GENERALIZED TRANSLATION OPERATOR ON THE SIMPLEX

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The generalized translation operator, T_θ^κ, for the weight function

$$W_\kappa(x) = x_1^{\kappa_1} \cdots x_d^{\kappa_d}(1-|x|_1)^{\kappa_d}, \qquad |x|_1 = (1 - x_1 - \ldots - x_d),$$

on the simplex $T^d = \{x : x_1 \ge 0, \ldots, x_d \ge 0, 1 - |x|_1 \ge 0\}$ is defined in [2]. For $f \in L^2(W_\kappa, T^d)$, the operator T_θ^κ has the orthogonal expansion

$$T_\theta^\kappa f(x) \sim \sum_{k=0}^{\infty} \frac{P_k^{(\lambda-1/2,-1/2)}(\cos 2\theta)}{P_k^{(\lambda-1/2,-1/2)}(1)} \operatorname{proj}_n f(x),$$

where $\lambda = \sum_{i=1}^{d+1} \kappa_i + (d-1)/2$ and proj_n denotes the projection operator from $L^2(W_\kappa, T^d)$ onto Π_n^d, the space of polynomials of degree at most n in d variables. This operator is used to define a modulus of smoothness in [2], which leads to a characterization of the weighted best approximation by polynomials (the direct and the inverse theorems) on T^d [3].

The operator T_θ^κ is associated with a corresponding one for the weight function

$$U_\kappa(x) := |x_1|^{2\kappa_1} \ldots |x_d|^{2\kappa_d}(1 - \|x\|^2)^{\kappa_{d+1}-1/2}$$

on the unit ball $B^d = \{x \in \mathbb{R}^d : \|x\|^2 \le 1\}$, which in turn is associated to the weighted spherical means for the weight function $h_\kappa^2(x) = \prod_{i=1}^{d+1} |x_i|^{2\kappa_i}$ on the unit sphere S^d [1]. In the case of the classical weight function $U_\mu(x) = (1-\|x\|^2)^{\mu-1/2}$ on the unit ball, an integral formula is found for the generalized translation operator [3]. The formula takes the form

$$T_\theta^\mu f(x) = A_\mu \int_{B^d} f\left(\cos\theta x + \sin\theta\, sD(x)U^T(x)\right)(1-\|s\|^2)^{\mu-1} ds,$$

where $D(x) = \operatorname{diag}\{\sqrt{1-\|x\|^2}, 1, \ldots, 1\}$ is a diagonal matrix, $U(x)$ is a unitary matrix whose first column is $x/\|x\|$ and A_μ is a constant ($s \in \mathbb{R}^d$ is taken as a row vector). For $d = 1$, this becomes the classical generalized translation operator

$$T_s f(t) = b_{\lambda-1/2} \int_{-1}^{1} f\left(st + u\sqrt{1-s^2}\sqrt{1-t^2}\right)(1-u^2)^{\lambda-1} du$$

for the weight function $w_\lambda(t) = (1-t^2)^{\lambda-1/2}$ on $[-1, 1]$.

The open question calls for an integral formula for the generalized translation operator with respect to the weight function W_κ on the simplex. The definition of T_θ^κ is given implicitly by an integral relation in [2], relying on the intertwining operator V_κ of Dunkl's operators for $h_\kappa^2(x)$ given above. For this h_κ, the operator V_κ is an explicit integral transform, which suggests that an integral formula for T_θ^κ should exist.

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References

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