60 MIN TIME LIMIT.
NO NOTES. NO CALCULATORS.
GOOD LUCK!

Problem 1

(a) (5 points) Find the sum of the series

$$\sum_{n=0}^{\infty} \frac{3n+1}{n!} 8^n;$$

Hint: You may use differentiation of a certain power series to find the sum in terms of x and then plug in the appropriate value for x.

(b) (5 points) Prove that

$$\sum_{n=0}^{\infty} \frac{1}{n!(2n+1)} x^{2n+2} = x \int_0^x e^{t^2}\, dt, \;\; x \in \mathbb{R}.$$

Hint: You may find it easier to start with the right hand side; expand e^{t^2} in a power series and then use integration of power series term by term.

Problem 2

Consider the matrices

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 2 & 4 \\ 0 & 2 & -1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 1 & 1 & 2 \end{bmatrix}.$$

(a) (4 points) Compute $(AB)_{2,3} + 2(BA)_{2,1}$ without computing the whole matrix products AB and BA;

(b) (2 points) Define a *linear combination* of vectors;

(c) (4 points) Write the first column of AB as a linear combination of the columns of A.

Problem 3

(a) (4 points) State the convergence result related to differentiation of power series;

(c) (6 points) Find the radius and interval of convergence for the series

$$\sum_{n=1}^{\infty} n^2 3^{-n} (x-2)^n.$$

Problem 4

Let f be a transformation from $\mathbb{R}^2$ into itself given by

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - x_2 \\ -x_1 \end{bmatrix}.$$

(a) (5 points) Is f invertible? If so, find its inverse.

(b) (5 points) Compute the projection of $f(\mathbf{e}_1)$ onto $f(\mathbf{e}_1 + \mathbf{e}_2)$.

Problem 5

A linear transformation $f : \mathbb{R}^3 \to \mathbb{R}^3$ satisfies

$$f(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad f(\mathbf{e}_3) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad f\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right) = \mathbf{e}_2.$$

(a) (5 points) Write down the matrix of f;
Hint: One needs $f(\mathbf{e}_2)$ as well.

(b) (5 points) Write down the expression of the transformation $f \circ f$.
Hint: Recall that $A_{g \circ f} = A_g A_f$.