One-bit Sensing: Phase Transitions for the RIP Property

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Motivation

Linear compressive sensing involves dimension reduction on a high dimensional vector set on which you apply a short, fat Gaussian matrix to the vector x.

$$\begin{bmatrix} ------ Z_1^t ------ \\ ------ \vdots ------ \\ ------ Z_m^t ------ \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} Zx \end{bmatrix}$$

In one-bit sensing, we replace the vectors Zx with

$$\mathbf{B}_m x = sign(Zx).$$

One-bit sensing is an extreme form of non-linearity, one that has many practical applications.

Question

Can one-bit sensing effectively distinguish points with only a few bit measurements?

Background

- The Hamming cube is $\mathbb{H}_m = \{0,1\}^m$.
- $\mathbb{S}^{N-1}$ is the unit sphere $\in \mathbb{R}^N$.
- In one-bit, we can only know the direction of x, not the length.

Restricted Isometry Property

Let $0 < \delta < \frac{1}{2}$, for $\mathbf{X} \subset \mathbb{S}^{N-1}$, $\mathbf{B}_m$ satisfies δ-RIP for all pairs $x, y \in \mathbf{X}$ if:

$$|d_{\mathbb{H}_m}(\mathbf{B}_m x, \mathbf{B}_m y) - d_{geo}(x,y)| \le \delta.$$

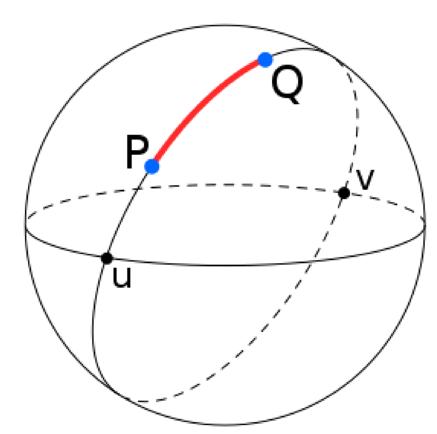
Linear Johnson-Lindenstrauss Lemma

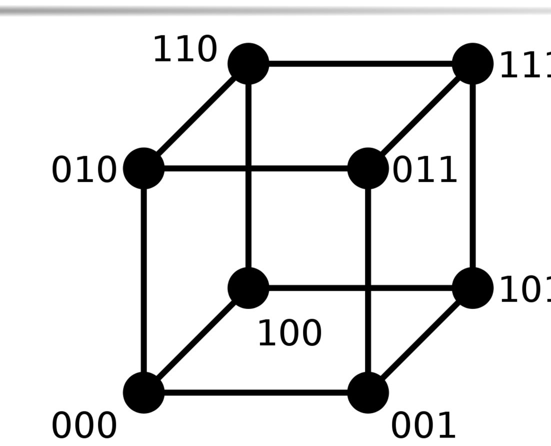
Let $0 < \delta < \frac{1}{2}$, for $\mathbf{X} \subset \mathbb{R}^N$, if $m > \frac{\ln n}{\delta^2}(1+4\delta)$, there exists a linear map $\mathbf{A}\colon \mathbb{R}^N \to \mathbb{R}^m$ such that for all $x, y \in \mathbf{X}$, $|\,\|\mathbf{A}x - \mathbf{A}y\| - \|x-y\|\,| < \delta\,\|x-y\|$.

One-bit Johnson-Lindenstrauss Lemma

Let $0 < \delta < \frac{1}{2}$, $\mathbf{X} \subset \mathbb{S}^{N-1}$, if $m > \frac{\ln n + \ln 2}{\delta^2}$, there exists a map $\mathbf{B}_m\colon \mathbb{S}^{N-1} \to \mathbb{H}_m$ which is δ-RIP.
This bound closely resembles the linear case.

Differences in Metrics





Phase Transition Theorems

1. If $\mathbf{X} \subset \mathbb{S}^{N-1}$ is n pairwise orthogonal unit vectors, $\mathbf{B}_m$ is one-to-one with probability $1 - e^{-k}$ when $m \ge 2\log_2 n + k$.

