**Introduction**

A line arrangement $\mathcal{A}$ is a finite collection of $n$ lines in $\mathbb{RP}^2$. A line arrangement is **simple** if every intersection point is made by two unique lines. Denote by $p_k$ the number of $k$-sided faces in the cell complex defined by $\mathcal{A}$.

- Roudneff proved in [1] that for every line $L$ of $\mathcal{A}$, there are at least three 4-gons or 5-gons having an edge on $L$, which implies:

\[
4p_4 + 5p_5 \geq 3n
\]

**Conjecture Improving the Bound**

- Let $\mathcal{A}$ be a simple arrangement of $n \geq 5$ lines. Then, $4p_4 + 5p_5 \geq 4n$. More precisely, for every line $L$ of $\mathcal{A}$, there exist at least four 4-gons and/or 5-gons having an edge on $L$.
- The conjecture is proven when there only exist 3, 4, and/or 5 gons in $\mathcal{A}$. It is unknown whether the conjecture is still true in other cases.

**Main Idea**

Let $F$ be a face of $\mathcal{A}$, $L$ and $L'$ be the lines of $\mathcal{A}$ defined by two edges of $F$. The face $F$ and the two lines $L'$ and $L'$ define four closed regions of $\mathbb{RP}^2$.

Let $E$ be either $E_1$ or $E_2$.

Let $s(E)$ be the number of edges on $L$ (equivalently $L'$) that are included in $E$.

Roudneff proved in [1] that if $s(E) \geq 2$, there exists at least one 4-gon or 5-gon of $E$ having an edge on $L$. We conjectured that if $s(E) \geq 3$, there exist at least two 4-gons and/or 5-gons of $E$ having an edge on $L$.

However, we found counterexamples to this guess.

**Counterexample**

For any arbitrary large $n$, there are infinitely many cases when $s(E) \geq n$ and there exist only one 4-gon or 5-gon of $E$ having an edge on $L$.

\[
s(E) = 7 \text{ and there is only one 5-gon inside } E \text{ having an edge on } L
\]

**Further Research**

- Prove or disprove whether the conjecture holds true when there exists one or more $k$-sided face(s) in $\mathcal{A}$, where $k \geq 6$
- Find other relationships among the $k$-gons (i.e. other inequalities among the $p_k$'s)

**Applications**

- Line arrangements in Architecture
- Oriented matroids
- Classification of structures in Computer Vision

**Acknowledgements**

We would like express our gratitude to Professor Josephine Yu for advising us during this project. Additionally, we would like to acknowledge James Anderson, Cvetelina Hill and Charles Wang for help and inspiration.