

Distribution of primes and a weighted energy problem

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Let $\pi(x)$ be the number of primes not exceeding x . The celebrated Prime Number Theorem (PNT), suggested by Legendre and Gauss, states that

$$\pi(x) \sim \frac{x}{\log x} \quad \text{as } x \rightarrow \infty.$$

Chebyshev [1] made the first important step towards the PNT in 1852, by proving the bounds

$$0.921 \frac{x}{\log x} \leq \pi(x) \leq 1.106 \frac{x}{\log x} \quad \text{as } x \rightarrow \infty.$$

Hadamard and de la Vallée Poussin independently proved the Prime Number Theorem in 1896, via establishing that $\zeta(s)$ does not have zeros on the line $\{1+it, t \in \mathbb{R}\}$. From a different perspective, Gelfond and Schnirelman (see [1, pp. 285–288]) proposed an interesting “elementary” method aimed at producing the PNT with a good error term, in 1936. It used polynomials with integer coefficients and the Chebyshev function $\psi(x) = \log \text{lcm}(1, \dots, x)$, $x \in \mathbb{N}$, together with the well known fact that the PNT is equivalent to $\psi(x) \sim x$ as $x \rightarrow +\infty$ (see [3]). Later work showed that the original Gelfond-Schnirelman method cannot give a proof of the PNT [4, Ch. 10]. However, Nair [5] and Chudnovsky [2] found a generalization based on an equivalent form of the Prime Number Theorem [3]

$$(1) \quad \int_1^x \psi(t) dt \sim \frac{x^2}{2} \quad \text{as } x \rightarrow +\infty.$$

Using the following weighted version of Vandermonde determinant

$$V_n^w(x_1, \dots, x_n) := \prod_{i=1}^n w^{n-1}(x_i) \prod_{1 \leq i < j \leq n} (x_i - x_j),$$

where $x_i \in [0, 1]$ and $w(x) = (x(1-x))^{\alpha_1}$, $\alpha_1 > 0$, they obtained the bound

$$(2) \quad \int_1^x \psi(t) dt \geq 0.99035 \frac{x^2}{2} \quad \text{as } x \rightarrow +\infty,$$

produced by the optimal choice $\alpha_1 \approx 0.195$. We developed the ideas of [5] and [2], and established a connection with the weighted capacity c_w (cf. [7]) of $[0, 1]$ corresponding to the weight w of the form

$$(3) \quad w(x) = \prod_{i=1}^k |Q_{m_i}(x)|^{\alpha_i}, \quad \alpha_i > 0, \quad i = 1, \dots, k,$$

where Q_{m_i} is a polynomial with integer coefficients of degree at most m_i . For $\alpha := \sum_{i=1}^k \alpha_i m_i$, we proved [6] that

$$(4) \quad \int_1^x \psi(t) dt \geq \frac{-2 \log c_w}{4\alpha + 3} \frac{x^2}{2} + O(x \log^2 x) \quad \text{as } x \rightarrow +\infty.$$

In particular, if $w(x) = x^{\alpha_1}(1-x)^{\alpha_2}$, $x \in [0, 1]$, $\alpha_1 = \alpha_2 = 0.195$, then $c_w \approx 0.1045575588$ and (2) holds true.

It is natural to try improving the bound (2) by choosing a weight with a proper combination of factors $Q_{m_i}(x)$ and exponents α_i . The most interesting question is, of course, whether one can find a weight $w(x)$ of the form (3) such that

$$\frac{-2 \log c_w}{4\alpha + 3} = 1?$$

It turns out this is impossible to achieve for any *fixed* weight of the type (3). The reason for such a conclusion transpires from the error term in (4), which is “too good.” Indeed, it is known from Littlewood’s theorem that the difference $\int_1^x \psi(t) dt - x^2/2$ takes both positive and negative values of the amplitude $cx^{3/2}$, $c > 0$, infinitely often as $x \rightarrow +\infty$. This is conveniently written in the notation

$$\int_1^x \psi(t) dt - \frac{x^2}{2} = \Omega_{\pm}(x^{3/2}) \quad \text{as } x \rightarrow +\infty$$

(cf. [3, pp. 91-92]). Hence the correct error term should be of the order $O(x^{3/2})$. Relating this to (1) and (4), we obtain in such an indirect way that

$$(5) \quad B(w) := \frac{-2 \log c_w}{4\alpha + 3} < 1.$$

We should also note that if the Riemann hypothesis is true, then

$$\int_1^x \psi(t) dt - \frac{x^2}{2} = O(x^{3/2}) \quad \text{as } x \rightarrow +\infty$$

(see Theorem 30 in [3, p. 83]). It would be very interesting to find a direct potential theoretic argument explaining (5). Although (4) cannot provide a proof of the PNT for a fixed weight w , this does not preclude the possibility that such a proof can be obtained by finding a *sequence* of weights w_n with $B(w_n) \rightarrow 1$, as $n \rightarrow \infty$. Thus one needs an insight into the nature of such factors, to address the following problem.

Problem. For $w(x)$ as in (3) and $\alpha = \sum_{i=1}^k \alpha_i m_i$, find

$$(6) \quad B := \sup_w \frac{-2 \log c_w}{4\alpha + 3}.$$

If $B = 1$ then find a sequence of weights that gives this value. If $B < 1$ then investigate whether B is attained for a weight of the form (3).

A solution of the minimum weighted energy problem associated with weights (3) is given in our paper [6]. This paper also contains more background material.

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