On Asymptotics of the Weighted Riesz Energy for Rectifiable Sets

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Let $d', d \in \mathbf{N}$, $d' \geq d$, A be a compact set in $\mathbf{R}^{d'}$ and $w : A \times A \to [0, \infty)$ be a bounded on $A \times A$ function which is continuous and strictly positive at every point (x, x), $x \in A$ (a *weight function on* A). For $s > 0$ the *(w, s)-energy of the set A* is defined by

$$E_s^w(A, N) := \inf_{\{x_1,\ldots,x_N\} \subset A} \sum_{1 \leq i \neq j \leq N} \frac{w(x_i, x_j)}{(x_i - x_j)^s}.$$

If $w(x, y) = 1$, then the *Riesz s-energy of the set A* is defined to be $E_s(A, N) := E_s^w(A, N)$. Exact asymptotics as $N \to \infty$ of the quantity $E_s(A, N)$ were obtained in [1] for $s = d$ and A being the unit sphere in $\mathbf{R}^{d+1}$, in [2] – for $s \geq 1$ and A being a finite union of rectifiable Jordan arcs in $\mathbf{R}^{d'}$, in [3] – for $s = d$ and A being a compact subset of a d-dimensional C^1-manifold in $\mathbf{R}^{d'}$, and for $s > d$ and compact sets $A \subset \mathbf{R}^{d'}$ contained in a finite union of bi-Lipschitz images of open bounded sets from $\mathbf{R}^d$. These papers also find asymptotic distribution of the energy minimizing points.

We generalize the results of [3] by substituting "bi-Lipschitz" with "Lipschitz" in assumptions about the set A for $s > d$, and extend this result to the case of the (w, s)-energy.

Denote by β_d the Lebesgue measure of the unit ball in $\mathbf{R}^d$ and by $H_d(\cdot)$ – the d-dimensional Hausdorf measure in $\mathbf{R}^{d'}$ normalized so that its restriction to $\mathbf{R}^d$ coinsides with the d-dimensional Lebesgue measure.

Theorem. *Let A be a compact set in $\mathbf{R}^{d'}$, $H_d(A) > 0$, and w be a weight function. If $s > d$ and A is a finite union of Lipschitz images of bounded sets from $\mathbf{R}^d$, then*

$$\lim_{N \to \infty} \frac{E_s^w(A, N)}{N^{1+s/d}} = C_{s,d} \left(\int_A (w(x, x))^{-d/s} dH_d(x) \right)^{-s/d},$$

where $C_{s,d} > 0$ is a constant independent of A. If A is a subset of a d-dimensional C^1-manifold in $\mathbf{R}^{d'}$, then

$$\lim_{N \to \infty} \frac{E_d^w(A, N)}{N^2 \ln N} = \beta_d \left(\int_A (w(x, x))^{-1} dH_d(x) \right)^{-1}.$$

In both cases any asymptotically energy minimizing sequence of N-point configurations on A has the limit distribution with density $\gamma_{s,d}(w(x, x))^{-d/s}$, $x \in A$, where $\gamma_{s,d}$ is a normalization constant.

[1] A.B.J. Kuijlaars, E.B. Saff, Asymptotics for minimal discrete energy on the sphere, *Trans. Amer. Math. Soc.* **350** 2 (1998) 523–538.

[2] A. Martinez-Finkelstein, V. Maymeskul, E.A. Rakhmanov, E.B. Saff, Asymptotics for minimal discrete Riesz energy on curves in $\mathbf{R}^d$ (2002, to appear).

[3] D.P. Hardin, E.B. Saff, Minimal Riesz energy point configurations for rectifiable d-dimensional manifolds, to appear in *Advances in Mathematics* (2004).