obtains quadrature formulas of nearly the highest order. The questions are: (1) If $\mathcal{C}$ is an extremal system of points with respect to some energy problem, does there exists an interpolatory quadrature formula based at these points? (2) If an interpolatory quadrature formula exists, then is it necessarily based on the extremal points for some discretized energy problem? In this connection, it is noteworthy that Jürgen Prestin has recently obtained interpolatory quadrature formulas, some of which can be thought of as based on a set of tensor product Fekete points with respect to a suitable energy functional.

References

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