

A conjecture on the strong asymptotics of Bergman orthogonal polynomials

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Let G be a bounded simply-connected domain in the complex plane \mathbb{C} , whose boundary $L := \partial G$ is a Jordan curve and let $\{P_n\}_{n=0}^\infty$ denote the sequence of *Bergman polynomials* of G . This is defined as the sequence

$$P_n(z) = \gamma_n z^n + \cdots, \quad \gamma_n > 0, \quad n = 0, 1, 2, \dots,$$

of polynomials that are orthonormal with respect to the inner product

$$(f, g) := \int_G f(z) \overline{g(z)} dm(z),$$

where dm stands for the 2-dimensional Lebesgue measure. Also, let $\Omega := \overline{\mathbb{C}} \setminus \overline{G}$ denote the exterior (in $\overline{\mathbb{C}}$) of \overline{G} . Then, the *exterior* conformal map Φ associated with G is the conformal map $\Phi : \Omega \rightarrow \Delta := \{w : |w| > 1\}$, normalised so that

$$\Phi(z) = cz + \mathcal{O}(1), \quad z \rightarrow \infty, \quad c > 0.$$

The constant

$$\text{cap } L = 1/c,$$

is called the (logarithmic) capacity of L .

With respect to the strong asymptotics of the leading coefficient γ_n and of $P_n(z)$, for $z \in \Omega$, we consider the following two formulas:

$$\gamma_n = \sqrt{\frac{n+1}{\pi}} \frac{1}{\text{cap } L^{n+1}} \{1 + \alpha_n\},$$

and

$$P_n(z) = \sqrt{\frac{n+1}{\pi}} \Phi'(z) \Phi^n(z) \{1 + \beta_n\}, \quad z \in \overline{\Omega}.$$

If the boundary L of G is an *analytic* Jordan curve, then a result due to T. Carleman gives respectively,

$$\alpha_n = \mathcal{O}(\rho^{2n}) \text{ and } \beta_n = \mathcal{O}(\rho^n), \quad n \rightarrow \infty,$$

for some $\rho < 1$; see e.g. [1, pp. 12–13]. In the case where L is *smooth*, typically $L \in C(p+1, s)$, where $p+1 \in \mathbb{N}$ and $p+s > \frac{1}{2}$, then a result of P.K. Suetin (see [2, Thms 1.1 and 1.2]) gives,

$$\alpha_n = \mathcal{O}\left(\frac{1}{n^{2(p+s)}}\right) \text{ and } \beta_n = \mathcal{O}\left(\frac{\log n}{n^{p+s}}\right), \quad n \rightarrow \infty.$$

Apart from the above very important results, we haven't been able to find, in the relevant literature, any similar result concerning the behaviour of the two sequences $\{\alpha_n\}$ and $\{\beta_n\}$, associated with more general Jordan curves.

Accordingly, our conjecture is concerned with boundary curves that encountered very frequently in the applications, namely with piecewise analytic Jordan curves. It is based on certain theoretical results and strong numerical evidence and can be stated as follows.

Conjecture. *Assume that the boundary L of G is a piecewise analytic Jordan curve without cusps. Then,*

$$\gamma_n = \sqrt{\frac{n+1}{\pi}} \frac{1}{\text{cap } L^{n+1}} \left\{1 + \mathcal{O}\left(\frac{1}{n^2}\right)\right\}, \quad n \rightarrow \infty,$$

and

$$P_n(z) = \sqrt{\frac{n+1}{\pi}} \Phi'(z) \Phi^n(z) \left\{1 + \mathcal{O}\left(\frac{1}{n}\right)\right\}, \quad z \in \Omega, \quad n \rightarrow \infty.$$

References

- [1] Dieter Gaier. *Lectures on complex approximation*. Birkhäuser Boston Inc., Boston, MA, 1987. Translated from the German by Renate McLaughlin.
- [2] P. K. Suetin. *Polynomials orthogonal over a region and Bieberbach polynomials*. American Mathematical Society, Providence, R.I., 1974. Translated from the Russian by R. P. Boas.