2. Let $\mathbf{X} \subset \mathbb{S}^{N-1}$ be n pairwise orthogonal unit vectors. Set $\Pi(\delta, m) = \mathbb{P}(\mathbf{B}_m \text{ is } \delta\text{-RIP})$, $M_{\delta,k} = \frac{\ln n}{\delta^2} + \frac{\ln k}{\delta^2}$.
$\Pi(\delta,m) \ge 1 - e^{-k}$ for all $m > M_{\delta,k} + \mathrm{O}\left(\ln(n+k) + \frac{\ln\ln(n+k)}{\delta^2}\right)$
$\Pi(\delta,m) \le 1 - e^{-k}$ for all $m < M_{\delta,k} - \mathrm{O}\left(\ln(n+k) + \frac{\ln\ln(n+k)}{\delta^2}\right)$.

The $\delta-$RIP Simulations

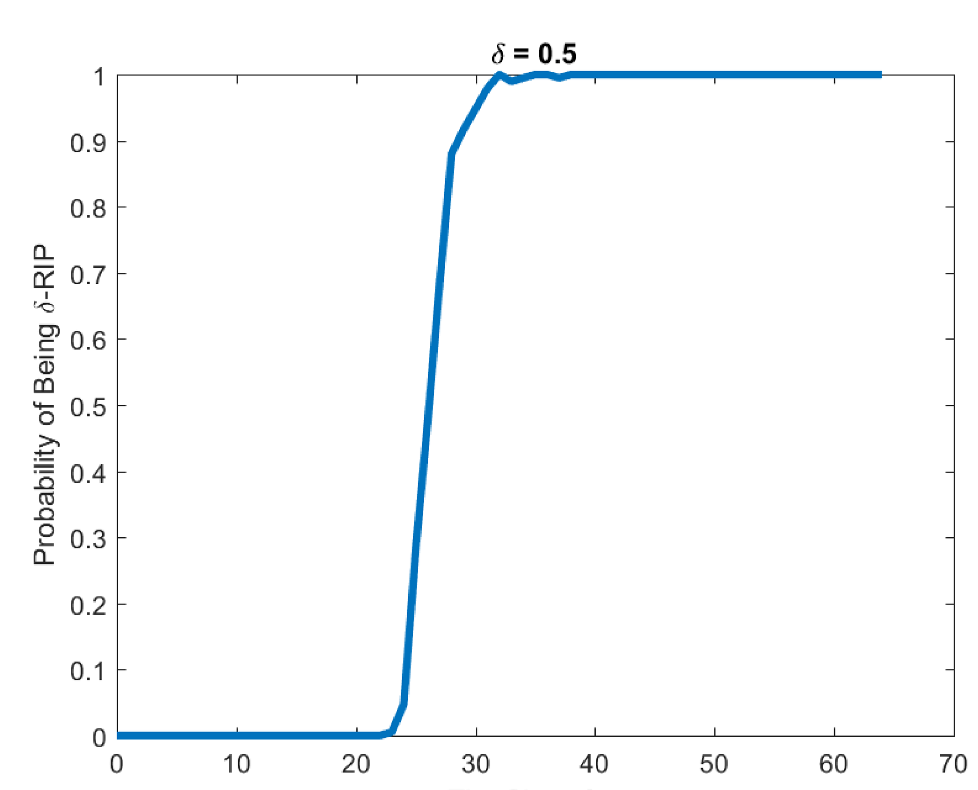


Figure 1: This is the special case of the $\frac{1}{2}$-RIP.

