SINGULARITIES AND CHERN-WEIL THEORY

Blaine Lawson

Two classical theorems concerning the Euler characteristic $\chi(\Sigma)$ of a compact orientable surface Σ are:

The Gauss-Bonnet Theorem.

$$\chi(\Sigma) \;=\; \frac{1}{4\pi}\int_{\Sigma} K\, dA$$

The Hopf Theorem. *If V is a vector field with isolated singularities $p_1, ..., p_n$ on Σ, and if n_i is the index (or winding number) of V at p_i, then*

$$\chi(\Sigma) \;=\; \sum_{i=1}^{n} n_i.$$

These two results are actually quite closely related. Using the vector field V one can construct a family of curvature 2-forms Ω_t on Σ with the property that:

$$\sum_i n_i \delta_{p_i} \quad \overset{0 \leftarrow t}{\longleftarrow} \quad \Omega_t \quad \overset{t \to \infty}{\longrightarrow} \quad K\, dA$$

where δ_p denotes the delta function at p. Furthermore, from this family one can construct a canonical 1-form T with L^1-coefficients on Σ such that

$$K\, dA \;-\; \sum_i n_i \delta_{p_i} \;=\; dT,$$

that is, the difference between these two generalized 2-forms is exact.

This all follows from a general theory which relates geometric singularities (such as the zeros of a vector field), to characteristic forms. I shall present an introduction to this theory.