A Conjecture on the Number of Quadrilaterals and Pentagons in Simple Line Arrangements

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Introduction

- A line arrangement $\mathcal{A}$ is a finite collection of n lines in $\mathbb{RP}^2$. A line arrangement is ***simple*** if every intersection point is made by two unique lines. Denote by p_k the number of k-sided faces in the cell complex defined by $\mathcal{A}$.
- Roudneff proved in [1] that for every line $\mathcal{L}$ of $\mathcal{A}$, there are at least three 4-gons or 5-gons having an edge on $\mathcal{L}$, which implies:

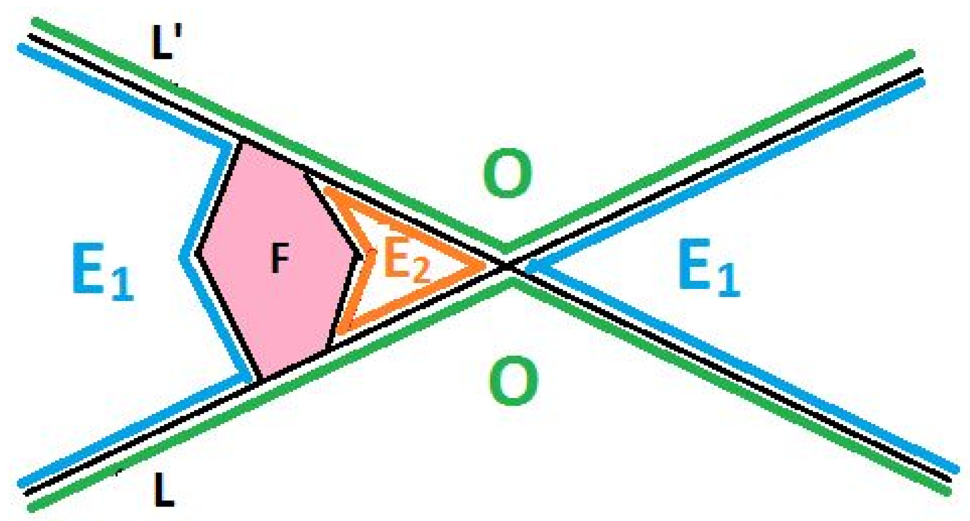
$$4p_4 + 5p_5 \geq 3n$$

Conjecture Improving the Bound

- Let $\mathcal{A}$ be a simple arrangement of $n \geq 5$ lines. Then, $\mathbf{4p_4 + 5p_5 \geq 4n}$. More precisely, for every line $\mathcal{L}$ of $\mathcal{A}$, there exist at least four 4-gons and/or 5-gons having an edge on $\mathcal{L}$.
- The conjecture is proven when there only exist 3, 4, and/or 5 gons in $\mathcal{A}$. It is unknown whether the conjecture is still true in other cases.

Main Idea

Let $\mathcal{F}$ be a face of $\mathcal{A}$, $\mathcal{L}$ and $\mathcal{L}'$ be the lines of $\mathcal{A}$ defined by two edges of $\mathcal{F}$. The face $\mathcal{F}$ and the two lines $\mathcal{L}'$ and $\mathcal{L}'$ define four closed regions of $\mathbb{RP}^2$.



Let $\mathcal{E}$ be either $\mathcal{E}_1$ or $\mathcal{E}_2$.

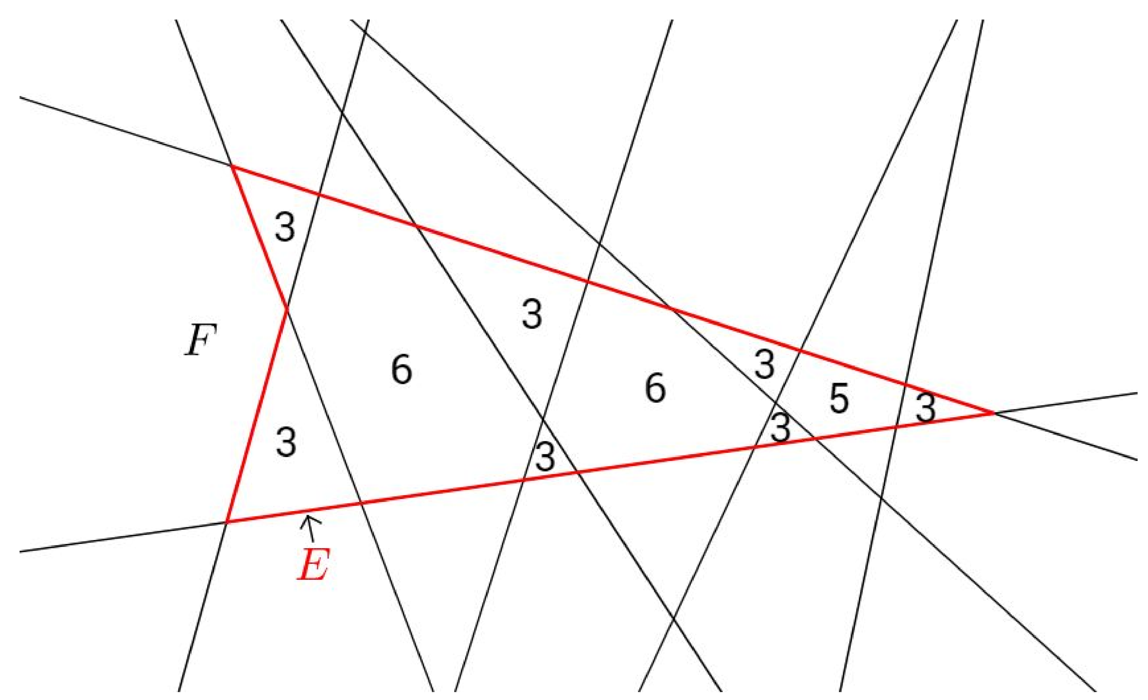
Let $s(\mathcal{E})$ be the number of edges on $\mathcal{L}$ (equivalently $\mathcal{L}'$) that are included in $\mathcal{E}$.

Roudneff proved in [1] that if $s(\mathcal{E}) \geq 2$, there exists at least one 4-gon or 5-gon of $\mathcal{E}$ having an edge on $\mathcal{L}$. We ***conjectured*** that if $s(\mathcal{E}) \geq 3$, there exist at least two 4-gons and/or 5-gons of $\mathcal{E}$ having an edge on $\mathcal{L}$. However, we found counterexamples to this guess.

Counterexample

For any arbitrary large n, there are infinitely many cases when $s(\mathcal{E}) \geq n$ and there exist only one 4-gon or 5-gon of $\mathcal{E}$ having an edge on $\mathcal{L}$.

$s(\mathcal{E})$ = 7 and there is only one 5-gon inside $\mathcal{E}$ having an edge on $\mathcal{L}$



Further Research

- Prove or disprove whether the conjecture holds true when there exists one or more k-sided face(s) in $\mathcal{A}$, where $k \geq 6$
- Find other relationships among the k-gons (i.e. other inequalities among the p_k's)

Applications

- Line arrangements in Architecture
- Oriented matroids
- Classification of structures in Computer Vision

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[1] Roudneff, J. P. (1987). Quadrilaterals and pentagons in arrangements of lines. *Geometriae Dedicata*, *23*(2), 221-227.