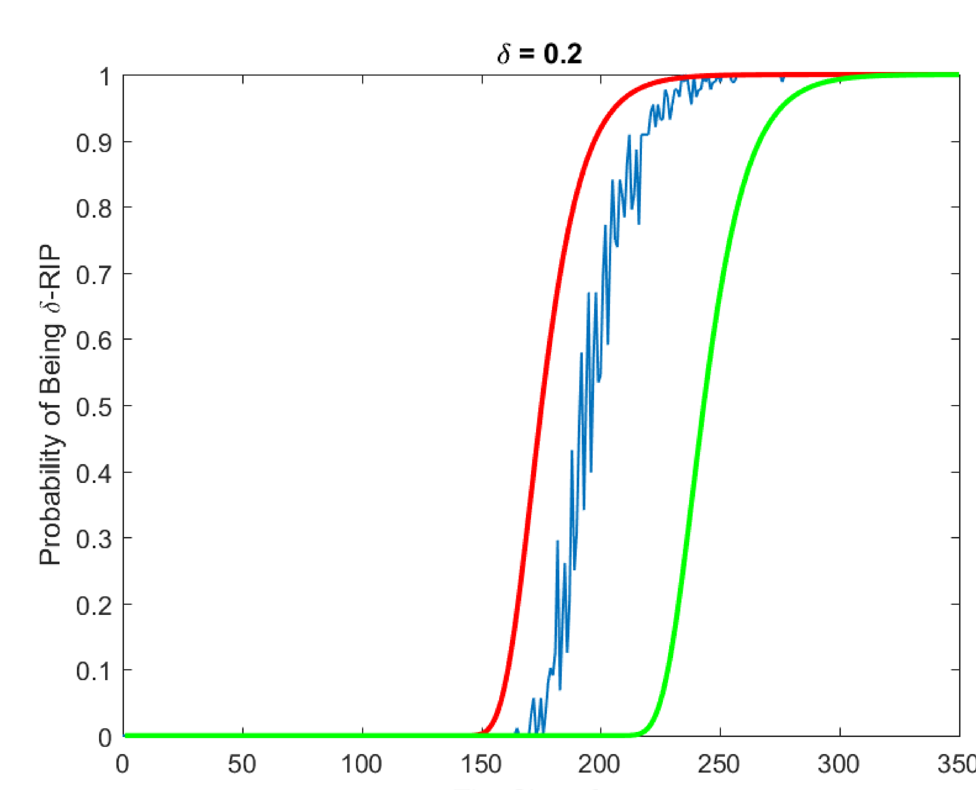
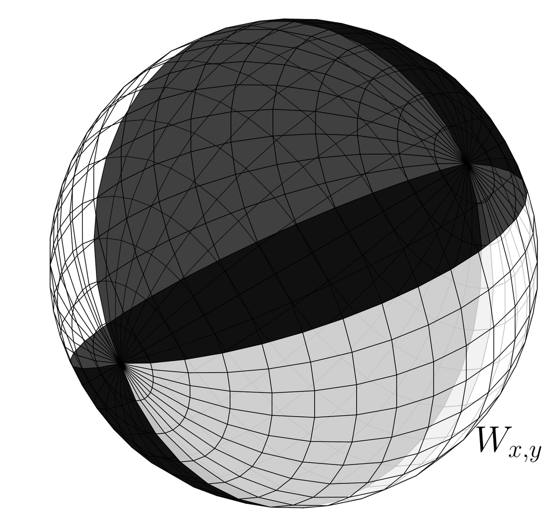


Figure 2: This is a visual representation of Theorem 2.The red and green curves effectively bound the probability curve.

The Wedge Properties



The wedge $W_{x,y}$ is defined as: $\{\theta \in \mathbb{S}^{N-1} : \operatorname{sgn}(x \cdot \theta) \ne \operatorname{sgn}(y \cdot \theta)\}$. These are the θ which distinguish between points x and y under $\mathbf{B}_m$.
Important fact:

$$\mathbb{P}(W_{x,y}) = \frac{\cos^{-1}(x \cdot y)}{\pi} = d_{geo}(x,y).$$

These observations allow us to use delicate properties of Bernoulli distributions to prove the main results.

Conclusions

The bounds in the one-bit case are the same as the bounds in the linear case, even though the one-bit case uses less information. The probability that B_m satisfies δ-RIP passes through a phase transition. It changes from zero to one in a tight window of m. Thus, we can conclude that it *is* possible to distinguish between points in a one-bit context, and preserve pairwise distances with only a few measurements.

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